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Methods of Series Analysis. I. Comparison of Current Methods Used in the Theory of Critical Phenomena*

D. L. Hunter and George A. Baker, Jr.
Applied Mathematics Department, Brookhaven National Laboratory, Upton, New York 11973
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We survey the principal types of methods of series analysis which have been used in the study of critical phenomena with a view to determining their accuracy and applicability to the treatment of the critical-point singularities. These methods include the ratio method and its variants, such as the Neville-table method; the Padé-approximant procedures; and the procedures based on the generalized-Polya theorems of Thompson *et al.* We show that the actual procedures of Thompson *et al.* are mathematically equivalent to certain of the Padé-approximant procedures. We give a general error analysis for the series-analysis procedures and derive a relation between the expected magnitude of the errors in the parameters of $A(1-yx)^{-\gamma}$, namely, $\Delta y:\Delta\gamma:\Delta A$ as $1:J:J\ln J$, where J is the order of the last term of the series analyzed. This relation is briefly illustrated by numerical data. We further give procedures for establishing estimates of the magnitude of errors in the parameters that have the same type of validity as those commonly used to determine the accuracy of a truncated Taylor series. We discuss the commonly occurring but anomalous case of "defects" (errant close pole and zero) in Padé-approximant procedures. We show them to be related to Padé's block structure of the approximant table and emphasize the artificial nature of the apparently rapid convergence that they cause. By numerical investigation of many test functions, similar in structure to those believed to appear in problems of critical phenomena, we have illustrated the following conclusions. First, for series where there is only a simple algebraic singularity, closer to the origin and well separated from any other singularity, the ratio method, perhaps with Neville-table improvement, is the most effective procedure. Second, for series where there are interfering singularities close to the one considered, or where there are singularities either closer than or nearly at the same distance from the origin as the one considered, the Padé procedures are best. Finally, for not exactly representable singularity structures of the type just decscribed, the convergence of even the Padé-approximant procedures are significantly slowed. None of the general methods described does a very impressive job in computing the γ value if the function is in fact of the form $A(1-yx)^{-\gamma}\ln|1-yx|$.

I. INTRODUCTION AND SUMMARY

Many important results in the theory of critical phenomena have been obtained by deducing the asymptotic behavior of functions from their series expansions. Relatively few rigorous results are available for the many models that have been introduced to describe systems in the region of a phase transition. In the absence of many exact results, considerable effort has been directed toward calculating series expansions, often of considerable length, for the model partition functions, and thence the thermal and magnetic (or a counterpart) functions. Various methods have been used to extract estimates of the critical points and critical exponents from the series. Many of the results that have been obtained in this way are summarized in two review articles[1] and in a recent book.[2] Other developments that have heightened interest in the field have been the derivation of the rigorous exponent inequalities,[3] the predictions of scaling theory,[4] and, more recently, the attempts to deduce equations of state that reflect all the critical properties.[5]

However, while many expansions have proved amenable to accurate estimation of the critical parameters using the current analysis techniques, others subjected to the same analysis have yielded results which are too uncertain to reach definite conclusions regarding, for example, the validity of scaling theory for a particular model. In many cases, the series expansions are not long enough to make accurate predictions; in other cases, the series would appear to be long enough, but the structure of the function seems to be too complicated for current methods to treat accurately. This is especially true of the low-temperature series for the three-dimensional Ising model, where there are known to be other singularities in the functions closer to the origin than the one corresponding to the physical transition. On the other hand, there has not been any rigorous way to assess the reliability of the results obtained in the analysis of the series expansions.

If for a function $f(x)$,

$$f(x) \sim A(1-yx)^{-\gamma} \text{ as } x \to y^{-1} , \tag{1.1}$$

then y defines the critical (singular) point, γ is the critical exponent, and A is the amplitude. The many methods and variations which have been used to estimate these parameters for one or more singularities in a function for which only a finite number of terms in the Taylor series is known may, broadly speaking, be considered in three groups. The ratio method was first used successfully to estimate both critical points and exponents by Domb and Sykes.[6] The use of Padé approximants was demonstrated by Baker,[7] who verified earlier ratio predictions and obtained new results as well. The third group we describe as being based on generalized Polya theorems and are due to Parks[8] and Thompson, Guttmann, and Ninham.[9] The method of critical-point renormalization[10,11] yields results by what is essentially the ratio method and will be discussed under that heading, but Padé approximants may also be used to study the renormalized series.

The purpose of this paper is threefold: to review these methods that are currently available, to consider the assessment of the reliability of the results obtained using the methods, and to numerically investigate their effectiveness on known functions whose form is different from the simple form usually considered. In a subsequent publication[12] we will introduce some new procedures for analyzing series that are particularly suited to studying special forms of the function.

In Sec. II we review the three groups of methods. Guttmann's[9(b)] Eq. (4) suggests that their procedure is capable of treating a very general form of the function. However, the low order of approximation upon which subsequent analysis is based means, as he points out, that the form reduces to the usual simple product of factors of the form of the right-hand member of (1.1). We show that this third group is mathematically equivalent to certain commonly used Padé-approximant procedures. Because of this equivalence, we do not consider the third group in the numerical studies of Sec. IV.

In assessing the reliability of estimates of the critical parameters, it is important to consider their interdependence. For example, one must be careful to distinguish between estimates for one critical parameter which are independent of any assumed value for another parameter (unbiased estimates) and those which do assume a value for another parameter (biased estimates). In Sec. II we are careful to distinguish between biased and unbiased procedures.

In Sec. III we consider three topics relevant to assessing the reliability of results. With regard to the interdependence of the parameters, we argue that uncertainties in estimates of the critical point, critical exponent, and amplitude, in that order, are roughly in the proportion $1 : J : J \ln J$, where J is the order of the last coefficient used. Further, we consider the question of fixing these uncertainties on an absolute scale for different methods. The other two points pertain to the use of Padé approximants and indicate that caution is sometimes required in interpreting Padé results that are apparently quite well converged. Approximants which contain a nearly coincident zero-pole pair (with small residue) closer to the origin than the singularity of interest are likely to be anomalously similar to the approximant of degree 1 less in the numerator and denominator in all other respects, and hence should not be considered in examining the Padé table for convergence. In addition, we show that the difference between contiguous Padé approximants for sufficiently small values of the argument behaves as some high power of the argument, but that at some distance from the origin, this regular behavior breaks down and the difference remains relatively constant. Thus, if the point of interest is outside the well-behaved region, the variation between contiguous approximants, and hence the uncertainty assigned to estimates based on these approximants, should be greater than it appears to be.

In deducing the form of the function and the values of the critical parameters using the various techniques we describe, it is desirable to know something about their effectiveness for different forms, and the rate at which the estimates they provide converge toward the true values. Applying these techniques to the expansions of known functions is valuable in making such assessments. For the two-dimensional Ising model in zero applied field, the free energy, specific heat, and spontaneous magnetization are known exactly, and the asymptotic form of the magnetic susceptibility is known almost rigorously. Application of the series-analysis techniques to these series has justified their application in related problems, and has provided a valuable yardstick for comparison with the analysis of other series. However, the examination of other forms than are represented by the two-dimentional Ising functions is necessary.

In Sec. IV we apply the ratio and Padé techniques to 11 test functions which, in addition to the dominant branch point closest to the origin, have other additive singularities distributed in different ways in the complex plane. We feel that such an additive form is an obvious one to investigate for application to the low-temperature Ising-model series. In this case, Padé approximants have not resolved the parameters associated with the various singularities as well as one would expect if the functions were little different from a simple product of singular factors. For several of our test series we find that our best estimates have errors of a few percent. This is the order of the error one finds in estimates of the critical point from low-temperature Ising series. In addition, to illustrate the treachery of logarithmic factors which might multiply the usual singular factor (1.1), we include a test function of that form. We present details of the analysis for four of the functions so that the problems encountered in assessing a ratio sequence or Padé table are clear. We illustrate the points raised in Sec. III with actual examples using the test functions. The results for all the functions are summarized in a way that facilitates comparisons between the methods and illustrates the degree and rate of convergence of the estimates to the known values of the parameter. The rate and manner in which these estimates tend toward the true value of the parameter, as asymptotically they must, is best studied using test functions, but the examples we give indicate that the considerations of Sec. III will be valuable in assessing reliability in real situations where the asymptotic values are not known.

II. METHODS OF ESTIMATING CRITICAL PARAMETERS

A. Ratio Method and Its Variants

The ratio method may be used to study only that singularity which determines the radius of convergence of the series expansion. However, we will also consider the case where the singularity of interest is not originally the closest one to the origin, but where an Euler transformation may be used to map it closer than any other singularity. The effectiveness of the ratio method is affected by the proximity of other singularities to the circle of convergence; ways of compensating for the effects of other singularities will be considered also.

If the function $f(x)$ has the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.1}$$

and the singularity closest to the origin in the complex x plane has the form (1.1), then asymptotically the contribution of this singularity will dominate all other contributions to the coefficients a_n, regardless of whether the singularity (1.1) is a factor of $f(x)$. We may write

$$a_n \sim (-1)^n A\binom{-\gamma}{n} y^n \equiv A\binom{n+\gamma-1}{n} y^n , \tag{2.2}$$

where the $\binom{\alpha}{\beta}$ are binomial coefficients and the amplitude A is defined by

$$A = \lim f(x)(1-yx)^{+\gamma} \quad \text{as } x \to y^{-1} \tag{2.3}$$

and may be estimated from (2.2) once y and γ have been estimated. Using (2.2), we can write the ratio r_n of successive coefficients as

$$r_n \equiv a_n/a_{n-1} \sim [1+(\gamma-1)/n]y . \tag{2.4}$$

Equation (2.4) is the basis of the ratio method. The limit of the sequence $\{r_n\}$ is y and one attempts to estimate that limit given a finite number of terms in the sequence. The point at question usually is whether the sequence is long enough to indicate its asymptotic behavior. Often, as few as six terms, and occasionally more than 35 terms are available. The irony is that sometimes a short sequence might give more clues to its asymptotic behavior than might a much longer sequence for a different function. Plotting the ratios against $1/n$ gives a curve that will have intercept y and limiting slope $(\gamma-1)y$. One may obtain an unbiased estimate of y from graphical extrapolation to the intercept, and a biased estimate of γ from the slope (biased because it depends upon the value chosen for the intercept). In many problems, the potential precision of the method surpasses the accuracy attainable using graphical methods, so one may use more accurate numerical extrapolations of the sequence $\{r_n\}$. Linear extrapolants

$$y_n = nr_n - (n-1)r_{n-1} \tag{2.5}$$

are most commonly used although higher-order extrapolants in the form of a Neville table[13] are often valuable. If γ is known, the modified ratio sequence

$$\beta_n = na_n/(n+\gamma-1)a_{n-1}$$

proposed by Domb and Sykes[6(b)] will approach the limit y with zero slope, making possible a more precise extrapolation.

To estimate γ one forms the associated sequence $\{\gamma_n^{(b)}\}$ from $\{r_n\}$, where

$$\gamma_n^{(b)} = nr_n y^{-1} - n + 1 . \tag{2.6}$$

In this form the estimates γ_n are necessarily biased, but one could use the unbiased sequence

$$\gamma_n^{(u)} = nr_n y_n^{-1} - n + 1, \tag{2.7}$$

where y_n from (2.5) replaces the best estimate of the limit y in going from (2.6) to (2.7). The sequence $\gamma_n^{(u)}$ must converge to the correct limit, since y_n converges to y, but the convergence is slower than for $\gamma_n^{(b)}$.

A possible variation[14] is to consider the logarithmic derivative of the series $f(x)$ [this transformation is frequently used with the Padé-approximant analysis of $f(x)$; see Sec. II B],

$$\frac{d}{dx} \ln f(x) = \sum a_n^* x^n \sim \frac{\gamma y}{1-yx} , \tag{2.8}$$

from which it is obvious that the new series coefficients behave as

$$a_n^* \sim \gamma y^{n+1} . \tag{2.9}$$

The ratios of the transformed coefficients again form a sequence of estimates for y, and this time biased estimates for γ are obtained directly from the coefficients by dividing by y^{n+1}.

Let us consider corrections to (2.4) which would arise for different forms of $f(x)$. First, if $f(x)$ has two strong additive singularities $|y_1| \geq |y_2|$,

$$f(x) = A_1(1 - y_1 x)^{-\gamma_1} + A_2(1 - y_2 x)^{-\gamma_2} , \tag{2.10}$$

then the coefficients have the form

$$a_n = A\binom{n+\gamma_1-1}{n} y_1^n[1 + O(y_2^n/y_1^n)] . \tag{2.11}$$

For y_2 not on the circle of convergence, the corrections rapidly tend to zero and r_n [Eq. (2.4)] should be nearly linear in $1/n$, even for relatively small values of n. We note that if y_1 is on the positive real axis and y_2 is at or near $-y_1$, then the corrections alternate in sign and the ratios will have a regular oscillation of period 2 in n. This type of oscillation occurs in the high-temperature series for loose-packed lattices where $y_2 = -y_1$ corresponds to the antiferromagnetic singularity. If the singularity in $f(x)$ is a factor

$$f(x) = (1 - yx)^{-\gamma} g(x), \tag{2.12}$$

where $g(x)$ is analytic at $x = y^{-1}$, then the coefficients will have the form

$$a_n = \binom{n+\gamma-1}{n} y^n[A + O(1/n) + \cdots] . \tag{2.13}$$

There would be additional corrections of $O(y_2^n/y_1^n)$ if $g(x)$ were singular at $x = y_2^{-1}$, but as we saw, this effect falls off rapidly when $|y_2| < |y_1|$. Substitution of (2.13) into (2.4) gives

$$r_n = [1 + (\gamma - 1)/n + O(1/n^2)]y , \tag{2.14}$$

which in turn implies

$$\gamma_n = nr_n y^{-1} - n + 1 = \gamma + O(1/n) + \cdots . \tag{2.15}$$

This suggests that $1/n$ plots might also be useful in extrapolating the sequence $\{\gamma_n\}$.

Since oscillations frequently arise in the sequences $\{r_n\}$ and $\{\gamma_n\}$ when dealing with high-temperature series (the most common application of the ratio method), we will discuss possible variations one might use when oscillations are present. The most obvious course is to take linear extrapolants of alternate ratios: either graphically taking the envelope of the ratio plot or numerically using

$$y_n^{(a)} = \tfrac{1}{2}[nr_n - (n - 2)r_{n-2}] \tag{2.16}$$

instead of (2.5). However, another alternative is also suggested. In many model situations in zero field, the partition function is an even function in the interaction J and hence in the usual expansion variable $v = \tanh(J/kT)$. These expansions in even powers of v only are certainly the extreme in series with oscillations. That they are readily analyzable in terms of the variable $x = v^2$ suggests the possibility of defining

$$\rho_n = (a_n/a_{n-2})^{1/2} \tag{2.17}$$

and extrapolating $\{\rho_n\}$ either consecutively or alternately to study series with a less pronounced oscillation. Any one of (2.16) or either of the suggested linear extrapolants of (2.17) could be substituted into (2.7) in place of y_n to obtain unbiased estimates of γ suitable for series with an oscillation.

A common assumption for a more general form of singularity than (1.1) is

$$f(x) \sim (1 - yx)^{-\gamma} |\ln(1 - yx)|^{\lambda} ; \tag{2.18}$$

in fact, the two-dimensional Ising-model specific heat in zero field is known exactly to behave as (2.18) with $\gamma = 0$, $\lambda = 1$. Moore[15] has considered the effect of such an additional factor on the ratio estimates for the critical index. Asymptotically our previous discussion is still valid, but significant corrections for small n slow the convergence to the asymptotic value. Moore finds that the effective value of γ seen for finite n is given by

$$\gamma_{\text{eff}} \simeq \gamma + \lambda/\ln(cn) , \tag{2.19}$$

where c is an undetermined constant. Even for large c the corrections can be substantial when, as is usual, λ and γ are of order unity.

