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Ferromagnetoelastic Resonance in Thin Films. I. Formal Treatment*

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We have undertaken a theoretical study of the effect of magnetoelastic interactions on ferromagnetic resonance in thin films as a possible mechanism for an appreciable anisotropy in resonance linewidth. This study is based on a formalism developed by Tiersten, which is exact within the framework of the quasistatic approximation. The resonance frequency and linewidth are calculated from the simultaneous solution of the coupled magnetoelastic equations of motion under various magnetic and elastic boundary conditions. We find that the magnetoelasticity has an appreciable effect on the resonance linewidth only in the cases where an elastic wave undergoes a thickness resonance near the ferromagnetic-resonance frequency, which we call the "ferromagnetoelastic resonance condition." The above facts have made it possible to develop a self-consistent approximate method, which greatly simplifies the mathematical treatment without sacrificing the physical model. The agreement between the approximate and exact calculations is excellent.

I. INTRODUCTION

The phenomenological Landau–Lifshitz (LL) equation of motion[1] for the magnetization $\vec{M}$, a continuum variable, has been widely used in the study of ferromagnetic resonance (FMR) in thin-metal films.[2–5] The LL equation of motion is applicable to strong ferromagnets, in which the $|\vec{M}|$ can be assumed constant,[6,7] and in which the damping is isotropic.[8]

A similar phenomenological equation of motion can be written for the elastic deformation in the same sample. The two systems of equations can be coupled through a set of quite general magnetoelastic and elastomagnetic coupling terms, as has been shown by Tiersten.[9–11] Approximate solutions have often been obtained by treating the solutions of the uncoupled equations as perturbations upon each other. More realistic solutions are obtained, however, if the coupled sets of equations are solved simultaneously. At a later stage in the development, it becomes possible to treat two sets of polarizations as perturbations upon each other. This simplifies the computations, while retaining the essential features of the coupled system.[12]

In this paper we apply such a method to evaluating the effects of magnetoelasticity on the observed FMR line shape in ferromagnetic-insulator films, using the magnetic parameters corresponding to nickel. A more accurate representation of the nickel-metal film could be obtained by adding an anisotropy field to the LL equation,[12] and by including the effects of conductivity through Maxwell's equations.[2–5] The effects of these processes have been evaluated separately in the absence of magnetoelastic coupling. The influence of magnetocrystalline anisotropy upon the oblique-angle resonance linewidth is small for nickel (a few Oe at room temperature at K band).[12] The effect of conductivity has been calculated for oblique-angle resonance in thin-nickel films at K band and is on the order of 10 Oe.[5] In the absence of a resonance, the coupling between the magnetic and electronic structure of the metal is already accounted for phenomenologically in the LL damping parameter.[13]

Effective internal-field linewidth anisotropy in single-crystal disks and films has been reported several times,[14–16] and except for one case,[14] the results are adequately explained by the calculations of power absorption based on the LL equation of motion.[5] The intent of the calculation presented in this paper is to evaluate the influence that magnetoelastic properties of the material might be expected to have on the observed resonance line.

In Sec. II, we present a method of calculation, based on the coupled equations of motion developed by Tiersten,[9–11] and which are exact within the framework of the quasistatic approximation. We then demonstrate by the calculation of the power absorbed that the conditions under which the magnetoelastic coupling has an appreciable effect on the resonance linewidth are those for which the elastic wave undergoes a thickness resonance at or near the FMR frequency. We call this the "ferromagnetoelastic-resonance" (FMER) condition. When these conditions are not met, the magnetoelastic contribution to the linewidth is very small. However, the magnetoelastic interaction produces a small shift in the resonance field far from FMER.

In Sec. III, on the basis of the above results, we develop an approximate method, which is much simpler to use and yet gives nearly the same accuracy as the exact method. The approximate method is widely applicable to the generation of longitudinal or transverse magnetoelastic waves under arbitrary magnetic and elastic boundary conditions.

The detailed results of the approximate calculation applied to nickel will be given in the following paper.[17] The application to phonon generation is reported elsewhere.[18,19]

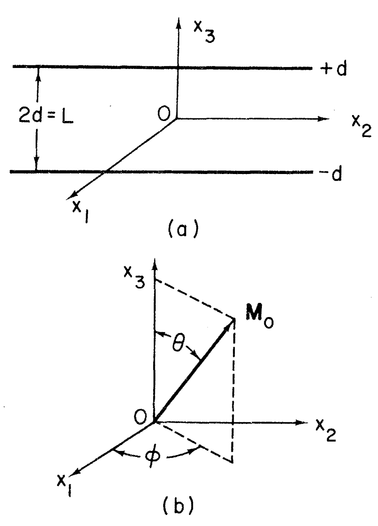

FIG. 1. (a) Infinite plate in coordinate system x_i ($i=1$, 2, 3) with surface at $x_3=\pm d$. (b) Polar-coordinate system defining the direction of M_0.

II. EXACT METHOD

A. Equations of Motion

Tiersten[9] has derived the differential equations and constitutive relations governing the macroscopic behavior of nonconducting magnetically saturated media undergoing finite deformations. He then specialized the resulting nonlinear equations to a linear approximation in the small-field variables for the important case of a small dynamic field superposed on a large biasing field. We have applied Tiersten's linearized equations to the study of magnetoelastic effects on the oblique-angle FMR in cubic-crystal thin films.

We consider an infinite plate with (100) surfaces placed in a rectangular Cartesian coordinate system x_i, $i=1$, 2, 3, with x_i along the crystallographic cube edges and with $x_3=\pm d$ defining the plate surfaces. The appropriate geometry is shown in Fig. 1(a). The static magnetization $\vec{M}_0$ is in an arbitrary direction with respect to the coordinate axes. The internal static field $\vec{H}_0$ is defined as the vector sum of the external biasing field, demagnetizing field, and local field[9] and lies parallel to $\vec{M}_0$. An rf field is then applied at normal incidence (wave vector parallel to x_3) upon both surfaces at right angles to $\vec{M}_0$. Tiersten's equations, then yield the following: the equations of motion for the rf magnetization m_i,

$$(1/\gamma)\dot{m}_i = e_{ijk}M_j^0[h_k^M + (2A/M_0^2)m_{k,ll} + h_k^L] - e_{ijk}m_kH_j^0 \; ; \quad (1a)$$

the equations of motion for the rf elastic displacement u_j,

$$\rho\ddot{u}_j = \tau_{ij,i} + M_i^0 h_{j,i}^M \; ; \quad (1b)$$

the equations of the quasistatic magnetic field,

$$h_{i,i}^M + 4\pi m_{i,i} - 4\pi M_j^0 u_{i,ji} = 0 \; , \quad (1c)$$

$$e_{ijk}h_{j,k}^M = 0 \; ; \quad (1d)$$

the linear magnetoelastic constitutive relations,

$$h_i^L = g_{ijk}u_{j,k} - (\lambda/\gamma M_0^2)e_{ijk}(M_j^0 h_k^E + m_j H_k^0) \,, \quad (1e)$$

$$h_k^E = h_k^M + (2A/M_0^2)m_{k,ll} + g_{klm}u_{l,m} \,, \quad (1f)$$

$$\tau_{ij} = c_{ijkl}u_{k,l} + \epsilon_{klj}m_k + \tfrac{1}{2}Z_{ijkl}(\dot{u}_{k,l} + \dot{u}_{l,k}) \,. \quad (1g)$$

