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Magnetic Field Dependence of the Cyclotron-Resonance Linewidth at Infrared Frequencies

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We have derived expressions for the cyclotron-resonance line-shape parameters by means of a new formalism. The formalism is based on a "damping iteration" for the exact expression of the conductivity tensor in the presence of static impurities. We have applied this theory to the case of a nondegenerate semiconductor in the quantum limit and have obtained an expression of the cyclotron-resonance linewidth for the case of dilute and weak scatterers. We have considered electron-impurity interactions with a screening radius, separately, of Gaussian and isotropically screened Coulomb forms. In both cases we have found that the cyclotron absorption line *broadens* and approaches a constant value with increasing magnetic field, in the limit of a strong enough field so that the screening radius is much larger than the cyclotron radius. This result is in contradiction to earlier theoretical calculations and a recent interpretation of some experiments. A discussion of the earlier theoretical works as well as of the experimental work of Apel *et al.* is given.

I. INTRODUCTION

The high-frequency cyclotron-resonance experiments[1–6] performed with the far-infrared lasers have opened up a new domain in semiconductor physics. Recently, not only the position of the absorption peak, but also the shape of the absorption line have been investigated. The absorption shape, and in particular the linewidth, depends sensitively on the scattering mechanisms, and thus such investigations can yield information on the detailed character of these mechanisms.[7] In particular, the magnetic field dependence of the cyclotron-resonance linewidth can complement that of the dc magnetoresistance, which is also sensitive to the scattering mechanisms, especially in the quantum limit.[8]

Of special interest in this regard is the recent experiment of Apel *et al.*[6] who measured the cyclotron-resonance linewidth in high-purity n-type InSb in the quantum limit at different values of the magnetic field. They have concluded that the linewidth *decreases* with increasing magnetic field, and cited the theoretical works of Kawamura *et al.*[3] and Kawabata,[9] which predict such a narrowing of the absorption line for the case of scattering by unscreened ionized impurities.

We have carried out a quantum-mechanical calculation of the cyclotron-resonance linewidth for a nondegenerate semiconductor in the quantum limit for a more realistic impurity potential. Specifically, we have considered a screened electron-impurity interaction, with a large screening radius, of the Gaussian or the screened-Coulomb type. By means of a new formalism[10,11] based on a *damping iteration* of the exact expression for the conductivity tensor,[12] we have obtained the resonance linewidth through a system of coupled integral equations. For the magnetic fields and temperatures of interest, we have found that, for sufficiently large screening radius, the linewidth of cyclotron resonance *increases* with increasing magnetic field and reaches a constant value. This contradicts the results of the work of Kawamura *et al.*[3] and Kawabata[9] for an unscreened impurity potential. In the limit of an infinite screening radius, our theory for the isotropically screened-Coulomb interaction predicts a linewidth that increases with the magnetic field (as for a finite screening radius), but its limiting value diverges. Interestingly, for the case of an unscreened-Coulomb interaction, we obtain a linewidth with a field dependence identical to that of Kawamura *et al.*,[3] but, again, with a divergent coefficient.

In Sec. II we formulate the problem and the method of calculation, the more technical steps being relegated to the Appendix. Application to the cases of an electron-impurity interaction of the Gaussian and the screened-Coulomb types is carried out in Sec. III and IV, respectively. In Sec. V we discuss our results in relation to those of Kawamura *et al.*[3] and Kawabata,[9] as well as the experimental results of Apel *et al.*[6] We conclude that more detailed theoretical and experimental work is desirable.

II. PROBLEM AND METHOD OF CALCULATION

The absorption of a circularly polarized electromagnetic wave of frequency ω is proportional to

$$\mathrm{Re}\sigma_{+-}(\omega)=\mathrm{Re}\,[\sigma_{xx}(\omega)+\sigma_{yy}(\omega)]+\mathrm{Im}\,[\sigma_{xy}(\omega)-\sigma_{yx}(\omega)], \tag{2.1}$$

where $\sigma_{ij}(\omega)$ $(i,j=x,y,z)$ is the complex conductivity tensor for the system. For a system of dynamically independent electrons of density n_e, the exact expression[13] for the conductivity tensor is

$$\sigma_{ij}(\omega)=(e^2n_e/\hbar\omega)\int_0^\infty dt\,e^{-izt}\langle[v_j,v_i(t)]\rangle, \tag{2.2}$$

where $z=\omega-i0^+$ (0^+ stands for a positive infinitesimal) and

$$v_i(t)=e^{iHt/\hbar}v_ie^{-iHt/\hbar}=e^{i\mathcal{L}t/\hbar}v_i \tag{2.3}$$

is the ith component of the velocity operator in the Heisenberg picture. Here $\mathcal{L}$ is the Liouville operator corresponding to the one-electron Hamiltonian

$$H=H_0+V, \tag{2.4}$$

where H_0 is the Hamiltonian of the electron in the presence of a static magnetic field $\vec{B}$ and $V(\vec{r})=\sum_i w(\vec{r}-\vec{R}_i)$ is the interaction of the electron with the static impurities. Thus

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1, \tag{2.5}$$

with

$$\mathcal{L}_0v_i=[H_0,v_i], \tag{2.6}$$

$$\mathcal{L}_1v_i=[V,v_i]. \tag{2.7}$$

In Eq. (2.2) $\langle\rangle$ denotes the thermal equilibrium average

$$\langle A\rangle=\mathrm{tr}(fA)/\mathrm{tr}f, \tag{2.8}$$

where $f=f(H)$ is the Fermi-Dirac distribution operator

$$f(H)=(e^{\beta(H-\zeta)}+1)^{-1}. \tag{2.9}$$

The chemical potential ζ is determined from $\mathrm{tr}f=n_e\Omega$, Ω being the volume of the sample and $\beta=1/k_BT$. Finally, though not explicitly indicated, expression (2.2) must be averaged over the random distribution of the impurities.