If series expansions in the same variable for two different functions known to be singular at the same point are available, then the method of critical-point renormalization[10,11] enables one to obtain an unbiased estimate for the difference between the critical exponents of the two functions. If we have

$$f_1(x) = \sum_n a_{1n} x^n \sim A_1(1 - yx)^{-\gamma_1} , \quad f_2(x) = \sum_n a_{2n} x^n \sim A_2(1 - yx)^{-\gamma_2} , \tag{2.20}$$

then the generating function

$$\mathfrak{F}(x, \gamma_1, \gamma_2) = \sum_n c_n x^n \sim (1 - x)^{-\gamma_1+\gamma_2-1} , \tag{2.21}$$

where

$$c_n \equiv a_{1n}/a_{2n} . \tag{2.22}$$

Using a different asymptotic form of (2.2), we can write

$$a_{1n} \sim n^{\gamma_1-1} y^n, \quad a_{2n} \sim n^{\gamma_2-1} y^n . \tag{2.23}$$

It follows from (2.22) and (2.23) that $c_n \sim n^{\gamma_1-\gamma_2}$ and this in turn implies the asymptotic relationship in (2.21). The critical point of the generating function has been renormalized to unity and the ratio method (2.6) applied to the coefficients c_n gives estimates of $\gamma_1-\gamma_2$ that are no longer biased by an assumed value for y. Moore *et al.*[11(a)] point out that better results are possible if one considers the logarithmic derivative of the generating function, since the amplitudes of certain types of competing singularities are reduced. From (2.8) and (2.21) we see that the coefficients themselves in the new series form a sequence of estimates for $\gamma_1-\gamma_2+1$. The critical-point renormalization procedure is particularly suitable for estimating the gap exponents Δ and Δ' and the correlation-length exponents ν and ν' given series for the derivatives of the free energy with respect to external field and the spherical moments, respectively.

We have discussed the use of linear extrapolations of sequences of estimates for y and γ. The Neville-table method[12] is one of nonlinear polynomial (Lagrange) interpolation which, when applied to sequences which apparently behave as $1/n$, can be used to extrapolate to $1/n=0$.

In that case, the Neville-table method enables one to form higher-order nonlinear extrapolants by making successive linear extrapolations. In general, given a sequence $\{f_n\}$ of values of a function at the points $x=\{a_n\}$, the jth-order interpolation polynomial through the points $(a_n, f_n),\ldots,(a_{n-j}, f_{n-j})$ is defined in terms of lower-order polynomials by

$$f(x|a_{n-j}, a_{n-j+1},\ldots, a_n)=\det\begin{vmatrix} f(x|a_{n-j}, a_{n-j+1},\ldots,a_{n-1}) & x-a_{n-j}\\ f(x|a_{n-j+1}, a_{n-j+2},\ldots,a_n) & x-a_n\end{vmatrix}\Bigg/(a_{n-j}-a_n)\ , \tag{2.24}$$

where the linear interpolation polynomials are

$$f(x|a_{n-1}, a_n)=\det\begin{vmatrix} f_{n-1} & x-a_{n-1}\\ f_n & x-a_n\end{vmatrix}\Bigg/(a_{n-1}-a_n)\ . \tag{2.25}$$

We can specialize to the case $a_n=1/n$ and then extrapolate to $x=a_\infty=0$. The sequence of jth-order extrapolants is then given by

$$\ell_n^{(j)}=f\left(0\left|\frac{1}{n-j},\ldots,\frac{1}{n}\right.\right)=\frac{1}{j}\det\begin{vmatrix}\ell_{n-1}^{(j-1)} & -n\\ \ell_n^{(j-1)} & -(n-j)\end{vmatrix}$$

$$=\frac{1}{j}\left[n\ell_n^{(j-1)}-(n-j)\ell_{n-1}^{(j-1)}\right]\ . \tag{2.26}$$

If we define $\ell_n^{(0)}\equiv f_n$, then the recurrence relation (2.26) is almost trivial to apply and provides extrapolants that may be carried to as high an order as the length of the series permits. As we can see from the $1/n$ dependence in (2.4) and (2.15), both $\{r_n\}$ and $\{\gamma_n\}$ are readily extrapolated using (2.26).

In theory, for an ideal sequence the accuracy of the extrapolation increases with the order of the extrapolant. However, the sequences we are concerned with approach the $1/n$ form only asymptotically; they will inevitably have "noise" in the early terms due to contributions which are not expandable in $1/n$. The noise in the early terms propagates forward as the order of the extrapolants increases, so that for finite sequences for sufficiently large j the apparent convergence of the Neville table breaks down. For series with an oscillation due to a nearby singularity on the negative real axis (exponential behavior in n), this breakdown occurs immediately—for the linear extrapolants. As before, we could circumvent this by extrapolating subsequences of alternate ratios. Making the appropriate modifications to (2.26), we get instead

$$\ell_n^{(j)}=(1/2j)\left[n\ell_n^{(j-1)}-(n-2j)\ell_{n-2}^{(j-1)}\right]\ . \tag{2.27}$$

Equation (2.27) is readily generalizable to account for oscillations in $\{f_n\}$ of different periods.

Another variation which has been used extensively, especially in conjunction with the ratio method when nonphysical singularities lie closer to the origin than the physically interesting one, is to first apply an Euler transformation to the series. The transformation is chosen so that in the complex plane of the transformed variable the interfering singularities lie beyond the circle defined by the singularity being studied. The method was first used by Danielian and Stevens[16] to study the antiferromagnetic singularity by transforming away the ferromagnetic one. Baker *et al.*[17] considered the effect of such Euler transformations and showed that they left certain Padé approximants invariant. Since then, that type of transformation has been used by others in a variety of contexts.[18]

If there is only one interfering singularity, it is possible to transform it to infinity, but even if there is more than one, it is possible to find an optimal Euler transformation for our present purposes (in the past, the transformations used in the latter case were chosen rather haphazardly). We will consider the Euler transformation defined by

$$w=(1+a)z/(1+az)\ , \tag{2.28}$$

where a is a real constant. We do not consider the more general bilinear transformation

$$w=(b+cz)/(1+az)\ , \tag{2.29}$$

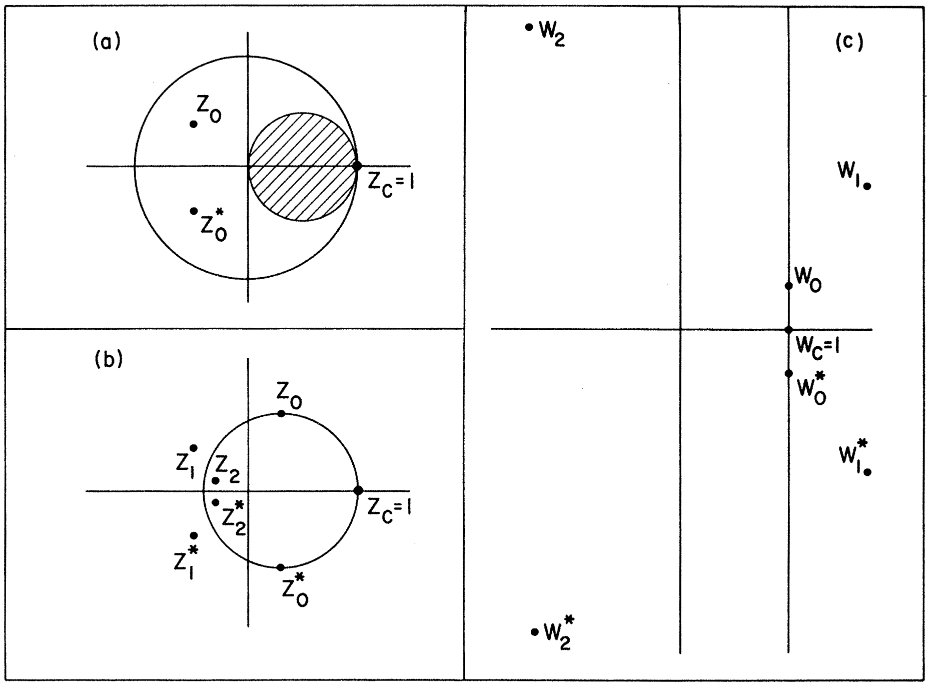

FIG. 1. (a) Complex z plane showing $z_c=1$, the physically interesting singularity, and a pair z_0, z_0^* of interfering singularities. The shaded region in the z plane is mapped into itself by the transformation (2.28) if $a>-1$. (b) The complex z plane for a case where there are three pairs of interfering singularities in the z plane. (c) The complex ω plane showing that the optimal transformation for case (b) is the one which maximizes $|\omega_0|$ even though $|z_0|>|z_1|>|z_2|$. The straight line $\mathrm{Re}(\omega)=1$ is the transform of the circle shown in (b).

since the transformation may be applied successfully to a truncated series only if $\alpha_0 = 0$ in the expansion

$$w=\sum_{n=0}^{\infty}\alpha_n z^n , \tag{2.30}$$

i.e., only if $b=0$ in (2.29). Equation (2.28) implies that the origin is a fixed point in the transformation, and we choose to make the physical singularity (for convenience located at $z=1$) the other fixed point. Thus (2.29) reduces to (2.28) for the case we will consider. The circular region of radius $\frac{1}{2}$ through $z=0$ and $z=1$ [shaded region in Fig. 1(a)] is mapped into itself by (2.28) when $a>-1$; so it must be assumed free of interfering singularities or we will not be able to find a satisfactory transformation. Let us assume, however, that there is a complex singularity at $z_0=x_0+iy_0$, $|z_0|<1$. Since the series coefficients are real, z_0^* is also a singular point. If (2.28) maps z_0 into w_0, then we define the optimum transformation as the one which maximizes

$$|w_0|^2=\frac{(1+a)^2 z_0 z_0^*}{(1+az_0)(1+az_0^*)} . \tag{2.31}$$

The condition

$$\frac{\partial}{\partial a}|w_0|^2=0 \tag{2.32}$$

yields

$$a=\frac{x_0-1}{x_0(1-x_0)-y_0^2} , \tag{2.33}$$

from which we see that

$$w_0=1+i\{[y_0^2-x_0(1-x_0)]/y_0\} . \tag{2.34}$$

Hence, the optimal transformation which leaves the physical transition unchanged maps the circle through the points 1, z_0, z_0^* into the straight line $\mathrm{Re}(w)=1$. If there is more than one complex-conjugate pair of interfering singularities, then it is not immediately obvious what the optimal transformation will be. Figure 1(b) illustrates a case where there are three pairs of singularities closer to the origin than z_c. The optimal transformation in this case is the one which maximizes $|w_0|$ despite the fact that z_1 and z_2 both lie closer to the origin than z_0. Figure 1(c) shows that after such a transformation $|w_1|$ and $|w_2|$, although not maximized, are much greater than $|w_0|$.

B. Padé-Approximant Methods

Padé approximants were first applied to this problem by Baker[7] and have been used extensively by many workers since. Comprehensive reviews of the theory and applications of Padé approximants have been given by Baker.[19,20] We shall outline some of the procedures which have been used successfully.

The $[N/D]$ Padé approximant to the function $f(x)$ is the rational polynomial expression

$$[N/D]=\frac{P_N(x)}{Q_D(x)}=\frac{p_0+p_1x+\cdots+p_Nx^N}{1+q_1x+\cdots+q_Dx^D}, \qquad (2.35)$$

where the coefficients $p_0, \ldots, p_N, q_1, \ldots, q_D$ are chosen to make the first $N+D+1$ terms of the expansion of (2.35) agree with the corresponding terms in the Taylor expansion of $f(x)$ about $x>0$. It is immediately obvious that the Padé approximant will represent all the singularities of $f(x)$ as simple poles, and that therefore it will most accurately represent a singularity if it is a simple pole. Therefore, it is customary to form Padé approximants to

$$F(x)=\frac{d}{dx}\ln f(x)=\sum_n b_n x^n . \qquad (2.36)$$

If $f(x)$ has the form (2.12), then $F(x)$ has a simple pole at $x=y^{-1}$, with residue $-\gamma$. We obtain an estimate of y and γ from the corresponding pole and residue of the Padé approximant.

For a series whose coefficients are known to order M, we can form a table of Padé approximants $[N/D]$ for all N, D such that $0\leq N+D\leq M$. The convergence properties of Padé approximants are not fully understood (see Baker[20] for currently known convergence properties). We will mention only the Padé conjecture[17] which, although unproven in general, probably does not even give the complete range of convergence. It says that there is at least an infinite subsequence of $[N/N]$ Padé approximants which converge uniformly to the function $F(x)$ provided $F(x)$ is regular for $|x|\leq R$ except for a finite number of poles within this circle, and possibly for one point on the circle $|x|=R$ at which continuity is assumed only when the point is approached from points interior to the circle; the domain of uniform convergence is $|x|\leq R$ with the interiors of small circles centered at the poles removed. Experience with estimating the critical parameters from Padé approximants shows that usually many more than the diagonal $(N=D)$ approximants apparently are converging, and that one might well consider not the whole table but a smaller triangular wedge symmetric about the main diagonal. Frequently one may find that estimates based on the $[n/d]$ and $[n+1/d+1]$ approximants are anomalously close together. Such irregularities in the apparent convergence are related to the existence of so called *defects* and are discussed in detail in Sec. III. Sometimes one may be tempted to believe that the Padé table is very well converged when in fact many of the highest-order approximants contain defects and the apparent convergence should be discounted entirely.

The following are the procedures involving Padé approximants which have been used extensively in obtaining estimates of the critical parameters y, γ, and A, if $f(x)$ is given asymptotically by

$$f(x)\sim A(1-yx)^{\gamma} \text{ as } x\rightarrow y^{-1} . \qquad (2.37)$$

(i) Form Padé approximants to

$$F_1(x)=\frac{d}{dx}\ln f(x)\sim\frac{-\gamma}{x-y^{-1}} \qquad (2.38)$$

and obtain unbiased estimates of y and γ by choosing the appropriate zero of the denominator and calculating the residue at that point, respectively. The γ estimate is unbiased in the sense that no assumed value of x_c is used. However, it is biased in the sense that if the pole is a poor approximation for y^{-1}, then the residue is an even poorer approximation for γ.

(ii) Form Padé approximants to

$$F_2(x)=(y^{-1}-x)\frac{d}{dx}\ln f(x)\sim\gamma \qquad (2.39)$$

assuming some value for y, and obtain biased estimates of γ by evaluating the Padé approximants at $x=y^{-1}$.

(iii) Form Padé approximants to

$$F_3(x)=f(x)^{1/\gamma}\sim A^{1/\gamma}/(1-yx) \qquad (2.40)$$

assuming a value for γ, and obtain biased estimates for y and A from the roots and residues of the approximants as in (i).

(iv) Form Padé approximants to

$$F_4(x)=(1-yx)^{\gamma}f(x)\sim A \qquad (2.41)$$

assuming values for y and γ, and obtain biased estimates for A by evaluating the approximants at $x=y^{-1}$.

(v) Form Padé approximants to

$$F_5(x)=\left(\frac{d}{dx}\ln\frac{d}{dx}f(x)\right)\Big/\frac{d}{dx}\ln f(x)\sim 1+\frac{1}{\gamma} \qquad (2.42)$$

and evaluate at $x=y^{-1}$ (assumed). This procedure was proposed by Baker *et al.*[21] because it is quite insensitive to the choice of y; technically, however, it must be considered a biased procedure. One drawback is that for large γ, the quantity calculated is relatively insensitive to γ as well!

(vi) In addition to (i)–(v), one may use Padé approximents in conjunction with critical-point renormalization. Forming Padé approximants to

$$F_6(x)=(1-x)\frac{d}{dx}\ln\mathfrak{F}(x,\gamma_1,\gamma_2)\sim\gamma_1-\gamma_2+1, \qquad (2.43)$$

where $\mathfrak{F}$ is the generating function (2.21), and evaluating at $x=1$ gives unbiased estimates for the difference between the two critical indices γ_1 and γ_2.

Procedures (i)–(vi) are all based on asymptotic relations which prevail as one approaches the singularity of interest; we emphasize that this need not be the singularity closest to the origin. If the Padé conjecture holds, then we are assured that as $N\to\infty$, a subsequence of the $[N/N]$ approximants converges to $F(x)$ arbitrarily close to the poles [thereby reflecting the asymptotic behavior of $F(x)$] provided we are in a region where the only singularities in $F(x)$ are poles. However, approximants are usually available for only relatively small N and we may not necessarily obtain good estimates for the parameters of interest using these procedures. For example, if we are studying the ith singularity from the origin in $F(x)$, the order of the denominator of the approximant must be at least of order i before there is hope of that singularity being reasonably represented in the approximant. Usually one would require denominators of even higher order, particularly if there are more than i singularities in $F(x)$ altogether. An exception to this would be the case

$$f(x)=\mathcal{a}(1-x/x_1)^{\gamma_1}\ldots(1-x/x_n)^{\gamma_n}\,, \qquad (2.44)$$

where the functions $F_1(x)$ and $F_2(x)$ would be exactly representable by $[n-1/n]$ and $[n-1/n-1]$-order Padé approximants, respectively. If (2.44) were multiplied by a nonvanishing analytic function $g(x)$, then the above and higher-order approximants would represent the function well, although not exactly. If $g(x)$ were allowed to vanish at a finite number of points, then we would need a higher-order denominator in the approximant to represent the singularities introduced by the $d\ln g(x)/dx$ terms in $F_1(x)$ and $F_2(x)$.