In Eqs. (1), the symbols are defined as follows: M_i^0 and H_i^0 are the x_i components of the static magnetization $\vec{M}_0$ and the internal static field $\vec{H}_0$, respectively. h_i^M, h_i^L, and h_i^E are the x_i components of the rf Maxwellian field, local field, and effective internal field, respectively. The local field is the sum of the magnetostrictive (elastomagnetic) field and the LL damping field. The effective internal field is the sum of the Maxwellian field, the exchange field, and the magnetostrictive field. It is easy to add the anisotropy field,[9,12] if so desired, to both the local field and the effective internal field. τ_{ij} is a component of the symmetric part of the stress tensor, which consists of the ordinary strain term, magnetoelastic term, and elastic-damping term. In magnetoelastic media, the stress tensor in general also contains an antisymmetric part. However, this is usually small compared with the symmetric part, so that it is not considered here. γ, ρ, A, and λ are the gyromagnetic ratio, mass density, exchange constant, and LL damping parameter, respectively. c_{ijkl}, ϵ_{kji}, g_{ikl}, and Z_{ijkl} are the elastic, magnetoelastic, elastomagnetic, and elastic-damping tensor constants, respectively. The symbol e_{ijk} represents the alternating tensor which is defined by

$$e_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ cyclic (123, 231, 312)} \\ 0 & \text{if any two indices are equal} \\ -1 & \text{if } ijk \text{ anticyclic (132, 213, 321)}. \end{cases}$$

An index preceded by a comma denotes differentiation with respect to a space coordinate, and the summation convention for repeated tensor indices is employed, as is the dot notation for differentiation with respect to time.

Since the magnetization is conserved and m is small compared to M_0, we can consider $\vec{m}$ to be perpendicular to $\vec{M}_0$, i.e.,

$$M_i^0 m_i = 0 \,. \quad (1h)$$

Hence, one of the three equations (1a) may be replaced by (1h).

It is worth noting the terms $M_i^0 h_{j,i}^M$ in (1b) and $-4\pi M_j^0 u_{i,ji}$ in (1c). The terms $\dot{M}_i^j h_{j,i}^M$ describe a body force produced by a nonuniform magnetic field.[9] The terms $-4\pi M_j^0 u_{i,ji}$ arise because the saturation magnetic moment per unit volume varies as the sample changes its volume.[9] It is terms of this sort, as well as the magnetoelastic terms in the stress tensor, which have been included by Tiersten[9] and by some subsequent authors,[20,21] which were neglected in earlier treatments. It will be seen later that these terms slightly modify the magnetoelastic coupling.

We now specify the material tensor constants. We can still consider the biased crystal as elastically cubic since the static deformation caused by the biasing field (magnetostriction) is small. Thus, the elastic constant takes the form

$$c_{ijkl} \equiv c_{pq} = \begin{pmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix}, \quad (2a)$$

where p and q are the abbreviated double indices. The elastic-damping constant Z_{ijkl} has exactly the same symmetry as c_{ijkl}.

Although the biased crystal is elastically cubic, it is not magnetically cubic since it is magnetized in an arbitrary direction with respect to the cube edges. Therefore, the magnetoelastic and elastomagnetic constants take the following form[9,12]:

$$\epsilon_{klj} \equiv \epsilon_{kp} = \begin{pmatrix} \epsilon_{11} & 0 & 0 & 0 & \epsilon_{15} & \epsilon_{16} \\ 0 & \epsilon_{22} & 0 & \epsilon_{15} & 0 & \epsilon_{26} \\ 0 & 0 & \epsilon_{33} & \epsilon_{16} & \epsilon_{26} & 0 \end{pmatrix} \,, \quad (2b)$$

where

$$\epsilon_{11} = (2B_1/M_0)\alpha_1 \,, \qquad \epsilon_{22} = (2B_1/M_0)\alpha_2 \,,$$

$$\epsilon_{33} = (2B_1/M_0)\alpha_3 \,, \qquad \epsilon_{15} = (B_2/M_0)\alpha_3 \,,$$

$$\epsilon_{16} = (B_2/M_0)\alpha_2 \,, \qquad \epsilon_{26} = (B_2/M_0)\alpha_1 \,,$$

and

$$g_{ijk} \equiv g_{ip} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{14} & g_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{14} \end{pmatrix}, \quad (2c)$$

where

$$g_{11}=-(2B_1/M_0)\alpha_1(1-\alpha_1^2), \quad g_{12}=(2B_1/M_0)\alpha_1\alpha_2^2, \quad g_{13}=(2B_1/M_0)\alpha_1\alpha_3^2,$$
$$g_{21}=(2B_1/M_0)\alpha_1^2\alpha_2, \quad g_{22}=-(2B_1/M_0)\alpha_2(1-\alpha_2^2), \quad g_{23}=(2B_1/M_0)\alpha_2\alpha_3^2,$$
$$g_{31}=(2B_1/M_0)\alpha_3\alpha_1^2, \quad g_{32}=(2B_1/M_0)\alpha_2^2\alpha_3, \quad g_{33}=(2B_1/M_0)\alpha_3(1-\alpha_3^2),$$
$$g_{14}=(2B_2/M_0)\alpha_1\alpha_2\alpha_3, \quad g_{15}=-(B_2/M_0)\alpha_3(1-2\alpha_1^2), \quad g_{16}=-(B_2/M_0)\alpha_2(1-2\alpha_1^2),$$
$$g_{24}=-(B_2/M_0)\alpha_3(1-2\alpha_2^2), \quad g_{26}=-(B_2/M_0)\alpha_1(1-2\alpha_2^2),$$
$$g_{34}=-(B_2/M_0)\alpha_2(1-2\alpha_3^2), \quad g_{35}=-(B_2/M_0)\alpha_1(1-2\alpha_3^2).$$

In the above B_1 and B_2 are the magnetoelastic coupling constants and α_1, α_2, and α_3 are the direction cosines of $\vec{M}_0$, which may be written in terms of the polar coordinates defined in Fig. 1(b) as

$$\alpha_1=\sin\theta\cos\phi, \quad \alpha_2=\sin\theta\sin\phi, \quad \alpha_3=\cos\theta. \tag{3}$$

Let us now assume the spatial variation to be along the x_3 direction only and consolidate Eqs. (1) into a set of differential equations for m_i and u_i by eliminating other variables. First of all, the quasistatic field equations (1c) and (1d) together with the electromagnetic boundary conditions (continuity conditions) simply yield

$$h_a^M=h_a^0, \tag{4a}$$

$$h_3^M=-4\pi m_3+4\pi M_3^0 u_{3,3}, \tag{4b}$$

where h_a^0 ($a=1, 2$) is the external rf field at the surfaces. Equation (4a) implies that the tangential component of the rf Maxwellian field is uniform across the plate. This is, of course, a result of the long-wavelength quasistatic approximation. Then, substituting the Eqs. (2) into the constitutive relations (1e)–(1g) and finally substituting the constitutive relations and Eqs. (4) into Eqs. (1a) and (1b), we obtain

$$(1/\gamma)\dot{m}_1=-\eta(1-\alpha_1^2)(H_0m_1-Dm_{1,33}-M_0h_1^0)+(\alpha_3+\eta\alpha_1\alpha_2)(H_0m_2-Dm_{2,33}-M_0h_2^0)$$
$$-(\alpha_2-\eta\alpha_3\alpha_1)[(H_0+4\pi M_0)m_1+Dm_{3,33}]-B_2[\alpha_1\alpha_2+\eta\alpha_3(1-2\alpha_1^2)]u_{1,3}$$
$$-B_2[(\alpha_2^2-\alpha_3^2)-2\eta\alpha_1\alpha_2\alpha_3]u_{2,3}-2(B_1-2\pi M_0^2)\alpha_3(\alpha_2-\eta\alpha_3\alpha_1)u_{3,3}, \tag{5a}$$