The spectral properties of the power absorption are thus determined, according to (2.2) and (2.3), by

$$(i\hbar)^{-1}\int_0^\infty dt\,e^{-izt}v_i(t)=\mathcal{R}(z)v_i, \tag{2.10}$$

where

$$\mathcal{R}(z)=(\mathcal{L}-\hbar z)^{-1} \tag{2.11}$$

is the resolvent for the Liouville operator $\mathcal{L}$. This Liouville form of the resolvent is related to the regular resolvent $R(z)\equiv(H-\hbar z)^{-1}$ by the integral[14]

$$\mathcal{R}(z)v_i=(\hbar/2\pi i)\int_{-\infty}^{+\infty}d\omega'\,R(\omega-\omega'-i0^+)\times v_iR(-\omega'+i0^+). \tag{2.12}$$

This can be proved most simply by considering its matrix elements in a representation that diagonalizes H. The resonance-line-shape parameters are thus determined rigorously by $\mathcal{R}(z)$ of (2.10). In the following we shall obtain them by generating a so-called "damping iteration" for $\mathcal{R}(z)$ and by keeping the leading terms.

The damping iteration can be carried out either directly for $\mathcal{R}(z)$,[10,15–17] or for the resolvents $R(z)$. In the latter case we obtain first the *complex level-shift* (i.e., energy shift and level broadening), but still need to carry out the ω' integration in (2.12), which is often complicated. By carrying out the damping iteration for $\mathcal{R}(z)$[10,15] directly, we avoid the ω' integration. We derive such a damping iteration for $\mathcal{R}(z)$, and derive its dominant terms for weak electron-impurity interaction in the Appendix. We find, according to (A16), that if $|a\rangle$, $|b\rangle$ are eigenstates of H_0 with corresponding energies ϵ_a, ϵ_b,

$$\langle a|\mathcal{R}(z)v_i|b\rangle=[\epsilon_{ab}+i\hbar\Gamma(z)_{(a,b)}-\hbar z]^{-1}\langle a|v_i|b\rangle, \tag{2.13}$$

where $\epsilon_{ab}\equiv\epsilon_a-\epsilon_b$. The *complex linewidth* $\Gamma(z)_{(a,b)}$ for the transition $b\to a$ is obtained from (A18), namely,

$$\Gamma(z)_{(a,b)}=\frac{i}{\hbar}\sum_c{}'\left(\frac{|\langle a|V|c\rangle|^2}{(\epsilon_{cb}+i\hbar\Gamma(z)_{(c,b)}-\hbar z}+\frac{|\langle c|V|b\rangle|^2}{\epsilon_{ac}+i\hbar\Gamma(z)_{(a,c)}-\hbar z}\right), \tag{2.14}$$

where the prime on the summation sign indicates exclusion of the diagonal matrix elements of V. An average over the random distribution of the scatterers is understood for $|\langle a|V|c\rangle|^2$ and $|\langle c|V|b\rangle|^2$. The real and imaginary parts of $\Gamma(\omega)_{(a,b)}$, namely,

$$\Gamma(\omega)_{(a,b)}=\gamma(\omega)_{ab}-i\Delta(\omega)_{ab}, \tag{2.15}$$

define the *linewidth* and *line shift*, respectively, for the resonance transition $b\to a$. From Eq. (2.14) we have for the positive definite linewidth $\gamma(\omega)_{ab}$,

$$\gamma(\omega)_{ab}=\frac{1}{\hbar^2}\,\mathrm{Re}\sum_c{}'\left(\frac{|\langle a|V|c\rangle|^2}{i(\omega-\omega_{cb})+\gamma(\omega)_{cb}}+\frac{|\langle c|V|b\rangle|^2}{i(\omega-\omega_{ac})+\gamma(\omega)_{ac}}\right) \tag{2.16}$$

and for the line shift $\Delta(\omega)_{ab}$,

$$\Delta(\omega)_{ab}=-\frac{1}{\hbar^2}\,\mathrm{Im}\sum_c{}'\left(\frac{|\langle a|V|c\rangle|^2}{i(\omega-\omega_{cb})+\gamma(\omega)_{cb}}+\frac{|\langle c|V|b\rangle|^2}{i(\omega-\omega_{ac})+\gamma(\omega)_{ac}}\right), \tag{2.17}$$

where

$$\omega_{ac}\equiv(\epsilon_a-\epsilon_c)/\hbar+\Delta(\omega)_{ac} \tag{2.18}$$

is the modified energy difference. Note that the line-shape parameters for the transition $b\to a$ depend on those of transitions $b\to c$ and $c\to a$ for all scattering states c.

The simplest, but not always justified, approximation to Eqs. (2.16)–(2.18) is to solve them by iteration and keep only the first iteration obtained by putting $\gamma=\Delta=0$ in the right-hand side and to evaluate them at the resonant frequency $\omega\cong\epsilon_{ab}/\hbar$. In such a case they reduce to the familiar "golden-rule" formulas of the simple perturbation theory,[18] i.e.,

$$\gamma_{ab}=\gamma_{ba}=\tfrac{1}{2}(\gamma_a+\gamma_b), \tag{2.19a}$$

$$\Delta_{ab}=-\Delta_{ba}=\Delta_a-\Delta_b, \tag{2.19b}$$

where

$$\gamma_a=\frac{2\pi}{\hbar}\sum_c{}'|\langle a|V|c\rangle|^2\delta(\epsilon_{ac}), \tag{2.20a}$$

$$\Delta_a=\frac{1}{\hbar}\sum_c{}'\frac{|\langle a|V|c\rangle|^2}{(\epsilon_{ac})_p}. \tag{2.20b}$$

More generally, however, the symmetry apparent in the approximate Eqs. (2.19) under the interchange $(a,b)\leftrightarrow(b,a)$ is broken in Eqs. (2.16) and (2.17). The same symmetry is restored, however, in $\gamma(\omega)_{ab}$ and $\Delta(\omega)_{ab}$ when evaluated at the peak frequency $\omega=\epsilon_{ab}/\hbar$ and at high enough magnetic fields so that $\hbar\Delta_{ab}\ll\epsilon_{ab}$. In the calculations to follow, we shall assume that this is the region of interest.