As another extreme, we consider the form

$$f(x)=A(1-yx)^{-\gamma}+g(x), \qquad (2.45)$$

where $g(x)$ is analytic at $x=y_1^{-1}$ but may be singular elsewhere. For this case

$$F_1(x)=\gamma y(1-yx)^{-1}-\gamma y(1-yx)^{\gamma-1}g(x)+\cdots\,. \qquad (2.46)$$

The Padé approximant is not able to represent the behavior of $F(x)$ at $x=y^{-1}$ nearly so well in this case, particularly if $\gamma<1$. Although approximants to F_2 should still converge to γ, the rate of convergence will be slowed considerably by the presence of the second term if $\gamma<1$. Higher-order terms in (2.46) would include ones proportional to $(1-yx)^{n\gamma-1}$ which will slow the convergence even further if γ is very close to 0. The Ising-model specific-heat index above the critical point is $\frac{1}{8}$[22] and the usual Padé-approximation procedures are extremely slow in converging to x_c and γ.[23]

Whereas the sequences $\{r_n\}$ and $\{\gamma_n\}$ for the ratio method converged to their limits as $1/n$, we have no such prior knowledge for the convergence of estimates from Padé approximants. Final estimates for the relevant parameters based on a table of Padé approximants therefore depend to some extent upon the opinion and experience of the person examining the table.

C. Methods Based on Generalized Polya Theorems

In this section we will discuss the methods proposed by Park[8] and by Thompson *et al.*,[9(a)] the latter generalized by Guttmann.[9(b)] We will indicate that the two methods are essentially equivalent, although Thompson *et al.* give arguments to justify using the approach for a much broader class of functions and, in addition, adapt the method to give biased index estimates given the location of the physical singularity. Whereas Park was able to solve the nonlinear equations by hand only for the case of two singularities, Guttmann, using computers, was able to solve them for many more singularities. However, even more important, we will show that these methods are essentially equivalent to the basic Padé-approximant procedures (i) and (ii).

Park begins by considering functions of the form

$$f_1(x)=\prod_{i=1}^{N}\left(1-\frac{x}{x_i}\right)^{-\gamma_i}\,, \qquad (2.47)$$

whereas Guttmann first considers

$$f_2(x)=x^m\prod_{i=1}^{N}\left[\alpha_i+A_i(x)\left(1-\frac{x}{x_i}\right)^{-\gamma_i}\right], \qquad (2.48)$$

where m is the order of the first nonzero coefficient in the expansion which $f_2(x)$ is to approximate. The above functions differ from those in the original papers in the convention regarding the sign of the γ_i's. Thompson *et al.* show that to leading order in n the coefficients b_n in

$$\frac{d}{dx}\ln[x^{-m}f_2(x)]=\sum_{n=0}^{N}b_nx^m \qquad (2.49)$$

and the coefficients in

$$\frac{d}{dx}\ln f_1(x)=\sum_{n=0}^{\infty}\frac{\gamma_i}{x_i^{n+1}}\,x^n \qquad (2.50)$$

are equal, in other words, that it might not be unreasonable to apply to a series estimation techniques which assume the form (2.47), when in fact the function which the series represents has the more general form (2.48).

Both methods reduce to finding solutions $x_i^{(N,j)}$ and $\gamma_i^{(N,j)}$ to the nonlinear equations

$$\sum_{i=1}^{N}\frac{\gamma_i}{x_i^{n+1}}=a_n,\quad n=j-2N+1,\ \ldots,\ j \qquad (2.51)$$

where the a_n are the coefficients of the logarithmic-derivative series being studied and we assume the a_n are determined to order $M_{\max}$. When we apply this procedure to a series, we are approximating its logarithmic derivative by a function of the form

$$F(x) = \frac{d}{dx} \ln[x^{-m} f(x)] \simeq \sum_{i=1}^{N} \frac{\gamma_i^{(N,j)}}{x_i^{(N,j)} - x} . \quad (2.52)$$

This is just the partial-fraction expansion of the rational function

$$\frac{P_{N-1}^{(j)}(x)}{Q_N^{(j)}(x)} = \sum_{i=1}^{N} \frac{\gamma_i^{(N,j)}}{x_i^{(N,j)} - x} , \quad (2.53)$$

where the subscripts denote the order of the polynomials $P(x)$ and $Q(x)$. The solutions to (2.51) using the first $2N$ coefficients (corresponding to $j = 2N - 1$) make the first $2N$ coefficients in the power-series expansion of (2.53) agree with the corresponding terms in the expansion of $F(x)$: therefore, by the uniqueness theorem, the approximant (2.53) is equal to the $[N - 1/N]$ Padé approximant to the logarithmic derivative $F(x)$. The correspondence with Padé approximants goes through for higher values of j since, using (2.51) and (2.53), the expansion of

$$c_0 + c_1 x + \cdots + c_{j-2N} x^{j-2N} + \frac{P_{N-1}^{(j)}(x)}{Q_N^{(j)}(x)} = \frac{\mathcal{P}_{j-N}(x)}{Q_N(x)} , \quad (2.54)$$

where

$$c_k = a_k - \sum_{i=1}^{N} \frac{\gamma_i^{(N,j)}}{(x_i^{(N,j)})^{k+1}}$$

will agree with the expansion of $F(x)$ to order j. Hence the solutions $x_i^{(N,j)}$ and $\gamma_i^{(N,j)}$ to Eqs. (2.51), $j = 2N - 1, \ldots, M_{\max}$, correspond exactly to the poles and residues of the $[j - N/N]$ Padé approximants to the logarithmic derivative of $f(x)$.

Thompson *et al.* did not propose using the method in this form. Rather, they assumed that the location of one of the singularities—say, x_N—was known. Equations (2.51) need now be solved for $2N - 1$ unknowns, n taking the values $j - 2N + 2, \ldots, j$. If we consider approximants to $(x_N - x)F(x)$ of the form

$$(x_N - x) F(x) \simeq \gamma_N^{(N,j)} + \sum_{i=1}^{N-1} \gamma_i^{(N,j)} \frac{x_N - x}{x_i^{(N,j)} - x} = \frac{P_{N-1}^{(j)}(x)}{Q_{N-1}^{(j)}(x)} , \quad (2.55)$$

then dividing (2.55) by $x_N - x$ would result in an approximation to $F(x)$ of the form (2.53) except that x_N is now specified. For $j = 2N - 2$, Eq. (2.55) becomes the $[N - 1/N - 1]$ Padé approximant to

$$(x_N - x) \frac{d}{dx} \ln[x^{-m} f(x)] . \quad (2.56)$$

An argument similar to that preceding (2.54) indicates that $\gamma_N^{(N,j)}$, $j = 2N - 2, \ldots, M_{\max}$, corresponds to the $[j - N + 1/N - 1]$ Padé approximants to (2.56) evaluated at x_N, while $x_i^{(N,j)}$ and $\gamma_i^{(N,j)}$, $i = 1, \ldots, N - 1$, are the poles and residues of those approximants. All of Guttmann's results could have been obtained from the Padé analysis of logarithmic derivatives.

The Padé-approximant formulation of the estimation procedure would seem to be preferable since the equations to be solved are linear. However, the nonlinear equations (2.51) may be linearized[9(b)] as one would expect seeing the equivalence of the approaches, but this constitutes an additional step. Park's work, coming as it did in 1956 before either the ratio or Padé methods were proposed, is a considerable achievement. It went largely unrecognized because, without the use of a computer, he was unable to carry it to its full potential.

In concluding this section, we will make reference to two procedures which do not properly come under any of the three subheadings in this section. The method of Guttmann *et al.*,[24] which we shall refer to as the *contour method*, is of limited applicability—namely, to a few functions such as the zero-field Ising susceptibility above T_c which appear to be very closely of the form

$$f(x) = (1 - x/x_c)^{-\gamma} g(x) , \quad (2.57)$$

where $g(x)$ is analytic for $|x/x_c| < 1$, except for $x = -x_c$, where $g(x)$ may behave as

$$g(x) \sim (1 + x/x_c)^{\alpha} . \quad (2.58)$$

Then for $n > -1 - \alpha$ the function

$$g_n(x) \equiv (1 + x/x_c)^n (1 - x/x_c)^{\gamma} f(x) \quad (2.59)$$

will have an expansion in x/x_c whose coefficients alternate in sign and decrease in magnitude for high enough powers m. As n increases, one must go to higher and higher orders $m \geq M(n)$ to see this behavior. The procedure is to find the range of values for x_c and γ for which, for given n and $M(n)$, this behavior is detectable in the available finite series for $g_n(x)$. The extremes of the ranges for given n form a contour in the (x, γ) plane. The size of the contour shrinks rapidly as n increases, but the cutoff in n for finite series of typical length is at about $n = 2$ or 3. If one of the parameters is known to high precision, then very accurate biased estimates of the other parameter can be obtained.

We also mention the procedure of Alexanian and Wortman.[25] As used by them for problems of statistical mechanics, their method seems to us to be equivalent to the integration of $(dp/dT)|_\rho$ from $T = \infty$, where the pressure p as a series expansion in the density ρ is known exactly to the desired value of T. In this integration $(dp/dT)|_\rho$ is expressed as a function of T and p by eliminating ρ in favor of p. The resultant p series is truncated at some fixed order in p. Any such approximation must, because of the analytic nature of the finite number of coefficients, lead to a compressibility which diverges as $(1 - T/T_c)^{-j}$, where j is an integer, probably unity, at the critical point. Inasmuch as the complete low-density expansion (information equivalent to the high-field expansion in the magnetic case) is required for this method, and it is clearly of the truncated Taylor-series type, we see little reason to employ it for the problems at hand and will not consider it further here.

III. VARIABILITY OF SERIES-SUMMATION METHODS

Let us suppose that we are trying to approximate the sum of a series which represents a function with a singularity

$$A(1-Yx)^{-G}, \qquad (3.1)$$

in which we are particularly interested. The approximate solution we have,

$$a(1-yx)^{-g}, \qquad (3.2)$$

will represent, together with the rest of our approximation scheme, the coefficients of the function used in the approximate summation, so that we have the equations

$$a\binom{-g}{j}y^j = A\binom{-G}{j}Y^j(1+\eta_j), \quad j=J, J+1, J+2 \;. \qquad (3.3)$$

The η_j are thought of as small percentage errors caused by some combination of other aspects of the function being approximated and the method of approximation. The reason for selecting percentage errors is that, if Y is very different from unity, the magnitude of the terms (3.3) can vary rapidly with j. If we now expand

$$a=A+\Delta A, \quad g=G+\Delta G, \quad y=Y+\Delta Y \qquad (3.4)$$

we can, substituting to leading order in (3.3), obtain the equations

$$\frac{\Delta A}{A}+\frac{j\,\Delta Y}{Y}+\sum_{k=0}^{j-1}\frac{\Delta G}{G+k}=\eta_j, \quad j=J, J+1, J+2 \;. \qquad (3.5)$$

These equations can easily be solved to yield

$$\Delta Y/Y=(2G+2J+1)\eta_{J+1}-(G+J+1)\eta_{J+2}-(G+J)\eta_J \;,$$
$$\Delta G=(G+J)\,(\eta_{J+1}-\eta_J-\Delta Y/Y)\,, \qquad (3.6)$$
$$\frac{\Delta A}{A}=\eta_J-J\,\frac{\Delta Y}{Y}-\left(\sum_{k=0}^{J-1}\frac{1}{G+k}\right)\Delta G \;.$$

From (3.6) we see that, relative to the size of the η's, $\Delta Y/Y$ is of order J. We consider J to be large (10 or more) because frequently this number of power-series terms of the function to be approximated is available. If the variation of η with j is smooth in the sense of a polynomial in j^{-1}, then the equation for $\Delta Y/Y$ will reduce the magnitude of $\Delta Y/Y$ to order J^{-2} relative to the magnitude of η by cancellation. We would expect this situation to arise when the only significant contribution to the η's is a confluent singularity and there is no other significant structure to the function. If the η's are caused by other singularities, then their behavior would be relatively geometric and then cancellation would not disturb the estimate that the relative magnitude of $\Delta Y/Y$ to the magnitude of the η's is of order J. It then follows that the magnitude of ΔG is J times that of $\Delta Y/Y$ or J^2 times that of the η's. It is this connection between the relative variation in the exponent and the location of the singularity which explains the widely observed phenomenon that the exponent is determined much less accurately than the location of the singular point. The variation in the multiplicative constant A, at least for J values which we have seen used in applications, is of roughly the order as that for ΔG. The sum in (3.6) is of order $\ln(G+J)$, so that the ΔA is of the order of $J^2\ln(G+J)$ times the magnitude of the η's.

Obtaining an accurate estimate of the magnitude of the η's is extremely difficult, except in special cases. For example, when using the ratio method and a clear even-odd effect is evident, the η's will be of the order of the difference between the odd and even ratio interpolations for the same coefficients. Generally, the principle we will use in estimating the magnitude of the η's is to use the summation method to predict an additional series coefficient and determine the magnitude of the η's from the accuracy of these predictions. We will now apply this principle to the Padé-approximant method. Empirically, one of the most reliable methods of assessing the error of $f(x_0)$ has been to compare the results for successive Padé approximants, and to notice that the difference behaves, at least for small x, like a high power of x. At some distance from the origin, this law breaks down and the magnitude of the fluctuation remains relatively constant beyond that point. A relatively conservative guide has been to simply extrapolate the small-x error law to x_0, the point of interest. Frequently, a much higher degree of consistency has been exhibited in the location of a singularity and the exponent value than would be warranted under the above guide. The temptation has been irresistible to quote much smaller purely *ad hoc* errors.

We will now consider the relation of the difference of successive Padé approximants to the magnitude of the η's. Let us denote

$$[N+j/N]\,(x)=P_N^{(j)}(x)/Q_N^{(j)}(x)\;, \qquad (3.7)$$

where we will normalize

$$Q_N^{(j)}(0)=1.0\;, \qquad (3.8)$$

which differs from the convention of Baker[19] where he used

$$Q_N^{(j)}(0)=D(1+j, N-1)\;. \qquad (3.9)$$

We use the notation

$$D(m,n)=\det\begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+n} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{m+n} & f_{m+n+1} & \cdots & f_{m+2n}\end{vmatrix}, \tag{3.10}$$

where the function being approximated has the expansion

$$f(x)=\sum_{j=0}^{\infty} f_j x^j . \tag{3.11}$$

Now it follows directly from the definition of the Padé approximants that

$$\frac{P_N^{(j)}(x)}{Q_N^{(j)}(x)}-\frac{P_N^{(j-1)}(x)}{Q_N^{(j-1)}(x)}=O(x^{2N+j}) . \tag{3.12}$$

Thus by cross multiplication we have

$$P_N^{(j)}(x)Q_N^{(j-1)}(x)-Q_N^{(j)}(x)P_N^{(j-1)}(x)=O(x^{2N+j}) . \tag{3.13}$$

But the left-hand side is a polynomial of degree $2N+j$. From the determinantal solution for the Padé approximants, we find directly that

$$\frac{P_N^{(j)}(x)}{Q_N^{(j)}(x)}-\frac{P_N^{(j-1)}(x)}{Q_N^{(j-1)}(x)}=\frac{D(j,N)x^{2N+j}}{D(j,N-1)Q_N^{(j)}(x)Q_N^{(j-1)}(x)} . \tag{3.14}$$

The expansion on the right-hand side gives the difference in the higher coefficients implied by the two successive Padé approximants. In particular, since the coefficient of x^{2N+j} is exact in the $[N+j/N]$ Padé approximant, the error in it made by the $[N+j-1/N]$ is

$$R_{N+j-1,N}=D(j,N)/D(j,N-1) . \tag{3.15}$$

It is to be noted that the denominator (3.15) is the $(N+1, N+1)$ minor of the numerator determinant. Equation (3.15) is, in addition to the product of the coefficients of the highest power of x in $P_N^{(j)}(x)$ and $Q_N^{(j-1)}(x)$, the last diagonal element in the triangularization of the matrix corresponding to $D(j,N)$, as can be easily seen from the fact that in forming an upper triangular matrix, every operation leaves the determinant unchanged. Equation (3.15) is perhaps computationally most simply given as the $(2N+j)$th coefficient in $f(x)Q_N^{(j-1)}(x)$. If the matrix corresponding to $D(j,N)$ is positive definite, then it can be proved[26] that $R_{N+j-1,N}\le 1$.