$$(1/\gamma)\dot{m}_2=-(\alpha_3-\eta\alpha_1\alpha_2)(H_0m_1-Dm_{1,33}-M_0h_1^0)-\eta(1-\alpha_2^2)(H_0m_1-Dm_{2,33}-M_0h_1^0)$$
$$+(\alpha_1+\eta\alpha_2\alpha_3)[(H_0+4\pi M_0)m_3-Dm_{3,33}]-B_2[(\alpha_3^2-\alpha_2^2)-2\eta\alpha_1\alpha_2\alpha_3]u_{1,3}$$
$$+B_2[\alpha_1\alpha_2+\eta\alpha_3(1-2\alpha_2^2)]u_{2,3}+2(B_1-2\pi M_0^2)\alpha_3(\alpha_1+\eta\alpha_2\alpha_3)u_{3,3}, \tag{5b}$$

$$\alpha_1m_1+\alpha_2m_2+\alpha_3m_3=0; \tag{5c}$$

$$\rho\ddot{u}_1=c_{44}u_{1,33}+Z_{44}\dot{u}_{1,33}+(B_2/M_0)(\alpha_3m_{1,3}+\alpha_1m_{3,3}), \tag{6a}$$

$$\rho\ddot{u}_2=c_{44}u_{2,33}+Z_{44}\dot{u}_{2,33}+(B_2/M_0)(\alpha_3m_{2,3}+\alpha_2m_{3,3}), \tag{6b}$$

$$\rho\ddot{u}_3=c_{11}u_{3,33}+Z_{11}\dot{u}_{3,33}+(2/M_0)(B_1-2\pi M_0^2)\alpha_3m_{3,3}. \tag{6c}$$

In Eqs. (5) $D=2A/M_0$ and $\eta=\lambda/\gamma M_0$. As was pointed out above, the terms $M_i^0h_{j,i}^M$ in Eq. (1b) and $-4\pi M_j^0u_{i,ji}$ in Eq. (1c) result in a correction, $-2\pi M_0^2$, to B_1 in the above equations. Thus, there is a set of five differential equations and an algebraic equation, with six unknowns, m_1, m_2, m_3, u_1, u_2, and u_3.

B. Plane-Wave Solution

Let us assume the external rf field $h_a^0=\underline{h}_a^0e^{i\omega t}$ and consider a solution of the form

$$m_i=m_i^0(t)+m_i'(x_3,t)$$
$$=(\underline{m}_i^0+\underline{m}_i'e^{-ikx_3})e^{i\omega t}, \tag{7}$$

$$u_i=\underline{u}_i e^{i(\omega t-kx_3)}.$$

Here $\underline{h}_a^0$, $\underline{m}_i^0$, and $\underline{u}_i$ are complex constants. If we substitute Eqs. (7) into Eqs. (5) and (6), the equations for $\underline{m}_i^0$, the $k=0$ uniform-precession mode, can be separated from those for $\underline{m}_i'$ and $\underline{u}_i$. The result is a relation between $\underline{m}_i^0$ and $\underline{h}_a^0$, the uniform-precession susceptibility. By equating the determinant of the coefficients of $\underline{m}_i'$ and $\underline{u}_i$ to zero, we obtain the dispersion relations between ω and k.

The uniform-precession susceptibility, χ_{ia}^0, is defined here such that

$$\underline{m}_i^0=\sum_{a=1}^{2}\chi_{ia}^0\underline{h}_a^0. \tag{8}$$

Then, χ^0_{ia} is given by

$$\chi^0_{11}=(M_0/\Delta^0)\,\{\beta[(1-\alpha_1^2)H_0+4\pi M_0\alpha_2^2] + i\eta(1-\alpha_1^2)(\omega/\gamma)\},$$
$$\chi^0_{12}=(M_0/\Delta^0)[-\beta\alpha_1\alpha_2(H_0+4\pi M_0) - i(\alpha_3+\eta\alpha_1\alpha_2)(\omega/\gamma)],$$
$$\chi^0_{21}=(M_0/\Delta^0)[-\beta\alpha_1\alpha_2(H_0+4\pi M_0) + i(\alpha_3-\eta\alpha_1\alpha_2)(\omega/\gamma)], \quad (9)$$
$$\chi^0_{22}=(M_0/\Delta^0)\,\{\beta[(1-\alpha_2^2)H_0+4\pi M_0\alpha_1^2] + i\eta(1-\alpha_2^2)(\omega/\gamma)\},$$
$$\chi^0_{3b}=-\frac{1}{\alpha_3}\sum_{a=1}^{2}\alpha_a\chi_{ab}, \quad b=1,\,2$$

where

$$\beta=1+\eta^2,$$
$$\Delta^0=\beta(\omega_0/\gamma)^2-(\omega/\gamma)^2 + 2i\eta[H_0+2\pi M_0(1-\alpha_3^2)](\omega/\gamma), \quad (10a)$$

and

$$(\omega_0/\gamma)^2=H_0[H_0+4\pi M_0(1-\alpha_3^2)]. \quad (10b)$$

The frequency ω_0 given in Eq. (10b) is that for uniform-precession resonance in the absence of magnetic damping. The LL damping shifts the frequency to $\beta^{1/2}\omega_0$ as can be seen from Eq. (10a). From Eq. (10a) the half-power-point linewidth of the uniform-precession resonance is

$$\Delta\omega/|\gamma|=2|\eta|[H_0+2\pi M_0(1-\alpha_3^2)]. \quad (11)$$

For the relatively simple geometry considered here, the dispersion relation does not depend upon α_1 and α_2 but on α_3 alone, since the magnetocrystalline anisotropy has been neglected. Using $\alpha_3=\cos\theta$, this becomes

$$\begin{aligned}&(\bar c_{11}k^2-\rho\omega^2)(\bar c_{44}k^2-\rho\omega^2)^2\{\beta D^2k^2+2[\beta(H_0+2\pi M_0\sin^2\theta)+i\eta(\omega/\gamma)]Dk^2+\Delta^0\}\\&-\sin^2 2\theta(B_1/M_0-2\pi M_0)^2(\bar c_{44}k^2-\rho\omega^2)[\beta(Dk^2+H_0)+i\eta(\omega/\gamma)]-(B_2/M_0)^2(\bar c_{11}k^2-\rho\omega^2)(\bar c_{44}k^2-\rho\omega^2)M_0k^2\\&\times\{(\sin^2\theta+\cos^2 2\theta\cos\theta)[\beta(Dk^2+H_0)+i\eta(\omega/\gamma)]+\beta\pi M_0\sin^2 2\theta\}\\&+\cos^2\theta\cos^2 2\theta(B_2/M_0)^4\beta(\bar c_{11}k^2-\rho\omega^2)M_0^2k^4+\cos^2\theta\sin^2 2\theta(B_2/M_0)^2(B_1/M_0-2\pi M_0)^2\beta(\bar c_{44}k^2-\rho\omega^2)M_0^2k^4=0,\end{aligned} \quad (12)$$

where

$$\bar c_{11}=c_{11}+i\omega Z_{11}, \quad \bar c_{44}=c_{44}+i\omega Z_{44}.$$

Equation (12) consists of five terms. The first represents the uncoupled spin-wave and elastic-wave dispersion relations. The second represents a coupling between the spin waves and the longitudinal (L) elastic wave (referred to as longitudinal magnetoelastic coupling); the third term a coupling between the spin wave and one polarization of the transverse (T) elastic wave (transverse magnetoelastic coupling). The last two terms represent couplings between the elastic waves.