Combining now Eqs. (2.1), (2.2), (2.10), and (2.13), we have that the power absorption is proportional to

$$\mathrm{Re}\sigma_{+-}(\omega)=\frac{e^2}{\hbar\omega}\,\mathrm{Re}\,\frac{1}{\Omega}\sum_{a,b}\frac{[f(\epsilon_b)-f(\epsilon_a)]\,|\langle a|v_+|b\rangle|^2}{i(\omega-\omega_{ab})+\gamma(\omega)_{ab}}, \tag{2.21}$$

where $v_+\equiv v_x+iv_y$. In obtaining (2.21) we have ignored the effect of the electron-impurity interaction on the thermal distribution of the electrons.

It is convenient to adopt the asymmetric gauge $\vec{A}=(0,Bx,0)$ for the magnetic field B in the z direction. The eigenstates $|a\rangle$ and eigenvalues ϵ_a of H_0 for a quasifree electron with effective mass m are then specified by $n=0,1,2,\ldots$, and $k=(k_y,k_z)$, a two-dimensional wave vector, so that

$$\langle\vec{r}|nk\rangle\propto\varphi_n(x-X_k)\,e^{i\vec{k}\cdot\vec{r}}, \tag{2.22}$$

$$\epsilon_{nk}=(n+\tfrac{1}{2})\,\hbar\omega_c+\hbar^2k_z^2/2m, \tag{2.23}$$

where $\varphi_n(x-X_k)$ are the eigenfunctions of a simple harmonic oscillator of frequency $\omega_c\equiv|e|B/mc$—the cyclotron frequency—centered at $X_k=-\hbar k_y/m\omega_c$. (As all interactions considered here are spin independent, we can ignore the spin quantum number.) In this representation the matrix elements of the operator $v_+=[p_x+i(p_y+m\omega_cx)]/m$ that enter the expression (2.21) for the power absorption obey the selection rule

$$\langle n'k'|v_+|nk\rangle=2i(\hbar\omega_c/2m)^{1/2}(n+1)^{1/2}\,\delta_{k',k}\,\delta_{n',(n+1)}. \tag{2.24}$$

For the determination of $\gamma(\omega)_{(n+1)k,nk}$ from Eq. (2.16) we need the matrix elements $|\langle n'k'|V|nk\rangle|^2$ of the impurity potential V, averaged over the random distribution of the scattering centers. We have[19]

$$|\langle n'k'|V|nk\rangle|^2=\frac{n_i}{\Omega}\sum_{\vec{q}}|w(\vec{q})|^2G(\mu)_{n'n}\,\delta_{k_y',k_y+q_y}\,\delta_{k_z',k_z+q_z}, \tag{2.25}$$

where n_i is the concentration of the impurities, $w(\vec{q})$ is the Fourier transform of the potential of a single impurity at the origin, and

$$G(\mu)_{n'n}=\left|\int_{-\infty}^{+\infty}dx\,e^{ixq_x}\varphi_{n'}[x+\lambda^2(k_y+q_y)]\times\varphi_n(x+\lambda^2k_y)\right|^2, \tag{2.26}$$

with $\mu=\lambda^2(q_x^2+q_y^2)$ and $\lambda=(\hbar/m\omega_c)^{1/2}$ being the cyclotron radius for the electron in its ground state. In particular, we have

$$G(\mu)_{00}=e^{-\mu/2},\quad G(\mu)_{11}=e^{-\mu/2}(1-\tfrac{1}{2}\mu)^2,\quad G(\mu)_{10}=G(\mu)_{01}=\tfrac{1}{2}\mu e^{-\mu/2}. \tag{2.27}$$

We note that the necessary matrix elements of V depend on $(k_y'-k_y)$, while $\epsilon_{n'k'}-\epsilon_{nk}$ are independent of k_y and k_y'. It follows that there exist solutions $\gamma(\omega)_{n'k',nk}$ of the system of coupled integral equations (2.16) that are independent of k_y, k_y'. These are the desired solutions, appropriate to a spatially uniform system.

As mentioned earlier, we are interested in the power absorption in high enough magnetic fields so that $\Delta(\omega_c)_{(n+1)k,nk}\ll\omega_c$. Furthermore, we consider a nondegenerate semiconductor at low temperatures so that $k_BT\ll\hbar\omega_c$. Thus the electrons

are in the $n=0$ Landau state and the resonance is dominated by the $(n=0)\rightarrow(n'=1)$ transition. In such a case, we have from (2.21), (2.18), and (2.24),

$$\mathrm{Re}\sigma_{+-}(\omega_c)\cong\frac{e^2}{\hbar\omega_c}\frac{1}{\Omega}\sum_k f(\epsilon_{0k})\frac{|\langle 1k|v_+|0k\rangle|^2}{\gamma(\omega_c)_{1k_z,0k_z}}$$

$$=\frac{2e^2n_e}{m}\left\langle\frac{1}{\gamma(\omega_c)_{1k_z,0k_z}}\right\rangle$$