The principles underlying the table-comparison method of error assessment become clear from (3.14). As long as we are away from roots of $Q_N^{(j)}(x)$ and $Q_N^{(j-1)}(x)$ and x is small compared to $[D(j,N-1)/D(j,N)]^{1/(2N+j)}$ the difference between successive approximants is small. Once this difference has been made small, the difference between these approximants (the sum of an infinite series of differences) and the function value will likewise be small. The justification is to be found in the theorems of de Montessus de Balliore[27] and Wilson[23] on the convergence of horizontal rows in the Padé table. The presumption is that when the successive differences are small and getting smaller, then we are in the convergent region described by their theorems which are valid for a wide class of functions, and convergence is then obtained in a manner analogous to the convergence of a Taylor series within its radius of convergence.

If we are analyzing a singularity by computing the Padé approximants to the logarithmic derivative of a function, then we can use a different reduction of (3.5) to estimate the relative errors in A and Y together with (3.15) to estimate the magnitude of the η's. In place of (3.6), we write

$$\begin{aligned}\Delta G &= (G+J+1)(g+J)(2\eta_{J+1}-\eta_J-\eta_{J+2}) ,\\ \Delta Y/Y &= -\Delta G(G+J)^{-1}+\eta_{J+1}-\eta_J ,\\ \frac{\Delta A}{A} &= \frac{-J\Delta Y}{Y}-\left(\sum_{k=0}^{J-1}\frac{1}{G+k}\right)\Delta G+\eta_J .\end{aligned} \tag{3.16}$$

Now for this analysis, $G=g=1.0$ or $\Delta G=0$; thus $\Delta Y/Y$ is of order η and $\Delta A/A$ is of the order of J times η. Since A plays the role of the exponent in the singularity, this ratio of errors confirms that found immediately following (3.6). We would estimate

$$\eta_{N+j-1,N}\approx A^{-1}Y^{-2N-j}D(j,N)/D(j,N-1) \tag{3.17}$$

here, since AY^{2N+j} is the coefficient in the singularity expansion.

Following the procedures given here leads to a relatively objective procedure of error assessment. It should have the same general strength and weaknesses as usual procedures for assessing the accuracy of the sum of a power series thought to be within its radius of convergence.

The above-described methods assume that one has reached such an order of approximation that the above-mentioned asymptotic convergence thoerems hold. *Prima facie* evidence that that assumption is false would be the appearance of a pole in a previously converging region of the complex plane where there is no singularity of the approximated function. As a practical matter, one frequently observes such poles, howbeit with very small residue. We call such a pair of one pole and one nearby zero a *defect*. For most problems so far encountered in the theory of critical phenomena, the residue of poles of physical interest is in the range 0.1–10. We have adopted an arbitrary, but serviceable, definition that a pole is a defect if it has a residue of absolute value less than 0.003 and lies in a circle centered approximately at the origin whose radius is geared to (usually equal to) the distance to the physically interesting singularity.

It will be helpful to examine the origin of these defects. First, the Padé approximants are determined by the solution of a set of linear equations. Suppose the determinant of the system of Padé equations vanishes. Then we know from the theory of linear simultaneous equations that the equations are either inconsistent, as, for example, are

$$x+y=2, \quad 2x+2y=5 \;, \tag{3.18}$$

or have an infinite number of solutions, as, for example, have

$$x+y=2, \quad 2x+2y=4 \;. \tag{3.19}$$

In the case of the Padé approximants, if the previous (N and D each 1 smaller) approximant had a nonzero determinant and the case where there are an infinite number of solutions holds, then it is easy to verify that

$$[N/D]=[N-1/D-1]\,(1+\alpha x)/(1+\alpha x) \;, \tag{3.20}$$

where α is arbitrary, is the infinite family of solutions. In the case where the determinant is not zero, but in some sense very small, a particular value of α will be selected. The α in the numerator will differ by very little from that in the denominator, provided the determinant is small enough. An examination of which $D(m,n)$ is implied small reveals that the $R_{N-1,D-1}$ of (3.15) is necessarily very small compared to the coefficient f_{D+N-1}. Thus, the relevant η's will be very small and the results of the $[N-1/D]$ and the $[N/D-1]$ will be very close to those for the $[N-1/D-1]$. By this line of reasoning, we will also expect that the poles of the $[N/D]$ will only be perturbed versions of those of the $[N-1/D-1]$. However, the appearance of a defect shows that the projection of f_{N+D} from $[N-1/D-1]$, $[N-1/D]$, or $[N/D-1]$ is not so good or, put another way, that the error in f_{N+D-1} projected from $[N-1/D-1]$ was abnormally small. It can, of course, happen that several successive coefficients will be projected with abnormally small error. Then we expect the appearance of structures analogous to Padé's blocks in the Padé table which he found in his investigations of normality.[29] Put succinctly, if the $[N/D]$ projects with abnormally small errors in the coefficients f_{N+D+j}, $j=1,\ldots,J$, then we will have a block in the Padé table with corners at $[N/D]$, $[N+J/D]$, $[N/D+J]$, and $[N+J/D+J]$. In this block, all the approximants $[N+k/D+l]$ with k, $l\le J$ will be very closely equal to $[N/D]$ and those with $k+l>J$ should have one or more defects. However, at most $\min(k,l)$ defects can occur anywhere in the block. Padé's results on block structure follow from his consideration of which determinants are necessarily zero (or nearly zero in our case).

The recognition of this structure in the Padé table is very important to the correct interpretation of the results. Instead of the theorems of de Montessus de Balliore and Wilson, we may base our analysis on Theorem 4 of Baker,[19] which leads to roughly geometric convergence for a bounded *subsequence* of the Padé approximants. By "bounded" we mean bounded over an area surrounding the origin and extending almost to the singularity of interest which is free of singularities in the approximated function. This subsequence is necessarily free of defects as a function is unbounded at a pole, no matter how small the residue. Thus, the procedure would be to compute the Padé table and examine it for either values at x_0 or the location of the relevant singularity as required. In addition, the appearance of defects should be noted. The table should next be decomposed into blocks as described above. The η's, computed according to (3.17), should be disregarded in the upper left half of all blocks. That is to say, if the $[N/D]$ projects an abnormally small error in the f_{N+D+j}, $j=1,\ldots,J$, then the $\eta_{N+k,D+l}$ for $0\le k+l<J$, $0\le k$, $0\le l$, should not be used in the assessment of the run of magnitude of the error in the coefficient projection. After this run of magnitude has been established, (3.16) is used to compute the variation in $\Delta Y/Y$ (the same as the η's) and $\Delta A/A$ (a factor of the number of coefficients greater).

If the Padé approximants have reached the stage where they are yielding a reasonably stable result which is in accord with the known physical structure of the function involved in the problem at hand, then these procedures are expected to yield sensible estimates of the remaining variation in the sought parameters.

IV. NUMERICAL RESULTS ON TEST SERIES

In this section we present the analysis of several test functions using the techniques described in Secs. II A and II B. These functions have been chosen to represent different functional forms than are usually considered. The Padé approximant, when applied to the logarithmic derivative of a function, is capable of exactly representing the situation where the original function is a simple product of branch-point singularities. Few physical series other than the two-dimensional Ising spontaneous magnetization are known to have that simple form for a small number of singularities. Most of our test functions have been chosen to illustrate a form that is effectively a sum of branch-point singularities. In all cases, the dominant singularity is at $x=x_1=1$ and has a critical exponent $\gamma_1=1.5$. The test functions illustrate cases where we add varying numbers of additional singularities situated on the negative real axis, along a circle centered close to the origin, and along

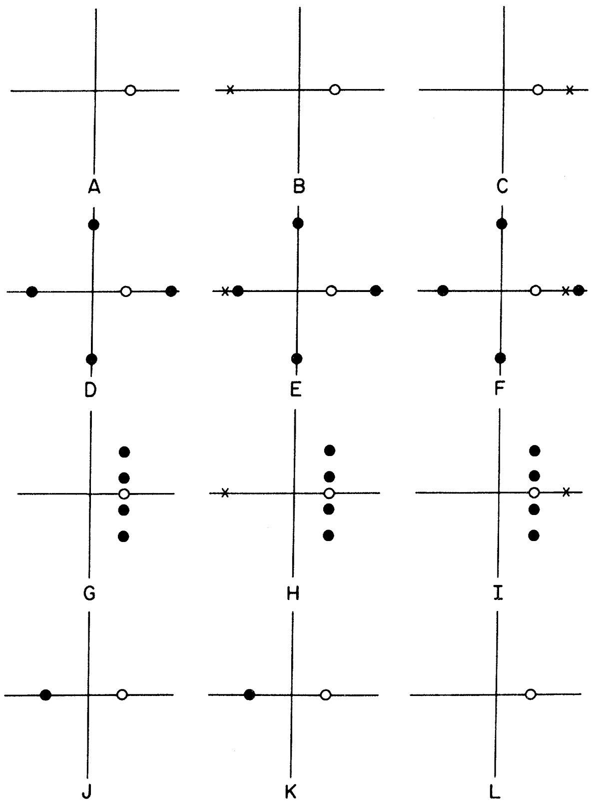

FIG. 2. Singular points in the test functions A–L: the singularity of interest (○), additive singularities at which the functions become infinite (•), and singularities which multiply the singularity of interest and at which the functions remain finite (×). The functions G, H, and I each have two more additive divergent singularities at $x=1 \pm 4.3816i$ which are not shown in the figure.

the line $\mathrm{Re}(x)=1$. In addition, we consider adding an entire function to the dominant singularity, and multiplying it by a function which is analytic at $x=1$ but singular at one other point in the complex x plane. In Table I we list test functions we have considered. The singularities in the complex plane are plotted for each of these functions in Fig. 2.

We have expanded these functions in a Taylor series to 20 terms. We have analyzed each one using the ratio method and the two variants suggested as appropriate if any oscillation is present in the ratios. For the ratio sequences and the biased and unbiased sequences for γ, we have formed the Neville extrapolants through sixth order; we have also formed Neville extrapolants using only alternate elements in the sequences. Finally we have formed complete tables of Padé approximants to the logarithmic derivatives of the series, examined the roots and residues, and also formed Padé approximants to $1-x$ multiplied by the logarithmic derivative and evaluated them at $x=1$, thereby obtaining critical-point estimates and unbiased and biased exponent estimates using Padé approximants.

To present details of the analysis using all these procedures for all 12 series would seem unnecessarily tedious. Rather, we have chosen to present details for just four of the series, selected so that most of the interesting features are illustrated. We then summarize the results for all the series in tabular form in a way that illustrates the improvement in the estimates as the number of terms used increases.

The functions we have chosen for close examination are C, G, K, and L. For C the ratio method with Neville extrapolation gives better results; for G and K the Padé method is most useful. The sequences for G are sufficiently irregular that Neville extrapolation is not indicated, but considerable improvement in the ratio results for the other series is obtained with Neville extrapolation. For C and K we find that the ρ_n variation of the ratio method is preferable. L is included to illustrate the universally poor results for γ when the logarithmic factor is present. Our discussion will be centered on the first three functions.

In Table II we present the ratio analysis for these four functions. Columns 2–5 show the sequences for the ratios r_n and for y_n, unbiased γ_n, and biased γ_n using (2.4), (2.5), (2.7), and (2.6), respectively. Columns 6–9 are similar but (2.17) for ρ_n replaces r_n throughout, and linear extrapolation of ρ_n's following (2.16) replaces y_n. For K, the oscillation in the ratios due to the strong

TABLE I. Functions A–L which we have used to study relative merits of current methods of series analysis. The functions A–K contain a dominant singularity at $x=1$ and additive terms which may have other singularities in various patterns in the complex plane. Function L is singular only at $x=1$ and is included only to show the effect of the logarithmic factor.

A	$(1-x)^{-1.5}+e^{-x}$
B	$(1-x)^{-1.5}(1+\frac{1}{2}x)^{1.5}+e^{-x}$
C	$(1-x)^{-1.5}(1-\frac{1}{2}x)^{1.5}+e^{-x}$
D	$(1-x)^{-1.5}+(1+\frac{1}{4}x^2)^{-1.25}+(1+\frac{15}{112}x-\frac{1}{4}x^2)^{-1.25}$
E	$(1-x)^{-1.5}(1+\frac{1}{2}x)^{1.5}+(1+\frac{1}{4}x^2)^{-1.25}+(1+\frac{15}{112}x-\frac{1}{4}x^2)^{-1.25}$
F	$(1-x)^{-1.5}(1-\frac{1}{2}x)^{1.5}+(1+\frac{1}{4}x^2)^{-1.25}+(1+\frac{15}{112}x-\frac{1}{4}x^2)^{-1.25}$
G	$(1-x)^{-1.5}+\{2(1-x)(2-x)^6/[(2-x)^7-x^7]\}^{1.25}$
H	$(1-x)^{-1.5}(1+\frac{1}{2}x)^{1.5}+\{2(1-x)(2-x)^6/[(2-x)^7-x^7]\}^{1.25}$
I	$(1-x)^{-1.5}(1-\frac{1}{2}x)^{1.5}+\{2(1-x)(2-x)^6/[(2-x)^7-x^7]\}^{1.25}$
J	$(1-x)^{-1.5}+(1+\frac{4}{5}x)^{-1.25}$
K	$(1-x)^{-1.5}+(1+\frac{4}{5}x)^{-1.25}+e^{-x}$
L	$-(1-x)^{-1.5}\ln(1-x)$

TABLE II. Ratio analysis of (a) C, (b) G, (c) K, and (d) L. Columns 3–5 show the sequences y_n, γ_n (unbiased), and γ_n (biased) calculated using the ratios r_n, while columns 7–9 show similar sequences calculated using $\rho_n = (r_n r_{n-1})^{1/2}$.

n	r_n	y_n	$\gamma_n^{(u)}$	$\gamma_n^{(b)}$	ρ_n	y_n	$\gamma_n^{(u)}$	$\gamma_n^{(b)}$
				(a) Test series C				
1	0.1250			−0.1250				
2	−5.3750	−10.6250	0.0118	−11.7500	0.8197			
3	0.5678	12.4535	−1.8632	−0.2965				
4	1.3773	3.8059	−1.5524	2.5094	0.8844	0.9490	0.7274	0.5375
5	1.0232	−0.3936	−16.9981	1.1158	1.1871			1.9356
6	1.0741	1.3287	−0.1496	1.4445	1.0483	1.3762	−0.4296	1.2899
7	1.0559	0.9470	1.8051	1.3915	1.0650	0.7596	3.8144	1.4547
8	1.0518	1.0228	1.2264	1.4143	1.0539	1.0705	0.8757	1.4308
9	1.0468	1.0072	1.3539	1.4215	1.0493	0.9945	1.4958	1.4438
10	1.0429	1.0075	1.3510	1.4291	1.0449	1.0089	1.3561	1.4487
11	1.0396	1.0060	1.3663	1.4351	1.0412	1.0049	1.3979	1.4535
12	1.0367	1.0051	1.3767	1.4403	1.0381	1.0044	1.4031	1.4575
13	1.0342	1.0044	1.3859	1.4446	1.0354	1.0036	1.4122	1.4608
14	1.0320	1.0038	1.3939	1.4484	1.0331	1.0031	1.4191	1.4636
15	1.0301	1.0033	1.4008	1.4517	1.0311	1.0027	1.4251	1.4661
16	1.0284	1.0029	1.4069	1.4546	1.0293	1.0023	1.4303	1.4682
17	1.0269	1.0026	1.4123	1.4572	1.0277	1.0020	1.4349	1.4701
18	1.0255	1.0023	1.4171	1.4595	1.0262	1.0018	1.4388	1.4718
19	1.0243	1.0020	1.4214	1.4616	1.0249	1.0016	1.4423	1.4733
				(b) Test series G				
1	0.4375			0.4375				
2	1.8304	3.2232	0.1357	2.6607	0.8949			
3	1.2957	0.2265	15.1635	1.8872	1.5400			
4	1.1664	0.7784	2.9939	1.6656	1.2294	1.5639	0.1444	1.9175
5	1.1143	0.9061	2.1491	1.5717	1.1401	0.5401	6.5533	1.7003
6	1.0881	0.9569	1.8224	1.5286	1.1011	0.8447	2.8216	1.6069
7	1.0760	1.0037	1.5047	1.5323	1.0821	0.9370	2.0835	1.5744
8	1.0681	1.0131	1.4350	1.5454	1.0721	0.9850	1.7076	1.5768
9	1.0596	0.9912	1.6216	1.5365	1.0639	1.0003	1.5722	1.5750
10	1.0504	0.9679	1.8531	1.5044	1.0550	0.9867	1.6925	1.5502
11	1.0424	0.9616	1.9243	1.4660	1.0464	0.9677	1.8948	1.5103
12	1.0365	0.9722	1.7935	1.4382	1.0394	0.9615	1.9724	1.4732
13	1.0330	0.9908	1.5532	1.4290	1.0348	0.9708	1.8569	1.4519
14	1.0313	1.0085	1.3157	1.4375	1.0321	0.9883	1.6210	1.4498
15	1.0305	1.0204	1.1486	1.4580	1.0309	1.0058	1.3748	1.4634
16	1.0302	1.0255	1.0730	1.4835	1.0304	1.0181	1.1928	1.4860
17	1.0299	1.0248	1.0847	1.5083	1.0301	1.0238	1.1037	1.5110
18	1.0293	1.0197	1.1702	1.5280	1.0296	1.0235	1.1067	1.5331
19	1.0284	1.0118	1.3118	1.5398	1.0289	1.0188	1.1883	1.5486
				(c) Test series K				
1	−0.1667			−0.1667				
2	−6.5500	−12.9333	0.0129	−14.1000	1.0448			
3	0.3789	14.2366	−1.9202	−0.8634				
4	2.5512	9.0681	−1.8747	7.2048	0.9832	0.9214	1.2677	0.9326
5	0.6766	−6.8219	−4.4959	−0.6171	1.3138			2.5690
6	1.5866	6.1366	−3.4487	4.5195	1.0361	1.1419	0.4439	1.2164
7	0.8114	−3.8394	−7.4794	−0.3199	1.1346	1.6867	5.5655	1.9425
8	1.3258	4.9265	−4.8470	3.6065	1.0372	1.0406	0.9736	1.2977
9	0.8926	−2.5732	−11.1219	0.0333	1.0878	0.9241	2.5950	1.7906
10	1.1995	3.9615	−5.9722	2.9949	1.0347	1.0248	1.0972	1.3472
11	0.9436	−1.6151	−16.4265	0.3797	1.0639	0.9560	2.2409	1.7027
12	1.1297	3.1769	−6.7328	2.5567	1.0325	1.0213	1.1318	1.3898
13	0.9750	−0.8822	−26.3674	0.6745	1.0495	0.9703	2.0603	1.6434
14	1.0887	2.5675	−7.0634	2.2420	1.0303	1.0170	1.1828	1.4237
15	0.9938	−0.3355	−58.4357	0.9065	1.0402	0.9795	1.9294	1.6023
16	1.0636	2.1108	−6.9380	2.0173	1.0281	1.0128	1.2417	1.4493