It can be seen from (12) that at $\theta=0$ the L elastic wave becomes uncoupled from the spin wave since the second and the last terms vanish and $\bar c_{11}k^2-\rho\omega^2$ factors out. At $\theta=45°$ one polarization of the T elastic wave becomes uncoupled since the fourth term vanishes and $\bar c_{44}k^2-\rho\omega^2$ factors out. At $\theta=90°$ the L elastic wave and one polarization of the T elastic wave become uncoupled since the second, the fourth, and the last terms vanish and $\bar c_{11}k^2-\rho\omega^2$ and $\bar c_{44}k^2-\rho\omega^2$ both factor out. At all other angles, all the elastic waves are coupled to the spin waves, and the dispersion relation is quintic in k^2 and quartic in ω^2. The L magnetoelastic coupling is strongest approximately at $\theta=45°$ since the second term has $\sin^2 2\theta$ dependence on θ. The angular variation of the T magnetoelastic coupling, represented by the third term, is more complicated. It is a maximum at $\theta=0$ and a minimum at some angle between 45° and 90° depending on the ratio of $4\pi M_0$ to $\omega_0/|\gamma|$.[17]

The secular equation (quintic in k^2) has five roots $k^2(\omega)$ at each value of ω_0. Each root yields two values of k, $\pm k(\omega)$. It is convenient to represent the full solution by the positive roots only. Thus, the solution of the secular equation is represented by five dispersion curves representing ten roots in all, k_r ($r=1,\,2,\,\ldots,\,10$). Furthermore, the existence of the spatially uniform solution (m_i^0) implies another $k=0$ branch coincident with the ω axis. If electromagnetic propagation is taken into account, this $k=0$ branch splits into two ($k\neq 0$) branches.[5]

In the absence of magnetic and elastic damping, the five values of k for every ω are either purely real or purely imaginary. Under this assumption, the dispersion curves for $\theta=0$, 45°, and 90° are plotted in Fig. 2 with the uncoupled branches indicated by dashed lines. The elastic-wave branches always lie in the real k-ω plane while the familiar spin-wave branch goes into the imaginary k-ω plane for $\omega<\omega_0$. There is also a nonpropagating spin-wave branch which lies entirely in the imaginary

k-ω plane.

If both magnetic and elastic damping are included, these branches no longer lie precisely in the real or imaginary k-ω planes, but come slightly out of these planes. As a result, the mixing of the elastic and propagating spin-wave branches at crossover points is modified.[12,22,23]

Each of the 10 values of k_r yields an independent plane-wave solution in the form of Eq. (7). Thus, the spatially nonuniform solution (m_i' and u_i) is a linear combination of ten independent plane waves,

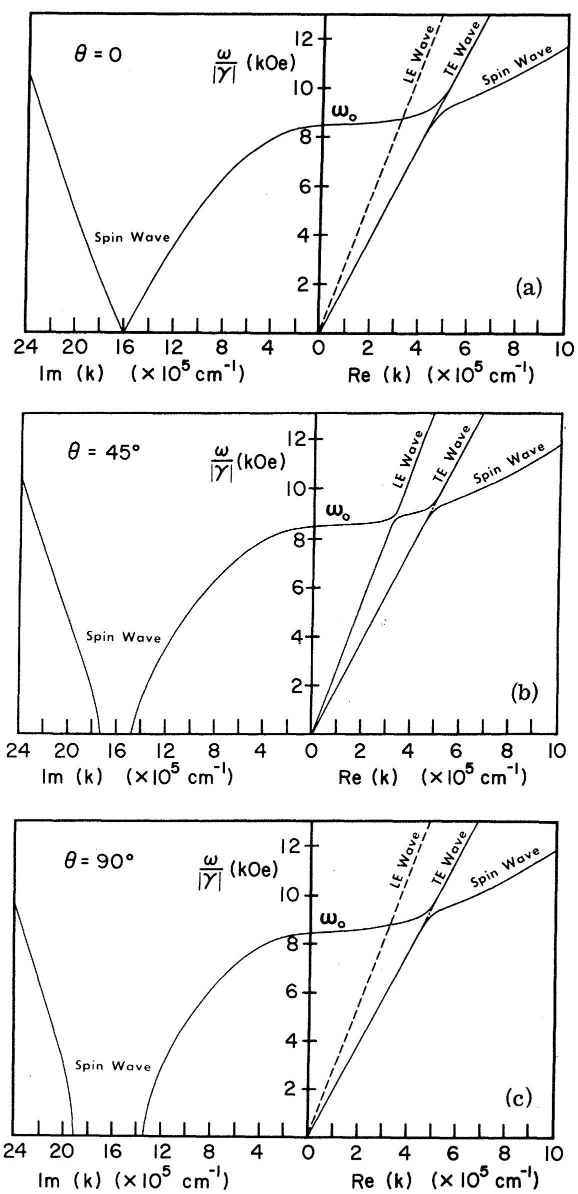

FIG. 2. Dispersion curves at $\omega_0/|\gamma| = 8432$ Oe, with zero damping. LE and TE stand for longitudinal elastic and transverse elastic. (a) $\theta = 0°$, (b) $\theta = 45°$, (c) $\theta = 90°$

whose amplitudes and phases are determined by boundary conditions.

C. Boundary Conditions, Power Absorption, Resonance Frequency, and Linewidth

The continuity conditions for the tangential component of the magnetic field and the normal component of the magnetic induction at the surfaces have already been taken into account and included in the equations of motion, Eqs. (5). The remaining boundary conditions are given by the surface spin and elastic constraints. Here, we consider two limiting cases of the spin-pinning condition[24]; spin unpinned (SU), and spin pinned[11] (SP), and two limiting cases of the elastic boundary condition; traction free[9] (TF) and deformation free[12] (DF). We then apply the four combinations of these boundary conditions, (i) SU-TF; (ii) SU-DF; (iii) SP-TF; (iv) SP-DF, to the two surfaces.

For symmetrical boundary conditions the solution for m_i must be even, with the five plane waves with positive roots paired with the five with negative roots. Thus, the solution takes the form

$$m_i = \left(\underline{m}_i^0 - \sum_{s=1}^{5} C_s \underline{m}_i^s \cos k_s x_3\right) e^{i\omega t} , \qquad (13a)$$

$$u_i = \left(\sum_{s=1}^{5} i C_s \underline{u}_i^s \sin k_s x_3\right) e^{i\omega t} , \qquad (13b)$$

where $\underline{m}_i^s$ and $\underline{u}_i^s$ are amplitude ratios[10,12] associated with k_s, and C_s are complex constants to be determined from the specific boundary conditions. As a result, it is sufficient to write out the boundary conditions at one of the surfaces (say $x_3 = d$): the SU condition at $x_3 = d$;

$$m_{1,3} = m_{2,3} = 0 , \qquad (14a)$$

the SP condition at $x_3 = d$;

$$m_1 = m_2 = 0 , \qquad (14b)$$

the TF condition at $x_3 = d$;

$$\tau_{31} = \bar{c}_{44} u_{1,3} + (B_2/M_0)(\alpha_3 m_1 + \alpha_1 m_3) = 0 ,$$

$$\tau_{32} = \bar{c}_{44} u_{2,3} + (B_2/M_0)(\alpha_3 m_2 + \alpha_2 m_3) = 0 , \qquad (14c)$$

$$\tau_{33} = \bar{c}_{11} u_{3,3} + (2B_1/M_0)\alpha_3 m_3 = 0 ,$$

the DF condition at $x_3 = d$;

$$u_1 = u_2 = u_3 = 0 . \qquad (14d)$$

We take (14a) and (14c) for the SU-TF condition, (14a) and (14d) for the SU-DF condition, (14b) and (14c) for the SP-TF condition, and (14b) and (14d) for the SP-DF condition.