$$\cong\frac{2e^2n_e}{m}\frac{1}{\gamma(\omega_c)_{10}}\,. \qquad (2.28)$$

Here $\langle\rangle$ indicates the thermal average with respect to the normalized Maxwell–Boltzmann distribution function proportional to $\exp(-\beta\hbar^2k_z^2/2m)$, and the last approximate equality follows from the fact that since $\langle\hbar^2k_z^2/2m\rangle=\frac{1}{2}k_BT\ll\hbar\omega_c$ for the temperatures of interest, we can approximate the thermal average of $1/\gamma(\omega_c)_{1k_z,0k_z}$ by its value at $k_z=0$ (or some mean value of k_z), to be denoted by $1/\gamma(\omega_c)_{10}=1/\gamma_{10}$. Clearly, for the experimentally interesting case of $(\omega_c/\gamma_{10})^2\gg1$, γ_{10} is the resonance linewidth.

Now, according to (2.16), (2.18), (2.23), and (2.25), we have for high magnetic fields $[\omega_c\gg\Delta(\omega_c)]$

$$\gamma_{10}=\frac{n_i}{\hbar^2}\,\mathrm{Re}\,\frac{1}{\Omega}\sum_{\vec{q}}|w(\vec{q})|^2$$

$$\times\sum_n\left(\frac{G(\mu)_{1n}}{i[(1-n)\omega_c+\omega_{q_z}]+\gamma_{nq_z,00}}+\frac{G(\mu)_{n0}}{i(n\omega_c+\omega_{q_z})+\gamma_{10,nq_z}}\right), \qquad (2.29)$$

where all γ's are evaluated at $\omega=\omega_c$, $\omega_{q_z}=\hbar q_z^2/2m$, and $G(\mu)_{nn'}$ are given by (2.26) with $\mu=\lambda^2(q_x^2+q_y^2)$, as before. The terms in (2.29) proportional to G_{nn} describe the contribution of the "adiabatic" scattering processes, while those proportional to $G_{nn'}$ $(n'\neq n)$ describe the "nonadiabatic" processes. It is clear that their contributions to γ_{10} are not simply additive.

In Secs. III and IV we shall consider two examples of an electron-impurity interaction with long-range shielding and solve the coupled Eq. (2.16) in order to obtain the magnetic field dependence of the cyclotron-resonance linewidth.

III. GAUSSIAN SCATTERING POTENTIAL

We consider here impurities of low concentration n_i and with single-center scattering potential of Gaussian form with effective range a. The Fourier transform $w(\vec{q})$ is then

$$w(\vec{q})=w_0e^{-a^2q^2}. \qquad (3.1)$$

The resonance absorption linewidth, according to (2.29) and (3.1), is given in this case by

$$\gamma_{10}=\frac{n_iw_0^2}{8\pi^2\hbar^2}\frac{1}{\lambda^2}\,\mathrm{Re}\int_0^\infty d\mu\int_{-\infty}^{+\infty}dq_ze^{-2a^2(q_z^2+\lambda^{-2}\mu)}$$

$$\times\sum_n\left(\frac{G(\mu)_{1n}}{i[(1-n)\omega_c-\omega_{q_z}]+\gamma_{nq_z,00}}+\frac{G(\mu)_{n0}}{i(n\omega_c+\omega_{q_z})+\gamma_{10,nq_z}}\right), \qquad (3.2)$$

where $\omega_{q_z}=\hbar q_z^2/2m$. The $\gamma_{nq_z,n'q_z'}$ in the right-hand side of (3.2) are, in turn, determined by (2.16), thus constituting a system of coupled integral equations.

However, since the main contribution to the q_z integral comes from values of $|q_z|<1/a$, one may approximate the γ's by their values at $q_z=0$, especially for large a, thereby supposing a smooth dependence of the linewidths on q_z. This approximation yields for the $\gamma_{nn'}$ a system of coupled algebraic equations. In order to obtain the coefficients of this system, it is consistent to ignore the $\omega_{q_z}=\hbar q_z^2/2m$ in the denominators of (3.2) for large enough a, i.e., ω_c, $\gamma\gg\hbar/4ma^2$, or equivalently $(a/\lambda)^2\gg1$ (since usually $\omega_c>\gamma$). We then have for the linewidths $\gamma(\omega_c)_{nn'}=\gamma_{nn'}$ the algebraic system of equations

$$\gamma_{nn'}=\mathrm{Re}\sum_m\left(\frac{\alpha_{nm}}{i(1-m+n')\omega_c+\gamma_{mn'}}+\frac{\alpha_{mn'}}{i(1-n+m)\omega_c+\gamma_{nm}}\right). \qquad (3.3)$$

Here $\alpha_{nn'}=\alpha_{n'n}$ are the real quantities

$$\alpha_{nn'}=[n_i/(2\pi)^3\hbar^2]\int d^3q\,|w(\vec{q})|^2G(\mu)_{nn'} \qquad (3.4a)$$

$$=A^2\int_0^\infty dy\,e^{-y}G[(\lambda^2/2a^2)y]_{nn'}, \qquad (3.4b)$$

with

$$A^2=n_iw_0^2/\hbar^2(8\pi)^{3/2}a^3 \qquad (3.5)$$

independent of the magnetic field.