TABLE II. (*Continued*)

n	r_n	y_n	$\gamma_n^{(u)}$	$\gamma_n^{(b)}$	ρ_n	y_n	$\gamma_n^{(u)}$	$\gamma_n^{(b)}$
				(c) Test series K				
17	1.0047	0.0634	253.4510	1.0807	1.0337	0.9857	1.8292	1.5737
18	1.0477	1.7775	−6.3909	1.8582	1.0260	1.0092	1.2986	1.4678
19	1.0109	0.3493	36.9860	1.2076	1.0291	0.9900	1.7517	1.5536
				(d) Test series L				
1								
2	2.0000			3.0000				
3	1.4792	0.4375	8.1429	2.4375	1.7200			
4	1.3099	0.8019	3.5335	2.2394	1.3919			2.5678
5	1.2270	0.8956	2.8499	2.1351	1.2678	0.5894	6.7541	2.3388
6	1.1782	0.9343	2.5668	2.0693	1.2024	0.8232	3.7633	2.2142
7	1.1462	0.9541	2.4091	2.0235	1.1621	0.8980	3.0590	2.1348
8	1.1237	0.9658	2.3072	1.9893	1.1349	0.9324	2.7371	2.0791
9	1.1070	0.9734	2.2353	1.9627	1.1153	0.9514	2.5502	2.0376
10	1.0941	0.9785	2.1815	1.9412	1.1005	0.9631	2.4270	2.0052
11	1.0839	0.9822	2.1394	1.9234	1.0890	0.9708	2.3391	1.9792
12	1.0757	0.9850	2.1054	1.9084	1.0798	0.9763	2.2728	1.9578
13	1.0689	0.9871	2.0773	1.8955	1.0723	0.9802	2.2209	1.9397
14	1.0632	0.9887	2.0536	1.8842	1.0660	0.9832	2.1789	1.9242
15	1.0583	0.9901	2.0333	1.8743	1.0607	0.9855	2.1442	1.9108
16	1.0541	0.9912	2.0156	1.8655	1.0562	0.9874	2.1149	1.8990
17	1.0504	0.9921	2.0001	1.8575	1.0523	0.9889	2.0899	1.8885
18	1.0472	0.9928	1.9863	1.8504	1.0488	0.9901	2.0681	1.8792
19	1.0444	0.9935	1.9740	1.8439	1.0458	0.9911	2.0491	1.8707

singularity on the negative axis is pronounced and the results using ρ_n are obviously much more accurate. Also, there is a slight advantage to using the ρ_n option for C, due to the alternating signs in the expansion of e^{-x}. There is no reason to suspect any advantage to using ρ_n in analyzing G and this is borne out by the sequences in Table II(b). There are six additional singularities in G and we note that a long-wavelength oscillation is evident in the ratio sequences.

Tables III and IV contain the first- through sixth-order Neville extrapolations of the sequences in Table II. The three blocks of sequences for each series in Table III are the extrapolants formed from columns 2 (r_n), 4 ($\gamma_n^{(u)}$), and 5 ($\gamma_n^{(b)}$) of Table II for the corresponding series, using (2.26). We have omitted the extrapolants for K because the oscillation in r_n is too great for any of the extrapolants to be meaningful. Table IV includes the extrapolants of alternate terms, using (2.27), of columns 6, 8, and 9 from Table II. The forward propagation of the "noise" in early terms of the series is evident in the Neville tables. For C, G, and K, which contain other singularities, extrapolants of order higher than 6 are of no value if only 20 terms are available. L has only one singular point and the indication is that higher-order extrapolants would continue the smooth, but extremely slow, convergence to the known values. For C we use the fourth-order extrapolants of sequences based on r_n; for G the higher-order extrapolants are all poor; for K we use the sixth-order extrapolant of the ρ_n sequences only; and for L we use the sixth-order extrapolants of all three r_n-based sequences. Blanks in the ratio and Neville tables indicate that (i) the quantity is not defined or (ii) the value is not real and finite.

Tables V–VII are Padé tables of roots of Padé approximants to the logarithmic derivatives of the test functions (Table V), residues of the corresponding roots of the same Padé approximants (Table VI), and evaluations at $x=1$ of Padé approximants to $1-x$ times the logarithmic derivatives of the test functions (Table VII). These are tables of estimates for y^{-1} and γ, both unbiased and biased, respectively. The Padé approximants do not, in general, give estimates for parameters of this type, which vary smoothly toward a limiting value. Rather, the Padé approximants have already extracted almost all of the evident trend so that an extrapolation of them is rarely profitable. In general, the approximants closest to the diagonal seem to be more reliable than those of the same order but removed from the diagonal. An exception to this would be the case where the function only has a finite number of simple-pole singularities; here all Padé approximants with denominators of order equal to or exceeding the number of singularities should be reliable. One chooses a value as representative of the Padé results, but the considerations of Sec. III should be taken into account as well as any apparent trend in the Padé table, before assigning confidence limits.

Superscripts in the Padé tables indicate the

TABLE III. Neville extrapolants of first through sixth order using consecutive terms in the r_n sequences for y_n, γ_n (unbiased), and γ_n (biased) for the series (a) C, (b) G, and (c) L. The sequences for the series K have such a marked oscillation that Neville extrapolants of this form are useless in analyzing the series.

n	$l_n^{(1)}$	$l_n^{(2)}$	$l_n^{(3)}$	$l_n^{(4)}$	$l_n^{(5)}$	$l_n^{(6)}$
			(a) y: $l_n^{(0)}=r_n$			
4	3.8059	−4.8417	−14.4532			
5	−0.3936	−6.6928	−7.9269	−6.2953		
6	1.3287	4.7731	16.2390	28.3220	35.2455	
7	0.9470	−0.0072	−6.3809	−23.3458	−44.0129	−57.2226
8	1.0228	1.2504	3.3462	13.0733	34.9248	61.2374
9	1.0072	0.9526	0.3571	−3.3794	−16.5415	−42.2747
10	1.0075	1.0088	1.1399	2.3142	8.0078	24.3740
11	1.0061	0.9994	0.9742	0.6841	−1.2720	−9.0052
12	1.0051	1.0005	1.0040	1.0636	1.5948	4.4616
13	1.0044	1.0003	0.9994	0.9891	0.8699	0.0242
14	1.0038	1.0002	1.0000	1.0016	1.0242	1.2299
15	1.0033	1.0002	1.0000	0.9998	0.9961	0.9540
16	1.0029	1.0001	1.0000	1.0000	1.0006	1.0080
17	1.0026	1.0001	1.0000	1.0000	0.9999	0.9988
18	1.0023	1.0001	1.0000	1.0000	1.0000	1.0002
19	1.0021	1.0001	1.0000	1.0000	1.0000	1.0000
			$\gamma^{(u)}$: $l_n^{(0)}=\gamma_n^{(u)}$			
4	−0.6200	4.3732	8.4747			
5	−78.7808	−196.0221	−329.6190	−414.1424		
6	84.0927	409.8397	1015.7015	1688.3617	2108.8625	
7	13.5340	−162.8627	−926.4658	−2383.0913	−4011.6725	−5031.7617
8	−2.8248	−51.9014	133.0341	1192.5340	3337.9093	5787.7699
9	2.3736	20.5684	165.5078	206.1000	−583.0472	−2543.5254
10	1.3246	−2.8714	−57.5643	−392.1725	−990.4450	−1262.0436
11	1.5199	2.3953	16.4398	145.9470	791.6903	2276.8031
12	1.4915	1.3527	−1.7750	−38.2046	−296.0167	−1383.7238
13	1.4968	1.5259	2.1033	10.8295	89.2841	538.8017
14	1.4972	1.4995	1.4028	−0.3485	−20.4689	−166.8062
15	1.4979	1.5022	1.5129	1.8157	6.1440	46.0653
16	1.4983	1.5014	1.4978	1.4525	0.6537	−8.4969
17	1.4986	1.5011	1.4998	1.5063	1.6352	3.4347
18	1.4989	1.5009	1.4997	1.4993	1.4811	1.1730
19	1.4991	1.5007	1.4998	1.5001	1.5025	1.5488
			$\gamma^{(b)}$: $l_n^{(0)}=\gamma_n^{(b)}$			
4	10.9271	−0.7563	−16.2095			
5	−4.4585	−27.5369	−45.3907	−52.6859		
6	3.0877	18.1802	63.8974	118.5414	152.7868	
7	1.0734	−3.9624	−33.4858	−106.5232	−196.5490	−254.7716
8	1.5742	3.0764	14.8075	63.1009	164.8753	285.3501
9	1.4794	1.1476	−2.7099	−24.6067	−94.7728	−224.5969
10	1.4970	1.5674	2.5470	10.4324	45.4715	138.9678
11	1.4957	1.4900	1.2835	−0.9276	−14.5597	−64.5857
12	1.4967	1.5019	1.5378	2.0463	6.2099	26.9796
13	1.4973	1.5001	1.4941	1.3959	0.3553	−6.4751
14	1.4977	1.5003	1.5007	1.5170	1.7349	3.5744
15	1.4980	1.5002	1.4998	1.4975	1.4586	1.0442
16	1.4983	1.5001	1.5000	1.5003	1.5064	1.5860
17	1.4985	1.5001	1.5000	1.5000	1.4992	1.4859
18	1.4987	1.5001	1.5000	1.5000	1.5001	1.5020
19	1.4988	1.5001	1.5000	1.5000	1.5000	1.4997

TABLE III. (*Continued*)

n	$l_n^{(1)}$	$l_n^{(2)}$	$l_n^{(3)}$	$l_n^{(4)}$	$l_n^{(5)}$	$l_n^{(6)}$
			(b) y: $l_n^{(0)}=r_n$			
4	0.7783	1.3383	2.1977			
5	0.9060	1.0976	0.9425	0.6288		
6	0.9569	1.0586	1.0196	1.0581	1.1440	
7	1.0037	1.1205	1.2030	1.3406	1.4536	1.5052
8	1.0131	1.0413	0.9093	0.6156	0.1805	−0.2438
9	0.9912	0.9144	0.6606	0.3497	0.1370	0.1153
10	0.9679	0.8747	0.7821	0.9644	1.5792	2.5406
11	0.9616	0.9332	1.0891	1.6263	2.4206	3.1218
12	0.9722	1.0256	1.3026	1.7297	1.8744	1.3282
13	0.9908	1.0932	1.3187	1.3548	0.7549	−0.5512
14	1.0085	1.1145	1.1928	0.8780	0.0199	−0.9601
15	1.0204	1.0978	1.0310	0.5860	0.0021	−0.0245
16	1.0255	1.0614	0.9033	0.5203	0.3757	0.9983
17	1.0248	1.0192	0.8227	0.5608	0.6580	1.1754
18	1.0197	0.9789	0.7769	0.6165	0.7612	0.9678
19	1.0118	0.9448	0.7631	0.7113	0.9767	1.4436
			$\gamma^{(u)}$: $l_n^{(0)}=\gamma_n^{(u)}$			
4	−33.5149	−112.2487	−172.3958			
5	−1.2299	47.1975	153.4950	234.9677		
6	0.1888	3.0264	−41.1447	−138.4646	−213.1511	
7	−0.4014	−1.8769	−8.4146	16.1330	77.9720	126.4926
8	0.9472	4.9928	16.4423	41.2991	56.3988	49.2077
9	3.1146	10.7008	22.1167	29.2097	19.5382	1.1079
10	3.9366	7.2245	−0.8868	−35.3921	−99.9940	−179.6821
11	2.6359	−3.2171	−31.0613	−83.8666	−142.0360	−177.0711
12	0.3546	−11.0520	−34.5566	−41.5471	17.7002	177.4365
13	−1.3304	−10.5980	−9.0849	48.2262	191.8636	395.0541
14	−1.7715	−4.4184	18.2403	86.5535	155.5425	107.1143
15	−1.1904	2.5869	30.6082	64.6199	20.7529	−181.4314
16	−0.0608	7.8463	30.6371	30.7236	−43.8482	−151.5168
17	1.2703	11.2539	27.1559	15.8422	−19.8732	24.0810
18	2.6241	13.4544	24.4568	15.0097	12.8453	78.2822
19	3.8605	14.3700	19.2535	−0.2586	−43.0100	−164.0296
			$\gamma^{(b)}$: $l_n^{(0)}=\gamma_n^{(b)}$			
4	1.0008	1.6614	2.8591			
5	1.1961	1.4890	1.3741	1.0029		
6	1.3133	1.5478	1.6066	1.7229	1.8669	
7	1.5543	2.1569	2.9690	3.9908	4.8980	5.4032
8	1.6370	1.8848	1.4313	−0.1064	−2.5648	−5.0524
9	1.4658	0.8666	−1.1699	−4.4213	−7.8732	−10.5274
10	1.2152	0.2130	−1.3121	−1.5255	1.3703	7.5326
11	1.0816	0.4803	1.1932	5.5775	14.1012	24.7103
12	1.1327	1.3882	4.1120	9.9496	16.0705	18.0399
13	1.3191	2.3443	5.5311	8.7241	6.7633	−4.0952
14	1.5482	2.9226	5.0432	3.8235	−4.9976	−20.6788
15	1.7438	3.0156	3.3875	−1.1659	−11.1445	−20.3649
16	1.8665	2.7256	1.4688	−4.2874	−11.1547	−11.1715
17	1.9050	2.1938	−0.2880	−5.9975	−10.1020	−8.1721
18	1.8628	1.5244	−1.8221	−7.1914	−10.2955	−10.6825
19	1.7523	0.8137	−2.9770	−7.3079	−7.6339	−1.8670