Upon substituting (13) into the above boundary conditions and using relation (8), each set of boundary conditions gives five equations to determine five unknowns (C_s) in terms of $\underline{h}_a^0$. It is easy to show that for the SU-DF condition, Eqs. (14)

yield a set of homogeneous algebraic equations for C_s. Consequently, $C_s = 0$ for all s, and the only solution is the uniform-precession mode, i.e.,

$$m_i = \underline{m}_i^0 e^{i\omega t} , \quad u_i = 0 . \tag{15}$$

The other three cases have nonzero solutions for C_s. Each of these cases yields a set of inhomogeneous algebraic equations. If we solve these equations for C_s in the form

$$C_s = \sum_{a=1}^{2} Y_a^s \underline{h}_a^0 , \tag{16}$$

then we can write the solution in the form

$$m_i = \sum_{a=1}^{2} \chi_{ia} h_a^0 , \tag{17a}$$

$$u_i = \sum_{a=1}^{2} \lambda_{ia} h_a^0 , \tag{17b}$$

where

$$\chi_{ia} = \chi_{ia}^0 + \chi_{ia}' \tag{18}$$

is the total magnetic susceptibility, the sum of the uniform-precession and spatially nonuniform susceptibilities, and λ_{ia} is the rf magnetostriction. χ_{ia}' and λ_{ia} can then be written in the form

$$\chi_{ia}' = \sum_{s=1}^{5} \underline{m}_i^s Y_a^s \cos k_s x_3 , \tag{19a}$$

$$\lambda_{ia} = \sum_{s=1}^{5} i \underline{u}_i^s Y_a^s \sin k_s x_3 . \tag{19b}$$

The power absorption per unit volume is now calculated from the formula

$$P = -\frac{\omega}{2} \operatorname{Im}\left(\frac{1}{2d}\int_{-d}^{d} \sum_{a=1}^{2} m_a h_a^{0*} dx_3\right) . \tag{20}$$

Using (17a) we can rewrite P as

$$P = -\tfrac{1}{2}\omega \operatorname{Im}\left(\sum_{a,b=1}^{2} \underline{h}_a^0 \chi_{ab}^A \underline{h}_b^{0*}\right) ,$$

where χ_{ab}^A is the average susceptibility defined by

$$\chi_{ab}^A = \frac{1}{2d}\int_{-d}^{d} \chi_{ab} dx_3$$

or, by virtue of (18) and (19a)

$$\chi_{ab}^A = \chi_{ab}^0 - \sum_{s=1}^{5} \underline{m}_a^s Y_a^s \sin k_s d / k_s d . \tag{21}$$

Suppose χ is an eigenvalue of χ_{ab}^A and $\underline{h}_a^0$ is the corresponding eigenvector. Then, we can write

$$\sum_{b=1}^{2} \chi_{ab}^A \underline{h}_b^0 = \chi \underline{h}_a^0 . \tag{22}$$

χ_{ab}^A has two eigenvalues and corresponding eigenvectors, which we denote by χ^+, $\underline{h}_a^+$ and χ^-, $\underline{h}_a^-$. Then,

$$\chi^{\pm} = \tfrac{1}{2}(\chi_{11}^A + \chi_{22}^A) \pm \tfrac{1}{2}[(\chi_{11}^A - \chi_{22}^A)^2 + 4\chi_{12}^A\chi_{21}^A]^{1/2} \tag{23}$$

and

$$\underline{h}_1^{\pm} = \frac{|\vec{h}^0|\,|\chi_{12}^A|}{(|\chi^{\pm} - \chi_{11}^A|^2 + |\chi_{12}^A|^2)^{1/2}} ,$$
$$\underline{h}_2^{\pm} = \frac{|\vec{h}^0|\,|\chi_{12}^A|(\chi^{\pm} - \chi_{11}^A)}{\chi_{12}^A(|\chi^{\pm} - \chi_{11}^A|^2 + |\chi_{12}^A|^2)^{1/2}} . \tag{24}$$

In Eqs. (24) $\underline{h}_a^{\pm}$ has been normalized such that

$$\sum_{a=1}^{2} \underline{h}_a^{\pm} \underline{h}_a^{\pm *} = |\vec{h}^0|^2 \tag{25}$$

and that $\underline{h}_1^{\pm}$ be real. Substituting these eigenvalues and eigenvectors in Eq. (20) yields

$$P^{\pm} = -\tfrac{1}{2}\omega\,|\underline{h}^0|^2 \operatorname{Im}(\chi^{\pm}) . \tag{26}$$

We find that χ^+ and χ^- correspond to the resonant and antiresonant responses, respectively, and that $\underline{h}_a^+$ and $\underline{h}_a^-$ correspond to the resonant and antiresonant polarizations, respectively.

In order to calculate resonance frequency and linewidth, we first fix the uniform-precession resonance frequency ω_0 and the magnetization direction, and calculate the corresponding internal field H_0 from Eq. (10b). We then vary the frequency ω with H_0 fixed, and calculate the derivative of the resonant power absorption P^+. The resonance frequency is given by the frequency at which $dP^+/d\omega$ vanishes. The linewidth is defined as the frequency difference between the two extrema of $dP^+/d\omega$.

D. Ferromagnetoelastic-Resonance Conditions

We have applied the above formalism to an insulator having otherwise as parameters those of nickel at room temperature and at K band. We find that the magnetoelasticity has no appreciable effect on the resonance linewidth unless an elastic wave undergoes a thickness resonance at or near the FMR frequency. This condition is referred to as ferromagnetoelastic resonance, FMER. The FMER conditions are summarized as follows: The film thickness L nearly equals an odd-integral number of half-wavelengths of either the L or the T elastic wave at the FMR frequency for the SU-TF or the SP-TF case; L nearly equals an even-integral number of half-wavelengths of either the L or the T elastic wave at the FMR frequency for the SP-DF case. These conditions can easily be understood by considering the fact that the TF and the DF cases sustain standing elastic waves[25] with the surfaces corresponding, respectively, to the antinodes and the nodes. Note, however, that the FMR frequency is different for different boundary conditions. These conditions will be examined in greater detail elsewhere.[17]

Under the FMER conditions the linewidth enhancement may be as much as 50% when the L elastic wave is in thickness resonance, and more than 100% when the T elastic wave is in thickness resonance. This effect is anisotropic, due to the anisotropy of the magnetoelastic coupling. Quantitative analyses of the linewidth enhancement and its angular and thickness dependences are reported elsewhere.[17,26]

III. APPROXIMATE METHOD

A. Introduction

In the preceding section we presented the exact method for calculating ferromagnetic-resonance parameters in the presence of magnetoelastic coupling. In principle, we can use this method for any material whose physical constants are known. In practice, however, the exact method is tedious to apply. This arises from the fact that the complexity of the magnetoelastic coupling produces a secular equation quintic in k^2. The numerical solution of this equation requires great accuracy. Otherwise, the accuracy of everything which follows becomes very poor. It is therefore desirable to simplify the method of calculation. Fortunately, an approximate method has been found[27] which is much simpler to use and yet gives all the essential information of the exact calculation.

The idea of the approximate calculation stems from the fact that a strong coupling exists only between the propagating spin wave and the elastic wave which undergoes a thickness resonance. By a series of transformations and approximations we decouple the equations for the nonpropagating spin wave and the elastic waves which are not in thickness resonance, leaving only the propagating spin wave and thickness-resonant elastic wave. The dispersion relation thus obtained becomes quadratic in k^2 and can readily be solved analytically. From this point on, the procedure is the same as for the exact method.

We have seen in Sec. II that in the absence of magnetocrystalline anisotropy the dispersion relation is invariant under rotation of the x_1 and x_2 axes about the x_3 axis. Therefore, without loss of generality we can put $\phi = 0$ (i.e., $\alpha_1 = \sin\theta$, $\alpha_2 = 0$, $\alpha_3 = \cos\theta$) in the equations of motion.