It is now clear that for large magnetic fields the contributions of the Landau states diminish with increasing quantum number m. Not only the denominators become larger, but also α_{nm} become smaller, since $G(y)_{nm}$ for $n,m>1$ have their maxima away from $y=0$, while the factor e^{-y} in (3.4b) weighs heavily the region around $y=0$, especially for $(a/\lambda)^2\gg1$. We shall thus keep in (3.3) only the states $m=0$, 1, thereby obtaining the system of three equations for γ_{00}, γ_{11}, and γ_{10}:

$$\gamma_{10}=\frac{\alpha_{00}+\alpha_{11}}{\gamma_{10}}+\alpha_{10}\left(\frac{\gamma_{00}}{\omega_c^2+\gamma_{00}^2}+\frac{\gamma_{11}}{\omega_c^2+\gamma_{11}^2}\right), \qquad (3.6a)$$

$$\gamma_{00}=2\alpha_{00}\frac{\gamma_{00}}{\omega_c^2+\gamma_{00}^2}+\alpha_{10}\left(\frac{1}{\gamma_{10}}+\frac{\gamma_{10}}{4\omega_c^2+\gamma_{10}^2}\right), \qquad (3.6b)$$

$$\gamma_{11} = 2\alpha_{11}\frac{\gamma_{11}}{\omega_c^2+\gamma_{11}^2} + \alpha_{10}\left(\frac{1}{\gamma_{10}} + \frac{\gamma_{10}}{4\omega_c^2+\gamma_{10}^2}\right), \tag{3.6c}$$

which are valid for aribtrary values of ω_c/γ. Now for the magnetic fields of interest we have $(\omega_c/\gamma)^2 \gg 1$. We find then from (3.6) for the linewidth γ_{10},

$$\gamma_{10} \cong \left[\alpha_{00}+\alpha_{11}+\alpha_{10}^2\left(\frac{1}{\omega_c^2-2\alpha_{00}} + \frac{1}{\omega_c^2-2\alpha_{11}}\right)\right]^{1/2}. \tag{3.7}$$

The $\alpha_{nn'}$ of interest are obtained from (3.4b) and (2.27), and are explicitly

$$\alpha_{00} = A^2(1+x)^{-1} \cong A^2(1-x+x^2-\cdots),$$
$$\alpha_{11} = A^2(1+x^2)(1+x)^{-3} \cong A^2(1-3x+7x^2-\cdots), \tag{3.8}$$
$$\alpha_{10} = A^2x(1+x)^{-2} \cong A^2x(1-2x+\cdots),$$

where the parameter

$$x \equiv \lambda^2/4a^2 = \hbar/4m\omega_c a^2 \tag{3.9}$$

is $\ll 1$ for the magnetic fields of interest. We note that $\alpha_{10} \ll \alpha_{00}$, α_{11}, and $\omega_c^2 \gg A^2$ for the fields of interest, and thus (3.7) gives

$$\gamma_{10} \cong (\alpha_{00}+\alpha_{11})^{1/2} + \alpha_{10}^2\,\omega_c^{-2}(\alpha_{00}+\alpha_{11})^{-1/2}. \tag{3.10}$$

Note that even in this limiting case γ_{10} does not separate into a simple sum of "adiabatic" and "nonadiabatic" contributions, although clearly the first term, described solely in terms of the "adiabatic" processes, dominates over the second, which involves "nonadiabatic" processes, as the magentic field increases.

We can now look at the magnetic field dependence of the resonance linewidth $\gamma(\omega_c)_{10} = \gamma_{10}$ for high magnetic fields. For such fields, $x = \lambda^2/4a^2 \ll 1$ and all approximations made so far are consistent. Thus for $x \ll 1$, we have from (3.10) and (3.8) that the "adiabatic" processes dominate and

$$\gamma_{10} \cong (\alpha_{00}+\alpha_{11})^{1/2} = A(\sqrt{2})(1-x), \tag{3.11}$$

where A, as given by (3.5), is independent of the magnetic field and varies as $a^{-3/2}$. Thus as the magnetic field B increases, the resonance line *broadens* like $[1-(\frac{1}{4}\hbar c/|e|a^2)B^{-1}]$ and reaches a limit $\gamma(\infty)_{10} = w_0(2n_i)^{1/2}/\hbar(8\pi a^2)^{3/4}$.

IV. SCREENED-COULOMB SCATTERING POTENTIAL

As a more realistic scattering potential of range a, we consider now the Coulomb potential screened according to the Debye–Thomas–Fermi approximation. In such a case we have

$$w(\vec{q}) = w_0/(q^2+a^{-2}), \tag{4.1}$$

where

$$w_0 = \pm 4\pi e^2 Z/K_L \tag{4.2}$$

for an ionized impurity of charge $\pm Z|e|$ and a crystal of dielectric constant K_L. According to the simple theory of screening of Argyres and Adams[20] for a nondegenerate semiconductor in the quantum limit, the screening length a in (4.1) is independent of the magnetic field and equal to the classical Debye screening length.

As in the previous case, the main contribution to the q_z integral for the γ's [see Eq. (3.2)] comes from $|q_z| < 1/a$, and thus for high magnetic fields, so that ω_c, $\gamma \gg \hbar/ma^2$, the $\gamma_{nn'}$ are given by the system of equations (3.3). The coefficients $\alpha_{nn'}$ are again obtained from (3.4a) with $w(\vec{q})$ now given by (4.1). We have explicitly

$$\alpha_{nn'} = C^2\int_0^\infty dy\,(y+a^{-2})^{-3/2}G(\lambda^2 y)_{nn'}, \tag{4.3}$$

where C^2 is independent of both the magnetic field and the range of the potential, namely,

$$C^2 = n_i\pi(e^2Z/\hbar K_L)^2. \tag{4.4}$$

As before, for high magnetic fields and for $a^2/\lambda^2 \gg 1$, we can ignore in (3.3) the terms $m > 1$, and thereby obtain the system of three equations (3.6) which determine the linewidth γ_{10}. Note for future reference that Eqs. (3.6) were derived without any assumption about the relative magnitudes of ω_c and γ_{10}, γ_{00}, γ_{11}.