TABLE III. (*Continued*)

n	$l_n^{(1)}$	$l_n^{(2)}$	$l_n^{(3)}$	$l_n^{(4)}$	$l_n^{(5)}$	$l_n^{(6)}$
			(c)			
			y: $l_n^{(0)}=r_n$			
4	0.8019	1.1664				
5	0.8956	1.0362	0.9494			
6	0.9343	1.0115	0.9868	1.0054		
7	0.9541	1.0039	0.9937	0.9989	0.9963	
8	0.9658	1.0009	0.9961	0.9984	0.9981	0.9987
9	0.9734	0.9997	0.9971	0.9985	0.9986	0.9988
10	0.9785	0.9991	0.9978	0.9987	0.9989	0.9990
11	0.9822	0.9988	0.9982	0.9989	0.9990	0.9992
12	0.9850	0.9987	0.9984	0.9990	0.9992	0.9993
13	0.9871	0.9987	0.9986	0.9991	0.9993	0.9994
14	0.9887	0.9987	0.9988	0.9992	0.9994	0.9995
15	0.9901	0.9988	0.9989	0.9993	0.9994	0.9995
16	0.9912	0.9988	0.9990	0.9993	0.9995	0.9996
17	0.9921	0.9989	0.9991	0.9994	0.9995	0.9996
18	0.9928	0.9989	0.9992	0.9994	0.9996	0.9996
19	0.9935	0.9990	0.9993	0.9995	0.9996	0.9997
			$\gamma^{(u)}$: $l_n^{(0)}=\gamma_n^{(u)}$			
4	−10.2947	−47.0179				
5	0.1156	15.7311	57.5637			
6	1.1512	3.2223	−9.2865	−42.7115		
7	1.4627	2.2415	0.9338	8.5991	29.1233	
8	1.5943	1.9890	1.5680	2.2022	−1.6359	−11.8889
9	1.6603	1.8912	1.6955	1.8549	1.5769	3.1834
10	1.6969	1.8433	1.7317	1.7860	1.7171	1.8106
11	1.7185	1.8159	1.7429	1.7626	1.7346	1.7491
12	1.7318	1.7984	1.7457	1.7512	1.7352	1.7357
13	1.7402	1.7861	1.7452	1.7440	1.7324	1.7292
14	1.7454	1.7769	1.7433	1.7386	1.7291	1.7246
15	1.7487	1.7697	1.7409	1.7343	1.7257	1.7207
16	1.7506	1.7638	1.7383	1.7307	1.7226	1.7174
17	1.7515	1.7589	1.7358	1.7274	1.7197	1.7144
18	1.7519	1.7546	1.7333	1.7246	1.7170	1.7117
19	1.7518	1.7509	1.7309	1.7219	1.7145	1.7092
			$\gamma^{(b)}$: $l_n^{(0)}=\gamma_n^{(b)}$			
4	1.6452	1.9780				
5	1.7177	1.8263	1.7251			
6	1.7406	1.7866	1.7469	1.7578		
7	1.7484	1.7677	1.7426	1.7393	1.7319	
8	1.7503	1.7559	1.7362	1.7298	1.7241	1.7215
9	1.7496	1.7473	1.7302	1.7228	1.7172	1.7137
10	1.7478	1.7406	1.7250	1.7171	1.7114	1.7076
11	1.7455	1.7351	1.7204	1.7123	1.7066	1.7026
12	1.7430	1.7304	1.7163	1.7082	1.7025	1.6984
13	1.7404	1.7263	1.7127	1.7047	1.6989	1.6947
14	1.7379	1.7227	1.7095	1.7015	1.6957	1.6915
15	1.7354	1.7195	1.7066	1.6986	1.6929	1.6887
16	1.7331	1.7166	1.7040	1.6960	1.6904	1.6861
17	1.7308	1.7139	1.7015	1.6937	1.6880	1.6838
18	1.7287	1.7115	1.6993	1.6915	1.6859	1.6817
19	1.7266	1.7093	1.6973	1.6895	1.0840	1.6798

number of defects in a particular approximant. Asterisks indicate that the approximant contains a pole on the real axis between the origin and the pole of interest, whose residue is large enough that it is not classed as a defect. Unless one expects a pole in that position, such approximants

TABLE IV. Neville extrapolants of first through sixth order using alternate terms in the ρ_n sequence for y_n, γ_n (unbiased) and γ_n (biased) for the series (a) C, (b) G, (c) K, and (d) L.

n	$l_n^{(1)}$	$l_n^{(2)}$	$l_n^{(3)}$	$l_n^{(4)}$	$l_n^{(5)}$	$l_n^{(6)}$
			(a)			
			y: $l_n^{(0)}=\rho_n$			
4	0.9491					
5						
6	1.3762	1.5898				
7	0.7596					
8	1.0705	0.7648	0.4897			
9	0.9945	1.2882				
10	1.0089	0.9166	1.0179	1.1499		
11	1.0049	1.0230	0.8020			
12	1.0044	0.9953	1.0739	1.1019	1.0923	
13	1.0036	1.0008	0.9749	1.0830		
14	1.0031	0.9998	1.0060	0.9550	0.8962	0.8635
15	1.0027	1.0000	0.9988	1.0196	0.9879	
16	1.0023	1.0000	1.0002	0.9944	1.0180	1.0586
17	1.0020	1.0000	1.0000	1.0013	0.9885	0.9887
18	1.0018	1.0000	1.0000	0.9998	1.0041	0.9971
19	1.0016	1.0000	1.0000	1.0000	0.9989	1.0050
			$\gamma^{(u)}$: $l_n^{(0)}=\gamma_n^{(u)}$			
4	0.4548					
5						
6	−2.7434	−4.3425				
7						
8	4.7916	12.3267	−17.8831			
9	−6.6195					
10	3.2777	1.0068	−6.5398	−12.6456		
11	0.9575	14.2174				
12	1.6380	−1.6415	−4.2898	−3.1647	−1.2685	
13	1.4905	2.6097	−10.7593			
14	1.5152	1.2884	5.0083	11.9818	18.0404	21.2585
15	1.5096	1.5623	−0.1288	9.1728		
16	1.5089	1.4900	1.9593	−1.0896	−8.9324	−17.9233
17	1.5078	1.5017	1.3908	3.1004	−1.1503	
18	1.5069	1.4997	1.5192	0.9690	2.6158	8.3899
19	1.5061	1.4998	1.4957	1.6399	0.3254	1.1862
			$\gamma^{(b)}$: $l_n^{(0)}=\gamma_n^{(b)}$			
4	0.4355					
5						
6	2.7947	3.9742				
7	0.2526					
8	1.8537	0.9127	−0.1078			
9	1.4054	2.8465				
10	1.5202	1.0200	1.0915	1.3913		
11	1.4975	1.6585	0.6684			
12	1.5013	1.4635	1.9070	2.3147	2.4994	
13	1.5007	1.5079	1.3322	1.7471		
14	1.5007	1.4992	1.5468	1.2766	0.8614	0.5884
15	1.5007	1.5004	1.4892	1.6265	1.5661	
16	1.5005	1.5001	1.5018	1.4567	1.5648	1.7993
17	1.5005	1.5001	1.4995	1.5112	1.4306	1.3741
18	1.5004	1.5001	1.4999	1.4977	1.5304	1.5132
29	1.5004	1.5000	1.4999	1.5004	1.4907	1.5258

TABLE IV. (*Continued*)

n	$l_n^{(1)}$	$l_n^{(2)}$	$l_n^{(3)}$	$l_n^{(4)}$	$l_n^{(5)}$	$l_n^{(6)}$
			(b)			
			y: $l_n^{(0)}=\rho_n$			
4	0.9215					
5						
6	1.1419	1.2521				
7	0.6867					
8	1.0406	0.9394	0.8351			
9	0.9241	1.2207				
10	1.0248	1.0009	1.0419	1.0936		
11	0.9560	1.0120	0.8380			
12	1.0213	1.0143	1.0277	1.0205	1.0059	
13	0.9703	1.0025	0.9915	1.0875		
14	1.0170	1.0063	0.9956	0.9715	0.9519	0.9430
15	0.9795	1.0046	1.0076	1.0217	0.9888	
16	1.0128	1.0002	0.9960	0.9844	0.9921	1.0055
17	0.9857	1.0058	1.0081	1.0087	0.9996	1.0042
18	1.0092	0.9969	0.9903	0.9907	0.9958	0.9977
19	0.9900	1.0061	1.0067	1.0048	1.0013	1.0023
			$\gamma^{(u)}$: $l_n^{(0)}=\gamma_u^{(u)}$			
4	1.5354					
5						
6	−1.2037	−2.5732				
7						
8	2.5628	6.3292	9.2966			
9	−7.8015					
10	1.5919	0.1355	−3.9936	−7.3161		
11	0.6468	15.4314				
12	1.3042	0.7289	1.3223	3.9802	6.2395	
13	1.0677	2.0147	−13.6381			
14	1.4894	1.9522	3.5833	5.2791	5.7986	5.7252
15	1.0779	1.1061	−0.2568	11.4518		
16	1.6538	2.1470	2.4717	1.3601	−0.9913	−3.2546
17	1.0776	1.0766	1.0225	2.4616	−3.8315	
18	1.7533	2.1015	2.0105	1.4340	1.4931	2.7353
19	1.0929	1.1501	1.3095	1.7041	1.0223	3.8537
			$\gamma^{(b)}$: $l_n^{(0)}=\gamma_n^{(b)}$			
4	0.7756					
5						
6	1.7841	2.2883				
7	0.3762					
8	1.5416	1.2992	0.9695			
9	1.2592	2.3629				
10	1.5453	1.5508	1.7186	1.9058		
11	1.3070	1.3909	0.5809			
12	1.6024	1.7168	1.8827	1.9648	1.9766	
13	1.3172	1.3400	1.2807	1.7181		
14	1.6276	1.6904	1.6551	1.4844	1.2922	1.1782
15	1.3354	1.3856	1.4541	1.6058	1.5497	
16	1.6282	1.6303	1.5302	1.4053	1.3578	1.3796
17	1.3587	1.4344	1.5238	1.6023	1.5999	1.6208
18	1.6158	1.5722	1.4561	1.3634	1.3299	1.3160
19	1.3832	1.4748	1.5624	1.6155	1.6275	1.6436

TABLE IV. (*Continued*)

n	$l_n^{(1)}$	$l_n^{(2)}$	$l_n^{(3)}$	$l_n^{(4)}$	$l_n^{(5)}$	$l_n^{(6)}$
			(c)			
			y: $l_n^{(0)}=\rho_n$			
4	1.5639					
5	0.5401					
6	0.8447	0.4851				
7	0.9370	1.2347				
8	0.9850	1.1253	1.3386			
9	1.0003	1.0794	1.0017			
10	0.9867	0.9893	0.8986	0.7886		
11	0.9678	0.9106	0.7700	0.6831		
12	0.9615	0.9112	0.8331	0.8003	0.8027	
13	0.9708	0.9777	1.0560	1.2348	1.4003	
14	0.9883	1.0552	1.2473	1.5579	1.8609	2.0372
15	1.0058	1.1020	1.2884	1.4918	1.6203	1.6752
16	1.0181	1.1076	1.1948	1.1424	0.8931	0.5705
17	1.0238	1.0825	1.0467	0.7747	0.2728	−0.2887
18	1.0235	1.0426	0.9127	0.5599	0.0940	−0.3056
19	1.0188	0.9999	0.8209	0.5105	0.2726	0.2725
			$\gamma^{(u)}$: $l_n^{(0)}=\gamma_n^{(u)}$			
4	−0.7111					
5	16.3832					
6	8.1758	12.6193				
7	−9.0911	−28.1968				
8	−1.6341	−11.4441	−19.4653			
9	−0.2171	10.8754	30.4114			
10	1.6321	6.5315	18.5152	28.0104		
11	3.3464	9.5826	8.5054	0.2906		
12	3.3720	6.8517	7.1718	1.5001	−3.8019	
13	1.6487	−2.1714	−15.8844	−31.1280	−40.5535	
14	−0.4873	−10.1353	−32.7845	−62.7518	−88.4526	−102.5611
15	−1.7593	−11.1312	−24.5709	−32.1717	−32.6935	−30.7286
16	−1.8049	−5.7576	1.5384	35.8613	95.0292	156.1899
17	−0.9294	1.7679	25.4163	81.6520	161.3286	242.1712
18	0.4180	8.1980	36.1094	79.3231	114.0925	123.6241
19	1.9074	12.5454	35.8966	50.3069	22.0962	−59.1227
			$\gamma^{(b)}$: $l_n^{(0)}=\gamma_n^{(b)}$			
4	3.0452					
5	0.3208					
6	0.9856	−0.0441				
7	1.2595	1.9635				
8	1.4867	1.9877	2.6649			
9	1.5770	1.9738	1.9789			
10	1.4437	1.3792	0.9736	0.5507		
11	1.2194	0.5936	−0.5565	−1.5072		
12	1.0884	0.3778	−0.6236	−1.4222	−1.8168	
13	1.1303	0.9298	1.3219	2.4959	3.6968	
14	1.3092	1.8613	3.8394	7.1866	10.6302	12.7046
15	1.5383	2.6603	5.2562	8.6987	11.8001	13.8259
16	1.7395	3.0302	4.9784	6.1175	5.4760	3.7580
17	1.8682	2.9405	3.4540	1.4266	−3.6639	−10.1072
18	1.9098	2.5062	1.4580	−2.9426	−10.1907	−18.0240
19	1.8677	1.8659	−0.4624	−5.8474	−12.3939	−17.4864

TABLE IV. (*Continued*).

n	$l_n^{(1)}$	$l_n^{(2)}$	$l_n^{(3)}$	$l_n^{(4)}$	$l_n^{(5)}$	$l_n^{(6)}$
			(d) y: $l_n^{(0)}=\rho_n$			
4						
5	0.5894					
6	0.8232					
7	0.8980	1.1294				
8	0.9324	1.0416				
9	0.9514	1.0182	0.9626			
10	0.9631	1.0091	0.9874			
11	0.9708	1.0048	0.9937	1.0054		
12	0.9763	1.0026	0.9961	1.0005		
13	0.9802	1.0014	0.9973	0.9995	0.9978	
14	0.9832	1.0006	0.9979	0.9993	0.9988	
15	0.9855	1.0002	0.9983	0.9993	0.9991	0.9994
16	0.9874	0.9999	0.9986	0.9993	0.9993	0.9994
17	0.9889	0.9997	0.9988	0.9993	0.9994	0.9994
18	0.9901	0.9996	0.9990	0.9994	0.9994	0.9995
19	0.9911	0.9995	0.9991	0.9994	0.9995	0.9995
			$\gamma^{(u)}$: $l_n^{(0)}=\gamma_n^{(u)}$			
4						
5	16.8852					
6						
7	−6.1787	−23.4766				
8	−0.3413					
9	0.7696	9.4549	25.9206			
10	1.1866	3.4785				
11	1.3889	2.4727	−3.3458	−14.3207		
12	1.5018	2.1320	0.7856			
13	1.5707	1.9797	1.4046	4.3736	9.9820	
14	1.6154	1.8996	1.5896	2.1925		
15	1.6458	1.8524	1.6615	1.8863	0.6426	−1.6923
16	1.6671	1.8223	1.6936	1.7977	1.5608	
17	1.6825	1.8018	1.7091	1.7626	1.6760	2.1067
18	1.6938	1.7871	1.7167	1.7455	1.7038	1.7753
19	1.7022	1.7761	1.7203	1.7358	1.7116	1.7323
			$\gamma^{(b)}$: $l_n^{(0)}=\gamma_n^{(b)}$			
4						
5	1.1071					
6	1.5072					
7	1.6246	2.0128				
8	1.6736	1.8400				
9	1.6974	1.7884	1.6762			
10	1.7100	1.7645	1.7142			
11	1.7168	1.7507	1.7192	1.7354		
12	1.7204	1.7413	1.7181	1.7200		
13	1.7222	1.7344	1.7155	1.7132	1.7066	
14	1.7229	1.7290	1.7126	1.7084	1.7038	
15	1.7228	1.7245	1.7097	1.7046	1.7003	1.6987
16	1.7223	1.7207	1.7069	1.7013	1.6970	1.6947
17	1.7216	1.7174	1.7044	1.6984	1.6941	1.6915
18	1.7206	1.7145	1.7020	1.6958	1.6914	1.6886
19	1.7195	1.7118	1.6998	1.6934	1.6890	1.6860

should also be disregarded. Blanks in the Padé tables indicate that no positive real pole exists for that approximant; this usually only happens when there are too few terms in either the numerator or denominator. Our assessment of these Padé tables is included in the summary (Table VIII).

In order to summarize the analysis of all 12 functions in such a way as to facilitate comparison

TABLE V. Padé-approximant table of estimates of x_c for functions (a) C, (b) G, (c) K, and (d) L, obtained from the simple poles in the Padé approximants to the logarithmic derivatives of the functions. Superscripts indicate the number of defects present in a particular Padé approximant, while asterisks indicate that there is a pole on the positive real axis between the origin and the pole tabulated.