Let us introduce the following transformation:

$$m_1 = m_\theta \cos\theta \ , \quad m_2 = m_\phi \ , \quad m_3 = -m_\theta \sin\theta \ , \tag{27}$$

where m_θ and m_ϕ are the transverse components of $\vec{m}$ with respect to $\vec{M}_0$. If we substitute (27) into Eqs. (5) and (6), and neglect the damping temporarily, we obtain

$$(1/\gamma)\dot{m}_\theta = H_0 m_\phi - D m_{\phi,33} - M_0 h_2^0 + B_2 \cos\theta u_{2,3} \ , \tag{28a}$$

$$(1/\gamma)\dot{m}_\phi = -(H_0 + 4\pi M_0 \sin^2\theta) m_\theta + D m_{\theta,33} + M_0 \cos\theta h_1^0 - B_2 \cos 2\theta u_{1,3} + (B_1 - 2\pi M_0^2) \sin 2\theta u_{3,3} \ , \tag{28b}$$

$$\rho \ddot{u}_1 = c_{44} u_{1,33} + (B_2/M_0) \cos 2\theta m_{\theta,3} \ , \tag{29a}$$

$$\rho \ddot{u}_2 = c_{44} u_{2,33} + (B_2/M_0) \cos\theta m_{\phi,3} \ , \tag{29b}$$

$$\rho \ddot{u}_3 = c_{11} u_{3,33} - (B_1/M_0) - 2\pi M_0 \sin 2\theta m_{\theta,3} \ . \tag{29c}$$

Note that Eq. (1c) is automatically satisfied. It is Eqs. (28) and (29) which we wish to solve.

B. Longitudinal Thickness Resonance

Let us now consider the case where the L elastic wave undergoes a thickness resonance (referred to as L thickness resonance). Since the wavelength of the L elastic wave is in general much shorter than that of the T elastic wave for a given ω, the T elastic wave is far from resonance when the L elastic wave is in resonance; that is, its response is negligible. Therefore, we neglect the T elastic wave in the equations of motion. If we again assume the external rf field

$$\begin{pmatrix} h_1^0 \\ h_2^0 \end{pmatrix} = h^0 = \underline{\vec{h}}^0 e^{i\omega t}$$

and a plane-wave solution

$$\begin{pmatrix} m_\theta \\ m_\phi \end{pmatrix} = m = (\underline{\vec{m}}^0 + \underline{\vec{m}}' e^{-ikx_3}) e^{i\omega t} \ ,$$

$$u_3 = \underline{u}_3 e^{i(\omega t - kx_3)} \ ,$$

Eqs. (28) and (29) become

$$\overleftrightarrow{A} \cdot \underline{\vec{m}}^0 = -M_0 \overleftrightarrow{Q} \cdot \underline{\vec{h}}^0 \ , \tag{30}$$

$$(Dk^2 \overleftrightarrow{I} - \overleftrightarrow{A}) \cdot \underline{\vec{m}}' = -iB_1 k \vec{R} \underline{u}_3 \ , \tag{31a}$$

$$(c_{11}k^2 - \rho\omega^2) \underline{u}_3 = i(B_1/M_0) k \vec{R}^* \cdot \underline{\vec{m}}' \ , \tag{31b}$$

where

$$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

$$\overleftrightarrow{A} = \begin{pmatrix} -(H_0 + 4\pi M_0 \sin^2\theta) & -i\omega/\gamma \\ i\omega/\gamma & -H_0 \end{pmatrix} \ , \tag{32}$$

$$\overleftrightarrow{Q} = \begin{pmatrix} \cos\theta & 0 \\ 0 & 1 \end{pmatrix} \ , \qquad \vec{R} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} \sin 2\theta \\ 0 \end{pmatrix} \ ,$$

and $\vec{R}^*$ is the transpose of $\vec{R}$.

Since $\overleftrightarrow{A}$ is a Hermitian matrix, there exists a unitary matrix $\overleftrightarrow{U}$ such that $\overleftrightarrow{U}^\dagger \cdot \overleftrightarrow{A} \cdot \overleftrightarrow{U}$ transforms $\overleftrightarrow{A}$ into diagonal form, which we denote by $\overleftrightarrow{A}'$. Here $\overleftrightarrow{U}^\dagger$ is the Hermitian adjoint of $\overleftrightarrow{U}$. $\overleftrightarrow{A}'$ and $\overleftrightarrow{U}$ are given, respectively, by

$$A'_{11} = -(H_0 + 2\pi M_0 \sin^2\theta) + \Omega \ ,$$

$$A'_{22} = -(H_0 + 2\pi M_0 \sin^2\theta) - \Omega \ , \tag{33}$$

$$A'_{12} = A'_{21} = 0 \ ;$$

$$U_{11} = (\omega/|\gamma|)/(2\Omega^2 + 4\pi M_0 \sin^2\theta\, \Omega)^{1/2} \ ,$$

$$U_{12}=i(\omega/|\gamma|)/(2\Omega^2-4\pi M_0\sin^2\theta\Omega)^{1/2}\ ,\qquad(34)$$

$$U_{21}=i(2\pi M_0\sin^2\theta+\Omega)/(2\Omega^2+4\pi M_0\sin^2\theta\Omega)^{1/2}\ ,$$

$$U_{22}=(2\pi M_0\sin^2\theta-\Omega)/(2\Omega^2-4\pi M_0\sin^2\theta\Omega)^{1/2}\ ;$$

where

$$\Omega=[(2\pi M_0\sin^2\theta)^2+(\omega/\gamma)^2]^{1/2}\ .$$

We note that $|U_{11}/U_{21}|=|U_{22}/U_{12}|=|(\omega/\gamma)/(2\pi M_0\sin^2\theta+\Omega)|$ corresponds to the ellipticity of spin precession at a given angle.

If we introduce the transformation

$$m=\overleftrightarrow{\mathrm{U}}\cdot\vec{\mu}\ ,\qquad(35)$$

where $\vec{\mu}=\vec{\mu}^0+\vec{\mu}'=(\vec{\underline{\mu}}^0+\vec{\underline{\mu}}'e^{-ikx_3})e^{i\omega t}$, then (30) and (31) become

$$\overleftrightarrow{\mathrm{A}}\cdot\vec{\underline{\mu}}^0=-M_0\overleftrightarrow{\mathrm{Q}}'\cdot\vec{\underline{\mathrm{h}}}^0\ ,\qquad(36)$$

$$(Dk^2\overleftrightarrow{\mathrm{I}}-\overleftrightarrow{\mathrm{A}}')\cdot\vec{\underline{\mu}}'=-i(B_1-2\pi M_0^2)k\vec{\underline{\mathrm{R}}}\,\underline{\mathrm{u}}_3\ ,\qquad(37\mathrm{a})$$

$$(c_{11}k^2-\rho\omega^2)\underline{\mathrm{u}}_3=i(B_1/M_0-2\pi M_0)k\vec{\underline{\mathrm{R}}}^*\cdot\vec{\underline{\mu}}'\ ,\qquad(37\mathrm{b})$$

where

$$\overleftrightarrow{\mathrm{Q}}'=\overleftrightarrow{\mathrm{U}}^\dagger\cdot\overleftrightarrow{\mathrm{Q}}=\begin{pmatrix}U_{11}\cos\theta & U_{21}^*\\ U_{12}^*\cos\theta & U_{22}\end{pmatrix}\ ,$$

$$\vec{\underline{\mathrm{R}}}=\overleftrightarrow{\mathrm{U}}^\dagger\cdot\vec{\mathrm{R}}=\begin{pmatrix}U_{11}\sin2\theta\\ U_{21}\sin2\theta\end{pmatrix}\ ,$$

$$\vec{\underline{\mathrm{R}}}^*=U_{11}\sin2\theta\quad U_{12}\sin2\theta\ .$$

It is more convenient to write Eq. (36) in explicit form

$$\underline{\mu}_1^0=-(M_0/A'_{11})(Q'_{11}\underline{\mathrm{h}}_1^0+Q'_{12}\underline{\mathrm{h}}_2^0)\ ,\qquad(38\mathrm{a})$$

$$\underline{\mu}_2^0=-(M_0/A'_{22})(Q'_{21}\underline{\mathrm{h}}_1^0+Q'_{22}\underline{\mathrm{h}}_2^0)\ .\qquad(38\mathrm{b})$$

From Eq. (33) we see that A'_{11} has a zero at $\omega/|\gamma|=[H_0(H_0+4\pi M_0\sin^2\theta)]^{1/2}$, whereas A'_{22} has no zeros. In fact, $|A'_{22}|$ increases monotonically with $\omega/|\gamma|$. It then follows from Eqs. (38) that $\underline{\mu}_1^0$ is the resonant mode, whose resonance frequency is

$$\omega_0/|\gamma|=[H_0(H_0+4\pi M_0\sin^2\theta)]^{1/2}\ ,\qquad(39)$$

while $\underline{\mu}_2^0$ is antiresonant.