The coefficients $\alpha_{nn'}$ of interest are obtained from (4.3) in terms of the incomplete Γ function, and for the fields of interest for which $x = \lambda^2/4a^2 \ll 1$, they are explicitly

$$\alpha_{00} \cong 2C^2a[1-(2\pi x)^{1/2}],$$
$$\alpha_{11} \cong 2C^2a[1-\tfrac{7}{4}(2\pi x)^{1/2}], \quad \alpha_{10} \cong C^2a(2\pi x)^{1/2}. \tag{4.5}$$

Now, for an impurity potential of arbitrary but finite a, the linewidths γ are finite and the magnetic fields of interest are high enough so that $(\omega_c/\gamma)^2 \gg 1$. Then, as in the previous case, Eqs. (3.6) can be solved simply, and γ_{10} is given by (3.10). For $x \ll 1$ the "adiabatic" processes dominate and

$$\gamma_{10} \cong (\alpha_{00}+\alpha_{11})^{1/2} = 2Ca^{1/2}[1-\tfrac{11}{16}(2\pi x)^{1/2}]. \tag{4.6}$$

Thus, as the magnetic field B increases the resonance line *broadens* as

$$1-\tfrac{11}{16}(\pi\hbar c/2|e|a^2)^{1/2}B^{-1/2}$$

and reaches a limit

$$\gamma(\infty)_{10} = [2(n_i\pi)^{1/2}Ze^2/\hbar K_L]a^{1/2}. \tag{4.7}$$

Thus the broadening of the resonance line and its "saturation" with increasing magnetic field for this case are similar to the results of Sec. III, except that the B dependence of the approach to the limiting value is different, owing to the different kind of screening.

Up to now the screening radius a has been taken

to be large but finite, so that $(\omega_c/\gamma)^2 \gg 1$. It is of some interest to inquire about the case of an unscreened impurity potential—unrealistic as this might be. This may be found as the limiting case of the screened potential by letting $a \to \infty$. Anticipating that in such a case γ_{10} might increase without limit, we cannot uncritically use expression (3.10) for γ_{10}, since it was derived under the assumption $(\omega_c/\gamma)^2 \gg 1$. However, from (3.6) we can see that as $a \to \infty$,

$$\gamma_{10} \cong (\alpha_{00} + \alpha_{11})^{1/2}, \tag{4.8}$$

as it easily follows from the structure of Eqs. (3.6) and the fact that, according to (4.5), α_{00}, $\alpha_{11} \gg \alpha_{10}$ as $a \to \infty$. Thus for the limiting case of $a \to \infty$, γ_{10} is again given by (4.6), which clearly diverges as $a \to \infty$.

If we consider from the start an unscreened-Coulomb interaction, the coefficients $\alpha_{nn'}$ are given by (4.3) with $a^{-2} = 0$, i.e.,

$$\alpha_{nn'} = C^2 \int_0^\infty dy\, y^{-3/2} G(\lambda^2 y)_{nn'}$$

$$= C^2 \lambda \int_0^\infty dy\, y^{-3/2} G(y)_{nn'}. \tag{4.9}$$

Then the quantity γ_{10} becomes

$$\gamma_{10} = (\alpha_{00} + \alpha_{11})^{1/2} = C' \lambda^{1/2} = C' (\hbar/m\omega_c)^{1/4}, \tag{4.10}$$

where

$$C' = C \{ \int_0^\infty dy\, y^{-3/2} [G(y)_{00} + G(y)_{11}] \}^{1/2} \tag{4.11}$$

is independent of B, but clearly infinitely large, as the integrals diverge. If one introduced an *ad hoc* cut-off parameter in order to make the integral convergent, then γ_{10} would *decrease* with increasing magnetic field according to $B^{-1/4}$.

It is clearly important for such a calculation how the screening, or, equivalently, an effective cutoff, is introduced into the theory. This is, in fact, the main purpose of the discussion in the last two paragraphs. The consideration of a screened-Coulomb potential with large screening radius a that we presented above is the most dependable approach. In another publication we shall, in fact, show that a more precise treatment of the screening in the presence of a magnetic field[21] yields a finite linewidth that increases and reaches a constant value with increasing B.

The divergence of the linewidth as $a \to \infty$ is most probably a consequence of the Born approximation for the cross section, and a better approximation can yield a finite linewidth. However, the discussion above indicates that an electron-impurity interaction with very large screening radius is an ineffective scattering mechanism insofar as the cyclotron-resonance linewidth in the quantum limit is concerned. Therefore other scattering mechanisms could dominate in this region, although they might *not* be the dominant mechanisms for the case of weak fields, or for the dc conductivity, with or without a magnetic field. However, for the parameters of the experiments of Apel *et al.*[6] we have calculated that the linewidth as given by this theory is roughly equal to the observed one.

V. DISCUSSION

The previous two model electron-impurity potentials with a finite, but large, screening radius yield, according to this theory, a cyclotron-resonance linewidth that *increases* and reaches a constant value with increasing magnetic field. This is in contradiction to the results of Kawamura *et al.*[3] and Kawabata,[9] which give a linewidth that *decreases* with increasing magnetic field for an unscreened electron-impurity interaction.

The work of Kawamura *et al.*[3] is of an intuitive, semiclassical nature, and thus it is difficult to compare it to ours. (See also the criticism of Miyake.[7]) However, we may observe that the field dependence of the linewidth in the quantum limit in the theory of Kawamura *et al.* is essentially determined by the *ad hoc* introduction of a cut-off length, taken to be equal to $\lambda = (\hbar/m\omega_c)^{1/2}$, for *short* distances of an unscreened-Coulomb interaction. The discussion in Sec. IV shows that the field dependence of the linewidth is critically dependent on the detailed character of the scattering potential, and that the *ad hoc* introduction of cut-off lengths is, therefore, untrustworthy.