(a)

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0.966 04	0.968 04^{1}		1.008 73	1.003 97	1.003 35	1.003 19^{1}	1.003 83^{1}	1.007 69*	0.997 39	0.999 70	1.000 07	1.000 15	1.000 17	1.000 17
3	0.968 07^{1}	0.965 26^{1}	0.994 10	1.002 43	1.003 05	1.002 30	1.006 51*	0.993 46	1.001 24	1.000 66	1.000 27	1.000 19	1.000 18	1.000 17	
4		0.997 72	1.004 71	1.003 17	1.002 80	1.003 91^{1}	1.027 82*	1.002 88^{2}	1.000 74	0.998 78	1.000 16	1.000 17	1.000 11		
5	1.014 31	1.003 76	1.003 28	1.001 70	1.000 72	1.000 40	1.000 33	1.000 28	1.000 24	1.000 20	1.000 18	1.000 17^{1}			
6	1.007 04	1.003 21	1.004 63^{1}	1.000 77	1.000 34	1.000 31	1.000 15	1.000 07	1.000 08	1.000 07	1.000 06				
7	1.006 08^{1}	1.002 03	1.000 15	1.000 45	1.000 31	1.000 35^{1}	1.000 07	1.000 08	1.000 08	0.999 84					
8	1.030 05*	1.001 15	1.000 42	1.000 37	1.000 21	1.000 07	1.000 09	1.000 08	1.000 10^{1}						
9	0.999 95	1.000 74	1.000 36	1.001 06*	1.000 12	1.000 09	1.000 08	1.000 05							
10	1.000 60	1.000 59	1.000 28	1.000 10	1.000 10	1.000 07	1.000 00								
11	1.000 59	1.000 61^{1}	1.000 21	1.000 10	1.000 10^{1}	1.000 05									
12	1.000 46	0.999 61	1.000 16	1.000 08	1.000 04										
13	1.000 33	1.000 12	1.000 13	1.000 07											
14	1.000 24	1.000 13	1.000 12												
15	1.000 18	1.000 12													
16	1.000 15														

(b)

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2			1.276 02	1.315 17^{1}	0.764 30	0.901 25	0.889 49^{1}		1.071 92	1.060 30^{1}	1.183 40^{1}	0.875 76	0.907 89	0.882 73^{1}	
3	1.040 76	0.926 46	1.033 83	1.046 60	1.034 60	1.023 99	1.013 21	0.999 91	0.984 88	0.970 32	0.963 13	0.965 49	0.974 88	0.990 35	
4	1.146 91^{1}	0.988 36^{1}	1.049 40	1.040 51	0.966 86	1.088 96*	1.047 51^{1}	1.080 74*	0.928 30	0.958 17	0.964 91	0.962 23^{1}	0.930 45*		
5	0.995 17	1.014 37	1.034 09	1.023 66	1.095 21*	1.008 38^{1}	1.060 39*	1.028 99^{2}	0.991 71	0.986 19	0.989 62	0.996 22			
6		1.041 84	1.029 38	1.045 12^{1}	1.050 60^{1}	1.064 05*	1.090 64^{2}*	0.967 55	0.984 83	0.988 16	0.979 32^{1}				
7	1.057 03	1.012 51	1.015 04	1.051 20^{1}	1.042 75^{2}	1.024 82	0.992 75	0.990 28	0.995 00	0.999 09					
8	1.102 15^{1}	1.015 12	1.012 76^{1}	1.099 63*	1.027 23^{2}	0.921 16	0.990 11	0.991 80^{1}	1.002 71						
9	0.988 31	0.981 96	0.918 48	0.952 30	1.007 13	0.992 14	0.994 95	0.998 12							
10	0.980 66	0.990 53^{1}	0.946 03	0.867 73^{1}	0.999 38	0.994 47	1.000 97								
11	1.034 04*	0.902 90	0.968 51^{2}	0.988 16	0.996 38	0.995 74									
12	0.936 39	1.008 45^{2}	0.980 41	1.001 57	0.995 89										
13	0.975 48	0.985 49	0.986 48	0.994 85											
14	0.977 98^{1}	0.986 64	0.985 10^{1}												
15	0.974 68^{1}	0.996 51													
16	0.946 84*														

TABLE V. (*Continued*)

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
								(c)							
2	1.035 43	1.164 22	1.103 29	1.025 02	0.964 27	0.967 98	0.987 09	1.006 19	1.010 42	1.006 89	1.000 98	0.998 24	0.998 21	0.999 57	1.000 58
3	1.235 52	1.125 94	1.228 38*	0.897 04	0.967 73	0.963 44[1]	2.000 00	1.011 83	1.008 47	1.016 18[1]	0.995 58	0.998 21	0.998 24[1]	1.004 06	
4	1.046 74	1.007 21	0.953 15	1.050 17	0.991 60	1.016 25	0.994 33	1.003 31	0.996 95	1.001 80	0.999 74	1.000 96	1.000 09		
5	1.025 17	0.832 96	0.989 08	1.003 34	1.001 60	1.001 18	1.000 86	1.000 72	1.000 56	1.000 48	1.000 39	1.000 34			
6	0.982 27	1.006 83	0.999 52	1.001 84	1.001 02	0.999 85	1.000 61	1.002 04*	1.000 39	1.001 79*	1.000 26				
7	0.997 10[1]	1.001 17	1.001 03	1.001 41	1.000 28	1.000 56	1.000 40[1]	1.000 22	1.000 16	1.000 14					
8	1.002 06	1.001 04	1.001 13[1]	1.001 23	1.000 60	1.000 44[1]	1.000 13	1.000 14	1.000 13						
9	1.000 66	1.001 24	1.001 25	1.001 17[2]	1.000 01	1.000 26	1.000 14	1.000 14							
10	1.001 77	1.001 25	1.001 24[1]	1.000 91	1.000 33	1.000 20	1.000 13								
11	1.000 64	1.000 98	1.004 63*	1.000 50	1.000 03	1.000 16									
12	1.001 20	1.000 82[1]	1.000 12	1.000 27	1.000 19										
13	1.003 08[1]	1.000 63	1.000 25	1.000 23											
14	1.000 40[1]	1.000 50	1.000 22												
15	1.000 52	1.000 45[2]													
16	1.000 34														
								(d)							
2	1.009 62	1.006 17	1.004 43	1.003 41	1.002 73	1.002 26	1.001 92	1.001 66	1.001 45	1.001 29	1.001 16	1.001 05	1.000 95	1.000 87	1.000 80
3	1.006 20	1.003 73	1.002 67	1.002 06	1.001 65	1.001 37	1.001 16	1.001 00	1.000 88	1.000 78	1.000 70	1.000 63	1.000 58	1.000 53	
4	1.004 47	1.002 68	1.001 86	1.001 43	1.001 15	1.000 95	1.000 81	1.000 70	1.000 61	1.000 55	1.000 49	1.000 44	1.000 40		
5	1.003 45	1.002 06	1.001 43	1.001 08	1.000 87	1.000 72	1.000 61	1.000 53	1.000 46	1.000 41	1.000 37	1.000 33			
6	1.002 77	1.001 66	1.001 15	1.000 87	1.000 69	1.000 57	1.000 48	1.000 42	1.000 37	1.000 33	1.000 29				
7	1.002 30	1.001 38	1.000 96	1.000 72	1.000 57	1.000 47	1.000 40	1.000 35	1.000 30	1.000 27					
8	1.001 96	1.001 17	1.000 81	1.000 61	1.000 49	1.000 40	1.000 34	1.000 29	1.000 26						
9	1.001 69	1.001 01	1.000 70	1.000 53	1.000 42	1.000 35	1.000 29	1.000 25							
10	1.001 49	1.000 89	1.000 62	1.000 47	1.000 37	1.000 30	1.000 26								
11	1.001 32	1.000 79	1.000 55	1.000 41	1.000 33	1.000 27									
12	1.001 18	1.000 71	1.000 49	1.000 37	1.000 29										
13	1.001 07	1.000 64	1.000 45	1.000 34											
14	1.000 98	1.000 59	1.000 41												
15	1.000 89	1.000 54													
16	1.000 82														

TABLE VI. Padé approximant table of unbiased estimates of $-\gamma$ for the functions (a) C, (b) G, (c) K, and (d) L. These are the residues of the poles (Table V) to the logarithmic derivative of the functions. Superscripts and asterisks have the same meaning as in Table V.

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
(a)															
2	−1.2506	-1.2613^{1}		−1.6598	−1.5878	−1.5786	-1.5763^{1}	-1.5840^{1}	-1.5753^{*}	−1.4041	−1.4945	−1.5068	−1.5097	−1.5103	−1.5102
3	-1.2615^{1}	-1.2477^{1}	−1.4573	−1.5640	−1.5736	−1.5598	-1.5916^{*}	−1.2250	−1.5416	−1.5258	−1.5134	−1.5110	−1.5105	−1.5103	
4		−1.5016	−1.6015	−1.5757	−1.5695	-1.5841^{1}	-0.9585^{*}	-1.5760^{2}	−1.5280	−1.4504	−1.5099	−1.5103	−1.5075		
5	−1.7654	−1.5855	−1.5776	−1.5483	−1.5261	−1.5176	−1.5155	−1.5139	−1.5126	−1.5114	−1.5104	-1.5101^{1}			
6	−1.6402	−1.5764	-1.5963^{1}	−1.5272	−1.5156	−1.5149	−1.5093	−1.5058	−1.5064	−1.5060	−1.5054				
7	-1.6253^{1}	−1.5546	−1.5107	−1.5188	−1.5149	-1.5160^{1}	−1.5055	−1.5064	−1.5062	−1.4836					
8	-1.5717^{*}	−1.5361	−1.5181	−1.5168	−1.5116	−1.5059	−1.5065	−1.5062	-1.5070^{1}						
9	−1.5076	−1.5267	−1.5164	-1.5230^{*}	−1.5081	−1.5065	−1.5062	−1.5044							
10	−1.5232	−1.5228	−1.5139	−1.5069	−1.5069	−1.5058	−1.5011								
11	−1.5228	-1.5233^{1}	−1.5116	−1.5069	-1.5069^{1}	−1.5045									
12	−1.5194	−1.4839	−1.5098	−1.5063	−1.5037										
13	−1.5156	−1.5079	−1.5087	−1.5057											
14	−1.5125	−1.5086	−1.5083												
15	−1.5106	−1.5079													
16	−1.5093														
(b)															
2			−6.6900	-7.9331^{1}	−0.1042	−0.5157	-0.4597^{1}		−3.2305	-2.8246^{1}	-9.2240^{1}	−0.1845	−0.3315	-0.2183^{1}	
3	−1.8342	−0.9937	−1.9250	−2.0896	−1.9117	−1.7441	−1.5698	−1.3611	−1.1433	−0.9498	−0.8582	−0.8896	−1.0345	−1.3422	
4	-2.5755^{1}	-1.4416^{1}	−2.1338	−2.0041	−0.7457	-2.2628^{*}	-2.0419^{1}	-2.3666^{*}	−0.4591	−0.7921	−0.8815	-0.8475^{1}	-0.5997^{*}		
5	−1.5025	−1.6733	−1.9073	−1.7462	-2.2706^{*}	-1.4049^{1}	-2.1668^{*}	-1.8523^{2}	−1.3133	−1.2201	−1.2827	−1.4212			
6		−2.0375	−1.8396	-2.0283^{1}	-2.0818^{1}	-2.2003^{*}	-2.3446^{2*}	−0.9059	−1.1953	−1.2548	-1.1250^{1}				
7	−2.3546	−1.5926	−1.6284	-2.0883^{1}	-2.0094^{2}	−1.8270	−1.3426	−1.2982	−1.3910	−1.4823					
8	-3.7239^{1}	−1.6295	-1.5958^{1}	-2.5962^{*}	-1.8571^{2}	−0.3246	−1.2950	-1.3263^{1}	−1.5806						
9	−1.2329	−1.1473	−0.4053	−0.7667	−1.5916	−1.3344	−1.3890	−1.4576							
10	−1.1284	-1.2598^{1}	−0.6852	-0.1482^{1}	−1.4684	−1.3792	−1.5331								
11	-1.7253^{*}	−0.2693	-0.9804^{2}	−1.2761	−1.4164	−1.4044									
12	−0.5761	-1.5164^{2}	−1.1512	−1.5196	−1.4073										
13	−1.0898	−1.2280	−1.2426	−1.3865											
14	-1.1226^{1}	−1.2452	-1.2229^{1}												
15	-1.0847^{1}	−1.4354													
16	-1.0173^{*}														

TABLE VI. (*Continued*)

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
(c)															
2	-2.1909	-3.4401	-2.6220	-1.6820	-1.0944	-1.1291	-1.3489	-1.6365	-1.7147	-1.6437	-1.5215	-1.4637	-1.4632	-1.4955	-1.5216
3	-4.8096	-2.9456	-4.0417*	-0.5607	-1.1265	-1.0879[1]		-1.7446	-1.6765	-1.8101[1]	-1.4024	-1.4632	-1.4637[1]	-1.6396	
4	-1.9704	-1.5680	-1.0206	-2.2229	-1.3979	-1.8091	-1.4182	-1.5720	-1.4442	-1.5459	-1.4997	-1.5297	-1.5074		
5	-1.7611	-0.2097	-1.3892	-1.5612	-1.5366	-1.5302	-1.5246	-1.5221	-1.5189	-1.5172	-1.5151	-1.5139			
6	-1.3424	-1.5926	-1.5095	-1.5403	-1.5273	-1.5015	-1.5199	-1.5294*	-1.5149	-1.5156*	-1.5118				
7	-1.4873[1]	-1.5303	-1.5286	-1.5339	-1.5125	-1.5186	-1.5151[1]	-1.5102	-1.5086	-1.5078					
8	-1.5415	-1.5287	-1.5298[1]	-1.5313	-1.5196	-1.5161[1]	-1.5074	-1.5079	-1.5073						
9	-1.5234	-1.5315	-1.5315	-1.5304[2]	-1.5042	-1.5115	-1.5079	-1.5077							
10	-1.5397	-1.5315	-1.5315[1]	-1.5262	-1.5133	-1.5099	-1.5074								
11	-1.5212	-1.5270	-1.5030*	-1.5180	-1.5040	-1.5085									
12	-1.5314	-1.5242[1]	-1.5075	-1.5120	-1.5094										
13	-1.5635[1]	-1.5206	-1.5113	-1.5107											
14	-1.5156[1]	-1.5177	-1.5106												
15	-1.5181	-1.5167[2]													
16	-1.5140														
(d)															
2	-1.9157	-1.8794	-1.8554	-1.8379	-1.8242	-1.8132	-1.8040	-1.7962	-1.7895	-1.7835	-1.7783	-1.7735	-1.7692	-1.7653	-1.7618
3	-1.8798	-1.8431	-1.8223	-1.8071	-1.7953	-1.7858	-1.7778	-1.7710	-1.7652	-1.7600	-1.7554	-1.7513	-1.7476	-1.7442	
4	-1.8561	-1.8224	-1.8016	-1.7878	-1.7771	-1.7685	-1.7613	-1.7552	-1.7498	-1.7452	-1.7410	-1.7373	-1.7339		
5	-1.8387	-1.8073	-1.7878	-1.7741	-1.7642	-1.7562	-1.7495	-1.7438	-1.7389	-1.7345	-1.7307	-1.7272			
6	-1.8252	-1.7956	-1.7772	-1.7642	-1.7543	-1.7468	-1.7405	-1.7351	-1.7305	-1.7264	-1.7227				
7	-1.8143	-1.7862	-1.7686	-1.7562	-1.7468	-1.7392	-1.7333	-1.7282	-1.7237	-1.7199					
8	-1.8051	-1.7783	-1.7615	-1.7496	-1.7405	-1.7333	-1.7273	-1.7224	-1.7182						
9	-1.7974	-1.7715	-1.7554	-1.7439	-1.7352	-1.7282	-1.7224	-1.7175							
10	-1.7906	-1.7657	-1.7501	-1.7390	-1.7305	-1.7238	-1.7182								
11	-1.7847	-1.7605	-1.7454	-1.7347	-1.7265	-1.7200									
12	-1.7794	-1.7560	-1.7413	-1.7308	-1.7228										
13	-1.7747	-1.7519	-1.7376	-1.7274											
14	-1.7704	-1.7482	-1.7342												
15	-1.7665	-1.7448													
16	-1.7629														

TABLE VII. Padé-approximant table of biased estimates of γ for the functions (a) C, (b) G, (c) K, and (d) L. These are obtained by evaluating Padé approximants to $1-x$ times the logarithmic derivative to the functions at $x=1$. Superscripts have the same meaning as in Table V.