In the absence of magnetoelastic coupling, Eqs. (37a) and (37b) yield three independent dispersion relations

$$Dk^2-A'_{11}=Dk^2-A'_{22}=c_{11}k^2-\rho\omega^2=0\ ,$$

which correspond to the branches of the propagating spin wave, the nonpropagating spin wave, and the L elastic wave, respectively. Thus, μ'_1, μ'_2, and u_3 correspond, respectively, to the normal coordinates of the uncoupled system. The magnetoelasticity couples these waves, and therefore μ'_1, μ'_2, and u_3 are no longer the normal coordinates. However, the coupling between μ'_2 and u_3 is expected to be very weak since the corresponding branches are widely separated for all frequencies. Hence μ'_2, the nonpropagating spin wave, can still be considered a normal coordinate.

Considering the facts that μ_2^0 is antiresonant and that μ'_2 is nonpropagating, we can neglect $\mu_2(=\mu_2^0+\mu'_2)$. Thus, Eq. (36) reduces to (38a) and Eqs. (37) reduce to

$$(Dk^2-A'_{11})\underline{\mu}'_1+i(B_1-2\pi M_0^2)k\underline{\mathrm{R}}_1\underline{\mathrm{u}}_3=0\ ,\qquad(40\mathrm{a})$$

$$i(B_1/M_0-2\pi M_0)k\underline{\mathrm{R}}_1\underline{\mu}'_1-(c_{11}k^2-\rho\omega^2)\underline{\mathrm{u}}_3=0\ .\qquad(40\mathrm{b})$$

The corresponding dispersion relation becomes

$$(Dk^2-A'_{11})(c_{11}k^2-\rho\omega^2)-(1/M_0)\times(B_1-2\pi M_0^2)^2f_L(\theta)k^2=0\ ,\qquad(41)$$

where

$$f_L(\theta)=\underline{\mathrm{R}}_1^2=U_{11}{}^2\sin^22\theta\qquad(42)$$

characterizes the angular variation of the L magnetoelastic coupling.

If we include the LL damping, the matrix $\overleftrightarrow{\mathrm{A}}$ becomes non-Hermitian. As a result $\overleftrightarrow{\mathrm{U}}$ is no longer unitary. If the damping is small, however, we can still use, to a good approximation,[28] the same transformation with the same unitary matrix, provided we make the following corrections[29] to Ω in A':

$$\Omega'=[(2\pi M_0\sin^2\theta)^2+(1/\beta^2)(\omega/\gamma)^2]^{1/2}-i\eta(\omega/\gamma)\ ,\qquad(43)$$

where $\beta=1+\eta^2$. The L elastic damping can easily be reintroduced by allowing c_{11} to be complex, i.e., $c_{11}=c'_{11}+ic''_{11}$.

We see that (41) is a quadratic equation in k^2 and can readily be solved analytically. The solution is

$$k^2=\frac{1}{2Dc_{11}}\left\{c_{11}A'_{11}+\rho\omega^2D+\frac{1}{M_0}(B_1-2\pi M_0^2)^2f_L(\theta)\pm\left[\left(c_{11}A'_{11}+\rho\omega^2D+\frac{1}{M_0}(B_1-2M_0^2)^2f_L(\theta)\right)^2-4\rho\omega^2Dc_{11}A'_{11}\right]^{1/2}\right\}\qquad(44)$$

which yields two pairs of roots, $\pm k_s$ ($s=1,2$). Thus, the solution is given by

$$\mu_1=\left(\underline{\mu}_1^0+\sum_{s=1}^{2}(C^se^{-ik_sx_3}+C^{s+2}e^{ik_sx_3})\right)e^{i\omega t}\ ,\qquad(45\mathrm{a})$$

$$u_3=\left(\sum_{s=1}^{2}\underline{\mathrm{u}}_3^s(C^se^{-ik_sx_3}-C^{s+2}e^{ik_sx_3})\right)e^{i\omega t}\ .\qquad(45\mathrm{b})$$

Here the amplitude ratios $\underline{\mathrm{u}}_3^s$ are obtained by substituting k_s into Eq. (40a) or (40b) with $\underline{\mu}_1^s$ normalized to unity. This yields

$$\underline{u}_3^s = \frac{Dk^2 - A'_{11}}{-i(B_1 - 2M_0^2)k_s\underline{R}_1} = \frac{i(B_1/M_0 - 2M_0)k_s\underline{R}_1}{c_{11}k^2 - \rho\omega^2} . \tag{46}$$

The coefficients C^s are constants which are to be determined by boundary conditions. We shall treat these in Sec. III D. The stress component τ_{33} which was given in Eq. (14c) can now be written in terms of μ_1 as

$$\tau_{33} = c_{11}u_{3,3} - (B_1/M_0)U_{11}\sin 2\theta \mu_1 . \tag{47}$$

C. Transverse Thickness Resonance

Let us now consider the case in which the T elastic wave undergoes a thickness resonance (transverse thickness resonance). Hence, we neglect the L elastic wave. By assuming the same plane-wave solution for m_θ, m_ϕ, u_1, and u_2 as in the previous discussion, Eqs. (28) and (29) again yield Eq. (30) for the uniform part of $\vec{m}$, and

$$(Dk^2 \overleftrightarrow{I} - \overleftrightarrow{A}) \cdot \underline{\vec{m}}' - iB_2k\overleftrightarrow{S} \cdot \underline{\vec{u}}_t = 0 , \tag{48a}$$

$$(c_{44}k^2 - \rho\omega^2)\underline{\vec{u}}_t + i(B_2/M_0)k\overleftrightarrow{S} \cdot \underline{\vec{m}}' = 0 , \tag{48b}$$

for the nonuniform part. In Eqs. (48), $\underline{\vec{u}}_t$ and $\overleftrightarrow{S}$ are defined as

$$\underline{\vec{u}}_t = \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} , \qquad \overleftrightarrow{S} = \begin{pmatrix} \cos 2\theta & 0 \\ 0 & \cos\theta \end{pmatrix} , \tag{49}$$

and all other quantities have been previously defined. Using the transformation of Eq. (35) with the same unitary matrix as before, Eqs. (48) now yield

$$(Dk^2 \overleftrightarrow{I} - \overleftrightarrow{A}') \cdot \underline{\vec{\mu}}' - iB_2k\overleftrightarrow{S}'^{\dagger} \cdot \underline{\vec{u}}_t = 0 , \tag{50a}$$

$$(c_{44}k^2 - \rho\omega^2)\underline{\vec{u}}_t + i(B_2/M_0)k\overleftrightarrow{S}' \cdot \underline{\vec{\mu}}' = 0 , \tag{50b}$$

where $\overleftrightarrow{S}' = \overleftrightarrow{S} \cdot \overleftrightarrow{U}$ and $\overleftrightarrow{S}'^{\dagger} = \overleftrightarrow{U}^{\dagger} \cdot \overleftrightarrow{S}$ is the Hermitian adjoint of $\overleftrightarrow{S}'$. If we introduce new variables v_1 and v_2 such that

$$\begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix} = \underline{\vec{v}} = \overleftrightarrow{S}'^{\dagger} \cdot \underline{\vec{u}}_t , \tag{51}$$

then we can write (50) as

$$(Dk^2 \overleftrightarrow{I} - \overleftrightarrow{A}') \cdot \underline{\vec{\mu}}' - iB_2k\underline{\vec{v}} = 0 , \tag{52a}$$

$$(c_{44}k^2 - \rho\omega^2)\underline{\vec{v}} + i(B_2/M_0)k\overleftrightarrow{s}' \cdot \underline{\vec{\mu}}' = 0 , \tag{52b}$$

where

$$\overleftrightarrow{s} = \overleftrightarrow{S}'^{\dagger} \cdot \overleftrightarrow{S}' = \overleftrightarrow{U}^{\dagger} \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{U} = \begin{pmatrix} U_{11}^2\cos^2 2\theta + |U_{21}|^2\cos^2\theta & U_{11}U_{12}\cos^2 2\theta + U_{21}^*U_{22}\cos^2\theta \\ U_{11}U_{12}^*\cos^2 2\theta + U_{21}U_{22}\cos^2\theta & |U_{12}|^2\cos^2 2\theta + U_{22}^2\cos^2\theta \end{pmatrix} . \tag{53}$$