The work of Kawabata[9] is entirely quantum mechanical, but it is concerned only with the linewidth due to the "nonadiabatic" processes. These, as we have seen, make a negligible contribution to the linewidth in the quantum limit, especially for a large screening radius. Thus a comparison with our results is meaningless. We must note, however, that according to our theory the contributions of the "nonadiabatic" and "adiabatic" processes to the linewidths are *not* additive, as Kawabata's calculation seems to imply.

The experimental results of Apel *et al.*,[6] as analyzed by the same authors[6] on the assumption of considering the charged-impurity scattering as the dominant mechanism, are also at variance with the results of our calculations. It must be pointed out, however, that a somewhat involved analysis of the experimental data is necessary before one can ascertain the field dependence of the cyclotron-resonance linewidth; as the effects of plasma, freeze-out, size, etc., have to be taken into consideration. Apel *et al.* have considered such effects in their interpretation of the data, but perhaps a careful reexamination is desirable. In addition there are, unfortunately, only three experimental points, one of which may *not* be as accurate as the others, in view of the proximity of an impurity resonance line

to the cyclotron-resonance line for that value of the magnetic field. Be that as it may, more experimental data are clearly desirable.

Finally, the theory we presented above has its limitations. The neglect of the contribution of the nondiagonal part of the resolvent $\mathcal{R}(z)$ (see the Appendix) to the power absorption is certainly an approximation, which, however, is felt to be valid for sufficiently dilute and weak scattering centers, at least near resonance. Furthermore, as mentioned in the Appendix, the averaging over the random distribution of the impurities is *not* performed rigorously. The possible consequences of the approximate treatment presented above, insofar as the field dependence of the linewidth is concerned, are difficult to assess in the framework of this theory. The other convenient simplifications made—such as the energy spectrum of the electron as well as the momentum dependence of the denominators of the coupled equations (3.2)—do not appear to be crucial approximations for sufficiently large screening radius. It is clearly desirable that a more critical theoretical investigation be carried out for this and other possible scattering mechanisms.

APPENDIX A: DAMPING ITERATION FOR $\mathcal{R}(z)$ AND BASIC APPROXIMATIONS

Consider an operator O, which, as does $\mathcal{R}(z)$, $\mathcal{L}$, etc., operates in the space of the operators of the system, and define its *diagonal* part O_D so that for any ordinary operator A of the system we have

$$\langle a|O_D A|b\rangle = O_{D(a,b)}\langle a|A|b\rangle, \tag{A1}$$

where the states $|a\rangle$, $|b\rangle$ are the eigenstates of H_0, i.e.,

$$H_0|a\rangle = \epsilon_a|a\rangle. \tag{A2}$$

The remainder $O - O_D$ is called its *nondiagonal* part O_N. Thus in $\langle a|O_N A|b\rangle$ no elements involving the matrix element $\langle a|A|b\rangle$ exist. Therefore, the Liouville operator $\mathcal{L}_0$ is diagonal and $\mathcal{L}_{0(a,b)} = \epsilon_a - \epsilon_b \equiv \epsilon_{ab}$. In contrast, $\mathcal{L}_1 = \mathcal{L}_{1D} + \mathcal{L}_{1N}$ has diagonal and nondiagonal parts, where

$$\langle a|\mathcal{L}_{1D}A|b\rangle = (\langle a|V|a\rangle - \langle b|V|b\rangle)\langle a|A|b\rangle, \tag{A3}$$

$$\langle a|\mathcal{L}_{1N}A|b\rangle = \sum_c{}' (\langle a|V|c\rangle\langle c|A|b\rangle - \langle a|A|c\rangle\langle c|V|b\rangle), \tag{A4}$$

and the prime on the summation sign indicates that $c \neq a$ in the first term and $c \neq b$ in the second term.

We may represent O_D and O_N in terms of an operator $\mathcal{P}$, i.e.,

$$O_D = \mathcal{P}O, \tag{A5}$$

$$O_N = (1 - \mathcal{P})O \equiv \mathcal{P}'O. \tag{A6}$$

Clearly, $\mathcal{P}$ is a projection operator, i.e., $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P}\mathcal{P}' = \mathcal{P}'\mathcal{P} = 0$. Also, both $\mathcal{P}$ and $\mathcal{P}'$ commute with any diagonal operator, such as $\mathcal{L}_0$.

We consider now the equation

$$(\mathcal{L}_0 - \hbar z)\mathcal{R}(z) = 1 - \mathcal{L}_1\mathcal{R}(z), \tag{A7}$$

which follows from the definition of $\mathcal{R}(z) = (\mathcal{L} - \hbar z)^{-1}$. Operating on (A7) with $\mathcal{P}$ and separately with $\mathcal{P}'$ and using in the right-hand side

$$\mathcal{R}(z) = \mathcal{R}_D(z) + \mathcal{R}_N(z), \tag{A8}$$

we obtain

$$(\mathcal{L}_0 - \hbar z)\mathcal{R}_D(z) = 1 - \mathcal{P}\mathcal{L}_1\mathcal{R}_N(z), \tag{A9}$$

$$(\mathcal{L}_0 - \hbar z)\mathcal{R}_N(z) = -\mathcal{L}_1\mathcal{R}_N(z) - \mathcal{P}'\mathcal{L}_1\mathcal{R}_D(z). \tag{A10}$$