(a)

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1.3380	2.6519	1.5210	1.5306	1.5255	1.5058	1.4949	1.5039	1.5077	1.5055	1.5047	1.5045	1.5047^{1}	1.5064^{1}	1.4812
3	1.5994	1.5282	1.5320	1.5280	1.5442^{1}	1.4919	1.5000	1.5092	1.5061	1.5034	1.5041	1.5034	1.5009	1.5005	
4	1.5506	1.5318	1.5297^{1}	1.5133	1.5078	1.5063	1.5056	1.5051	1.5046	1.5041	1.5039	1.5069^{1}	1.5004		
5	1.5454^{1}	1.5240	1.4915	1.5080	1.5058	1.5049	1.5028	1.5024	1.5023	1.5021	1.5019	1.5017			
6	1.5852^{1}	1.5151	1.5068	1.5066	1.5050	1.4982	1.5024	1.5022	1.5029^{1}	1.5011	1.5012				
7	1.5001	1.5105	1.5066	1.5069^{1}	1.5034	1.5024	1.5023	1.5027^{1}	1.5005	1.5012					
8	1.5087	1.5088	1.5056	1.5018	1.5027	1.5022	1.5029^{1}	1.5013	1.5012						
9	1.5088	1.5087^{1}	1.5045	1.5026	1.5024	1.5018	1.5012	1.5011							
10	1.5075	1.4768	1.5037	1.5024	1.0425	1.5013	1.5012								
11	1.5061	1.5028	1.5032	1.5021	1.5013	1.5012									
12	1.5049	1.5032	1.5030	1.5019	1.5012										
13	1.5042	1.5029	1.5042^{1}	1.5017											
14	1.5036	1.5024	1.5010												
15	1.5032	1.5020													
16	1.5029														

(b)

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1.5508	1.4890	1.5861	1.5264	1.4497	1.4052	1.3784	1.3624	1.3626	1.3796	1.4076	1.4553	1.5231	1.5767	1.5936
3	1.4494	1.5288^{1}	1.5502	1.7401	1.2795	1.3036	1.3390	1.3626	1.3624^{1}	1.3384^{1}	1.2238^{1}	3.0870	1.6768	1.6019	
4	1.5384	1.5457	1.5358^{2}	1.3775	1.3059	1.2604^{1}	1.2497^{1}	1.5026	1.4423	1.4554	1.4851	1.5072	1.5136		
5	1.5472	1.5401^{2}	1.5754^{1}	1.2814	1.3394	1.2498^{1}	1.2614^{2}	1.4279	1.4518	1.4260^{1}	1.5498	1.5162			
6	1.5978^{1}	1.4471	1.4214	1.3790	2.0298	1.5174	1.4620	1.4723	1.4925	1.5037	1.5052				
7	1.2518	1.4238	1.5354	1.4073^{1}	1.5396	1.4364	1.4703	1.4316^{1}	1.5080	1.5054					
8	1.3841	1.3987	1.4104^{1}	1.4270^{2}	1.4877	1.4736	1.4859	1.5013	1.5050						
9	1.3936	1.3539^{1}	1.3806^{1}	1.5701	1.4784	1.4792	1.5070	1.5058							
10	1.3967^{1}	1.3787^{1}	1.3654^{2}	1.4613	1.4793	1.4782^{1}	1.5058								
11	1.3902^{1}	1.4273^{2}	1.4154^{2}	1.4891	1.4870	1.4434^{1}									
12	1.3674^{1}	1.4162^{2}	1.4239^{3}	1.4869	1.4887^{1}										
13	1.2027	1.4672	1.7675	1.4995											
14	1.8610	1.5193	1.5265												
15	1.6278	1.5255													
16	1.5925														

TABLE VII. (*Continued*)

D \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
							(c)								
2	1.9325	2.0191[1]	1.2262	1.4071	1.4368	2.1331	1.5497	1.5310	1.5512[1]	1.4829	1.5004	1.5016[1]	1.3392[1]	1.5069	1.5063[1]
3	1.5908	1.5002	1.4889	1.5576	1.5032	1.5205	1.5039	1.5123	1.5053	1.5087	1.5057	1.5061	1.5052	1.5050	
4	1.5704[1]	1.4876	1.4983[1]	1.5196	1.5142	1.5122	1.5102	1.5091	1.5077	1.5070	1.5060	1.5058[1]	1.5050		
5	1.5094	1.5253	1.5150	1.5157	1.5096	1.5054	1.5056	1.5023	1.5040	1.5021	1.5030	1.5018			
6	1.5125[1]	1.5178	1.5156	1.5152[1]	1.5061	1.5056	1.5054[1]	1.5036	1.5032[1]	1.5028	1.5026[1]				
7	1.5184	1.5165	1.5031	1.5210[1]	1.5056	1.5064[1]	1.5032	1.5029	1.5022	1.5014					
8	1.5148	1.5134	1.5228[1]	1.5407[1]	1.5039	1.5038	1.5029	1.5047[1]	1.5006						
9	1.5139	1.5155[1]	2.6446[1]	1.5098	1.5038	1.5039[1]	1.5024	1.5007							
10	1.5106	1.5091	1.5023	1.5048	1.5030	1.5030	1.5019								
11	1.5099	1.5119[1]	1.5041	1.5039	1.5030	1.5030[1]									
12	1.5050	1.5080	1.5039	1.5042[1]	1.5021										
13	1.5075	1.5068[1]	1.5033	1.4998											
14	1.5072[1]	1.5099[1]	1.5027												
15	1.5060	1.5045													
16	1.5057[1]														
							(d)								
2	1.8034	1.7727	1.7564	1.7447	1.7358	1.7287	1.7227	1.7178	1.7134	1.7096	1.7063	1.7033	1.7006	1.6981	1.6958
3	1.7845	1.7565	1.7400	1.7295	1.7215	1.7150	1.7097	1.7052	1.7013	1.6979	1.6948	1.6921	1.6896	1.6874	
4	1.7710	1.7449	1.7295	1.7190	1.7115	1.7056	1.7006	1.6964	1.6928	1.6897	1.6868	1.6843	1.6820		
5	1.7606	1.7361	1.7216	1.7116	1.7041	1.6985	1.6938	1.6899	1.6865	1.6835	1.6808	1.6784			
6	1.7523	1.7290	1.7152	1.7056	1.6985	1.6928	1.6884	1.6846	1.6814	1.6785	1.6760				
7	1.7455	1.7232	1.7099	1.7007	1.6938	1.6884	1.6840	1.6804	1.6772	1.6745					
8	1.7396	1.7182	1.7054	1.6966	1.6899	1.6847	1.6804	1.6767	1.6737						
9	1.7346	1.7139	1.7015	1.6930	1.6865	1.6814	1.6772	1.6737							
10	1.7302	1.7101	1.6981	1.6898	1.6835	1.6786	1.6745								
11	1.7263	1.7068	1.6951	1.6870	1.6809	1.6760									
12	1.7228	1.7038	1.6924	1.6845	1.6785										
13	1.7196	1.7010	1.6899	1.6822											
14	1.7168	1.6986	1.6877												
15	1.7141	1.6963													
16	1.7117														

TABLE VIII. Summary of analysis for all series using ratio (R), Neville extrapolation of ratio results (N), and Padé approximants (P). The parameter $\epsilon = -\log_{10}(\Delta p_n / p_{\text{exact}})$, where $p_n = y_n,\ \gamma_n^{(u)},\ \gamma_n^{(b)}$, is tabulated for $n = 10, 15, 20$. For a particular series and a particular method, Δp_n is the amount by which the best estimate for the critical parameter p using n terms of the series differs from p_{exact}.

Test series	No. of terms used	$y^{-1}=x_c$			γ (unbiased)			γ (biased)		
		R	N	P	R	N	P	R	N	P
	10	3.2		2.7	2.4		1.7	4.1		2.4
A	15	8.1		4.0	7.1		2.6	9.4		3.0
	20	13.9		4.8	12.8		3.0	15.4		3.6
	10	2.3	2.4	2.7	1.3		1.7	1.7	2.5	2.7
B	15	2.8	3.8	3.9	1.6	2.3	2.5	1.9	3.8	3.3
	20	3.1	5.4	5.1	1.7	3.4	3.3	2.0	5.8	3.7
	10	2.3		2.5	1.0		1.2	1.3	1.8	1.7
C	15	2.5	4.1	3.4	1.3	2.7	2.2	1.5	3.6	2.4
	20	2.9	5.7	4.0	1.4	3.7	2.4	1.6	5.4	3.1
	10	1.7		1.9	0.8		0.7	1.5		1.0
D	15	2.6		2.2	1.4		1.0	2.6		1.9
	20	3.6		3.5	2.0		1.9	3.7		2.3
	10	1.6		1.3	0.7		0.4	1.6		1.3
E	15	2.4		2.5	1.3		1.4	1.8		1.8
	20	3.0	3.9	3.7	1.7		2.2	2.0	2.8	2.4
	10	1.5		1.1	0.8		0.3	0.8		0.4
F	15	2.8		1.4	1.1		0.3	1.3		1.2
	20	3.0		2.7	1.5		1.2	1.6	1.8	1.8
	10	1.4		1.3	0.6		0.4	1.3		0.8
G	15	1.6		1.3	0.6		0.4	1.4		1.1
	20	1.9		2.7	0.9		1.3	1.5		2.0
	10	1.6		1.7	0.7		0.7	1.5		0.9
H	15	2.0		1.8	0.9		0.9	1.6		1.2
	20	2.2		2.4	1.3		1.3	1.6		1.7
	10	1.0		0.9	0.1		−0.4	0.8		0.6
I	15	1.2		1.4	0.1		2.3	0.8		0.8
	20	1.5		2.2	0.4		1.3	1.2		1.7
	10	1.1	1.5	2.2	0.2		1.4	0.7		1.7
J	15	1.7	2.2	3.5	0.5		2.2	1.2		2.7
	20	2.0	2.8	4.4	0.8		2.7	1.4		3.2
	10	1.1		0.7	0.1		0.1	0.7		1.4
K	15	1.7	2.2	3.0	0.5		1.7	1.2		2.0
	20	2.0	2.6	3.9	0.8		2.3	1.4		2.5
	10	1.6	2.9	2.7	0.3		0.7	0.5	0.8	0.8
L	15	1.9	3.3	3.2	0.4	0.8	0.8	0.6	0.9	0.9
	20	2.2	3.5	3.6	0.5	0.9	0.8	0.7	0.9	0.9

of the results for different methods, and to illustrate the improvement as the number of terms used in the analysis increases, we define the following parameter:

$$\epsilon_n = -\log_{10}(\Delta p / p_{\text{exact}}) , \qquad (4.1)$$

where Δp_n is the amount by which an estimate of the parameter p (either y or γ) using n terms of the series differs from the exact value p_{exact}. ϵ is effectively a measure of the number of significant figures in an estimate of a critical parameter p.

In Table VIII we summarize all our numerical results by tabulating ϵ_n, for $n = 10, 15, 20$, for the ratio method (R), Neville extrapolation of ratio results (N), and Padé-approximant method (P). Entries are made in the columns headed N only if Neville extrapolation improves the ratio results.

The first nine test functions divide naturally into groups of three according to the pattern of additive divergent singularities in the complex plane. The members of each group differ in the factor which multiplies the dominant singularity:

1, $(1+\frac{1}{2}x)^{+1.5}$, and $(1-\frac{1}{2}x)^{+1.5}$. The convergent singularity in the second factor is three times further from the dominant singularity than is the convergent singularity in the third factor. First, for the group A, B, and C, where the additive term is an entire function, we note that the ratio method with Neville extrapolation is clearly better than the Padé approximant in predicting the critical parameters. The very rapid convergence of the ratio method for A is merely due to the nature of the entire function we have chosen. The coefficients in the expansion of e^{-x} are proportional to $1/n!$; so in the absence of other singularities ratio convergence is rapid. In addition, there is evidence that the ratio method is much less affected by the position of the convergent singularity relative to the circle of convergence than are the Padé procedures. The Padé results for A and B are quite similar, indicating that when that singularity is far away, it does not affect the position of the dominant singularity; while the results for C are significantly poorer, indicating that the proximity of the two singularities is important. This observation is also substantiated by the results for D, E, and F. That it is not so noticeable in G, H, and I we attribute to the strong influence of the other divergent singularities, some of which are closer than the convergent singularity.

For D, E, and F, the ratio method (with Neville extrapolation in some instances) is just slightly better than the Padé-approximant method. Here the additive singularities lie on a circle centered just to the right of the origin. The radius of this circle is about double the radius of convergence and the distance from $x=1$ to the closest of these singularities is 1.29. The contribution of the additive terms to the coefficients falls off approximately as 2^{-n}. The success of the ratio method is not surprising.

For G, H, and I, the additive singularities lie on the line $\mathrm{Re}(x)=1$ and tend to pinch the dominant singularity. None of the methods of analysis is dramatically successful, but the Padé method clearly has the edge. The closest pair of additive singularities lies only 0.45 away from the dominant singularity, thereby exerting a strong influence on the Padé results, but the pair lies even closer to the circle of convergence, so the ratio method is more affected.

J and K form a separate group and differ only by the addition of an entire function. There is one strong additional singularity close to the circle of convergence but opposite the dominant singularity. As we would expect, the Padé results are much better: nearly two orders of magnitude for J and over one order for K. As we saw in A, the presence of the term e^{-x} has negligible effect on the ratio analysis, but is noticeable in the Padé results.

L has no relationship to the forms of the other functions we have considered. This form of singularity is not readily amenable to analysis to determine γ. The estimates of y_c are all of reasonable accuracy but convergence of the γ sequences is very slow. For the ratio method, (2.19) predicts the observed γ to be $(\ln cn)^{-1}$ larger than the true values; $(\ln 19)^{-1}=0.34$, so with c set equal to 1, the value $\gamma_{19}^{(b)}=1.84$ is in line with our expectations. Close examination of Tables II–VII will show that there is no apparent convergence to these higher effective γ's. Rather, the sequences are all smoothly decreasing to a considerably lower value.

We have not had to rely on the error-assessment procedures described in Sec. III because our aim was to make a comparison with known results. However, we will conclude by illustrating their effectiveness for our test functions. First we compare the values of ϵ for unbiased γ estimates with ϵ for corresponding estimates of x_c in Table VIII. For $n=20$, we expect them to differ by an amount approximately equal to $\log_{10}19$ (≈ 1.3). There is a variation from series to series, but generally the two values of ϵ differ by 1.0–1.6, a range centered at the predicted value. At worst, the relationship between the relative errors in γ and x_c, readily determinable for these test functions, differ by about a factor of 2 from the order-of-magnitude estimate.

We have illustrated that $(\Delta\gamma/\gamma)(\Delta y/y)^{-1}\sim J$; to estimate the error $\Delta y/y$, we expand $(1-y_{20}x)^{-g_{20}}$, where y_{20} and g_{20} are estimates based on 20 terms, and compare the coefficient of x^{19} with the similar coefficient in the expansion of $f(x)$. For example, for the function C we take the ratio estimates

$$y\approx 1.001\,788, \quad \gamma\approx 1.438\,846, \tag{4.2}$$

which we obtained using all but the last term. We expand this, choosing the amplitude so that the coefficient of x^{18} matches that in the expansion of C, and then compare the coefficient of x^{19} predicted in this way with the actual coefficient. We find

$$\eta\approx 0.0006\,, \tag{4.3}$$

TABLE IX. Comparison of two Padé approximants.

x	$[10/7](x)-[11/7](x)$
0.8	0.000 002 4
0.9	0.000 040 9
1.0	0.000 855 5
1.1	0.091 601 8
1.2	0.217 268 0
1.3	0.143 409 2

from which we note that

$$\frac{\Delta y}{y} \approx \tfrac{1}{6} J\eta \ , \quad \frac{\Delta \gamma}{\gamma} \approx \tfrac{1}{4} J^2 \eta \ , \quad \frac{\Delta A}{A} \approx J^2 \eta \ln J \ , \qquad (4.4)$$

illustrating the reliability of this guide to error assessment.

Finally, to briefly illustrate the "table-comparison" procedure for error assessment of Padé estimates, we will consider the [10/7] and [11/7] Padé approximants to $1-x$ times the logarithmic derivative of K. We have evaluated the approximants for values of x from 0.1 to 2.0 in steps of 0.1. The difference we find for some values of x are given in Table IX. For $x \leq 1.0$ the difference may roughly be approximated by Ax^m for $m \gtrsim 20$, but for $x > 1.0$ this obviously breaks down. This indicates that the small-x error law holds all the way out to $x \approx 1.0$, the point at which we wish to estimate the error. We could on the basis of this evidence alone conclude that the error in the evaluation of the [11/7] Padé approximant, 1.5014, is of the order of the difference tabulated above, i.e., $\sim 10^{-2}$.

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