Here we have absorbed the diagonal elements of V in H_0, i.e., $(\mathcal{P}\mathcal{L}_1) = 0$. Solving (A10) for $\mathcal{R}_N(z)$, we obtain

$$\mathcal{R}_N(z) = -(\mathcal{L}_0 + \mathcal{P}'\mathcal{L}_1 - \hbar z)^{-1}\mathcal{L}_1\mathcal{R}_D(z), \tag{A11}$$

which, upon substitution in (A9), yields an equation for $\mathcal{R}_D(z)$ with the solution

$$\mathcal{R}_D(z) = [\mathcal{L}_0 + i\hbar\Gamma(z) - \hbar z]^{-1}. \tag{A12}$$

Here $\Gamma(z)$ is the diagonal operator

$$\begin{aligned} i\hbar\Gamma(z) &= -\mathcal{P}\mathcal{L}_1(\mathcal{L} - \mathcal{P}\mathcal{L}_1 - \hbar z)^{-1}\mathcal{L}_1 \\ &= -\mathcal{P}[\mathcal{L}_1\mathcal{R}(z)\mathcal{L}_1 + \mathcal{L}_1\mathcal{R}(z)\mathcal{P}\mathcal{L}_1\mathcal{R}(z)\mathcal{L}_1 + \cdots] \end{aligned} \tag{A13}$$

expressed in the last line in terms of $\mathcal{R}(z) = \mathcal{R}_D(z) + \mathcal{R}_N(z)$. Now expanding

$$(\mathcal{L}_0 + \mathcal{P}'\mathcal{L}_1 - \hbar z)^{-1} = (\mathcal{L} - \mathcal{P}\mathcal{L}_1 - \hbar z)^{-1},$$

which appears in (A11), in powers of $\mathcal{P}\mathcal{L}_1$ and comparing with (A13), we obtain

$$\mathcal{R}_N(z) = -\mathcal{P}'\mathcal{R}(z)[\mathcal{L}_1 - i\hbar\Gamma(z)]\mathcal{R}_D(z). \tag{A14}$$

Replacing $\mathcal{R}(z)$ by the sum of its diagonal and nondiagonal parts, making use of the definition of $\mathcal{P}$ and $\mathcal{P}'$ and solving the resulting equation for $\mathcal{R}_N(z)$, we get

$$\begin{aligned} \mathcal{R}_N(z) &= -\mathcal{P}'\mathcal{R}_D(z)\mathcal{L}_1[1 + \mathcal{R}_D(z)\mathcal{L}_1]^{-1}\mathcal{R}_D(z) \\ &= -\mathcal{P}'[\mathcal{R}_D(z)\mathcal{L}_1\mathcal{R}_D(z) \\ &\quad - \mathcal{R}_D(z)\mathcal{L}_1\mathcal{R}_D(z)\mathcal{L}_1\mathcal{R}_D(z) + \cdots]. \end{aligned} \tag{A15}$$

From Eqs. (A8), (A12), and (A15) it is evident that for a weak electron-impurity interaction V the dominant contribution to $\langle a|\mathcal{R}(z)v_i|b\rangle$ is given by $\langle a|\mathcal{R}_D(z)v_i|b\rangle$, which clearly remains finite even at the resonant frequency. Thus for weak scattering we have

$$\begin{aligned} \langle a|\mathcal{R}(z)v_i|b\rangle &\cong \langle a|\mathcal{R}_D(z)v_i|b\rangle \\ &= [\epsilon_{ab} + i\hbar\Gamma(z)_{(a,b)} - \hbar z]^{-1}\langle a|v_i|b\rangle, \end{aligned} \tag{A16}$$

where $\epsilon_{ab}=\epsilon_a-\epsilon_b+\mathfrak{L}_{0(a,b)}$ as before. Although a detailed analysis of this approximation is desirable, it is roughly equivalent to ignoring "indirect" processes compared to the "direct" processes of absorption, but with the fundamental difference that in this formalism the "direct" processes are calculated among states which have already been shifted and broadened by the presence of the impurities.

Now, the diagonal operator $\Gamma(z)$ is given implicitly by (A8), (A12), (A13), and (A15). To the lowest order in V these equations give the following implicit equation for $\Gamma(z)$:

$$\Gamma(z)\cong(i/\hbar)\{\mathfrak{L}_1[\mathfrak{L}_0+i\hbar\Gamma(z)-\hbar z]^{-1}\mathfrak{L}_1\}_D. \quad \text{(A17)}$$

More explicitly we have, bearing in mind that $\Gamma(z)$ and $\mathfrak{L}_0$ are diagonal operators,

$$\Gamma(z)_{(a,b)}\cong\frac{i}{\hbar}\sum_c{}'\left(\frac{|\langle a|V|c\rangle|^2}{\epsilon_{cb}+i\hbar\Gamma(z)_{cb}-\hbar z}+\frac{|\langle c|V|b\rangle|^2}{\epsilon_{ac}+i\hbar\Gamma(z)_{ac}-\hbar z}\right), \quad \text{(A18)}$$

where the prime on the summation indicates exclusion of the diagonal elements of V.

Since the diagonal operator $\Gamma(z)$ modifies $\mathfrak{L}_0$, in the denominator $\mathfrak{R}_D(z)$ [Eq. (A12)], we may identify it as the *complex linewidth*. Thus the real part of $\Gamma(z)_{(a,b)}$ is the *linewidth* of the transition $b\to a$ and the imaginary part (multiplied by $-\hbar$) is the *lineshift* for the same transition.

Finally, we must average (A16) over the random distribution of impurities. Such an average is difficult to carry out rigorously, as the random quantities appear in the denominator. We shall make the approximation of replacing $\Gamma(z)$ in (A16) by its impurity average, and of finding it from Eq. (A18) by taking the impurity average separately of the numerator and denominator of (A18).

Such an approximation for the impurity average is also made in the calculation of Kawabata.[9] It is difficult to assess the effects of this approximation on the field dependence of the cyclotron-resonance linewidth.

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