Let us rewrite Eqs. (52) in the following matrix form:

$$\left(\begin{array}{cc|cc} Dk^2 - A'_{11} & -iB_2k & 0 & 0 \\ i(B_2/M_0)s_{11}k & c_{44}k^2 - \rho\omega^2 & i(B_2/M_0)s_{12}k & 0 \\ \hline 0 & 0 & Dk^2 - A'_{22} & -iB_2k \\ i(B_2/M_0)s_{21}k & 0 & i(B_2/M_0)s_{22}k & c_{44}k^2 - \rho\omega^2 \end{array}\right) \begin{pmatrix} \underline{\mu}'_1 \\ \underline{v}_1 \\ \hline \underline{\mu}'_2 \\ \underline{v}_2 \end{pmatrix} = 0 . \tag{54}$$

We see that the transformation (51) has brought (54) nearly into block diagonal form as indicated by the dotted lines.

At $\theta = 0$, (54) is exactly in block form since s_{12} and s_{21} vanish,[30] and the $(\underline{\mu}'_1, \underline{v}_1)$ and $(\underline{\mu}'_2, \underline{v}_2)$ pairs become independent of each other. Both $\underline{\mu}'_1$ and $\underline{v}_1$ correspond to positive circular polarization, and both $\underline{\mu}'_2$ and $\underline{v}_2$ correspond to the negative circular polarization. That is, our transformation separates those waves with the resonant polarization from those with the antiresonant polarization. At oblique angles, the off-block elements are nonzero so that there exists a coupling between the $(\underline{\mu}'_1, \underline{v}_1)$ and $(\underline{\mu}'_2, \underline{v}_2)$ pairs. This coupling, however, is of higher order than the coupling between $\underline{\mu}'_1$ and $\underline{v}_1$ or that between $\underline{\mu}'_2$ and $\underline{v}_2$. The $(\underline{\mu}'_1, \underline{v}_1)$ and $(\underline{\mu}'_2, \underline{v}_2)$ pairs have elliptical polarizations with opposite senses of rotation, but in each pair the spin wave and the elastic wave do not have the same ellipticity.

Since the coupling between the $(\underline{\mu}'_1, \underline{v}_1)$ and $(\underline{\mu}_2', \underline{\nu}_2)$ pairs is weak, and $\underline{\mu}'_2$ is nonpropagating, we ignore the $(\underline{\mu}'_2, \underline{v}_2)$ pair, as we ignored $\underline{\mu}'_2$ in Sec. II. Then Eqs. (52) or (54) reduce to

$$(Dk^2 - A'_{11})\underline{\mu}'_1 - iB_2k\underline{v}_1 = 0 , \tag{55a}$$

$$i(B_2/M_0)ks_{11}\underline{\mu}'_1 + (c_{44}k^2 - \rho\omega^2)\underline{v}_1 = 0 . \tag{55b}$$

The corresponding secular equation is

$$(Dk^2 - A'_{11})(c_{44}k^2 - \rho\omega^2) - (B_2^2/M_0)f_T(\theta)k^2 = 0 , \tag{56}$$

where

$$f_T(\theta) = s_{11} = U_{11}^2\cos^2 2\theta + |U_{21}|^2\cos 2\theta \tag{57}$$

characterizes the angular dependence of the T magnetoelastic coupling. We see that we have again reduced the secular equation to quadratic form.

The LL damping is reintroduced in the same fashion, as previously, by replacing Ω in A'_{11} by Ω', defined in Eq. (43). The T elastic damping is introduced by allowing c_{44} to be complex, i.e., $c_{44}=c'_{44}+ic''_{44}$.

The secular equation in k^2 has the solution

$$k^2=\frac{1}{2Dc_{11}}\left\{c_{44}A'_{11}+\rho\omega^2D+\frac{B_2^2}{M_0}f_T(\theta)\pm\left[\left(c_{44}A'_{11}+\rho\omega^2D+\frac{B_2^2}{M_0}f_T(\theta)\right)^2-4\rho\omega^2Dc_{44}A'_{11}\right]^{1/2}\right\}, \quad (58)$$

which yields two pairs of roots $\pm k_s$ ($s=1, 2$). Thus the solution for μ_1 can be written in the same form as (45a), and that for v_1 in the same form as (45b) with the amplitude ratios replaced by

$$\underline{v}_1^s=\frac{Dk_s^2-A'_{11}}{iB_2k_s}=\frac{i(B_2/M)k_s\,\mathrm{s}_{11}}{c_{44}k^2-\rho\omega^2} . \quad (59)$$

There are again four constants (C^s) to be determined by the boundary conditions.

The stress component associated with the strain $v_{1,3}$ can be obtained by transforming τ_{31} and τ_{32} given in Eq. (14c) as

$$\overline{\tau}_1=S'^{\dagger}_{11}\tau_{31}+S'^{\dagger}_{12}\tau_{32} . \quad (60)$$

Expanding (60) and neglecting μ_2 yields

$$\overline{\tau}_1=c_{44}v_{1,3}+(B_2/M_0)\,\mathrm{s}_{11}\mu_1 . \quad (61)$$

D. Boundary Conditions and Power Absorption

Because of the similarity between the L-thickness-resonance and T-thickness-resonance cases, we can treat the two boundary-value problems in a similar fashion. Let us introduce the following definitions: for the L-thickness-resonance case,

$$c=c_{11}, \quad w=u_3, \quad b=-(B_1/M_0)\,U_{11}\sin2\theta ; \quad (62)$$

for the T-thickness-resonance case,

$$c=c_{44}, \quad w=v_1, \quad b=(B_2/M_0)\,(U_{11}^2\cos^22\theta+|U_{21}|^2\cos^2\theta) . \quad (63)$$

We also omit the subscript of μ_1. Then the analysis given below holds for both cases.

We consider the same symmetrical boundary conditions as in Sec. II. The solution can then be written

$$\mu=\left(\underline{\mu}^0-\sum_{s=1}^{2}C_s\cos k_sx_3\right)e^{i\omega t} , \quad (64a)$$

$$w=\left(\sum_{s=1}^{2}iC_s\underline{\mathrm{w}}^s\sin k_sx_3\right)e^{i\omega t} , \quad (64b)$$

where $C_s=2C^s$, C^s being the arbitrary constants used in (45). The SU, SP, TF, and DF conditions at $x_3=d$ are given, respectively, by

$$\mu'_{,3}=0 , \quad (65a)$$

$$\mu^0+\mu'=0 , \quad (65b)$$

$$cw_{,3}+b(\mu^0+\mu')=0 , \quad (65c)$$

$$w=0 . \quad (65d)$$

The combinations (65a)–(65c), (65b)–(65c), and (65b)–(65d) give the conditions SU-TF, SP-TF, and SP-DF, respectively. We have omitted the SU-DF condition since we have shown above that it corresponds to the uniform-precession problem.

On substituting Eqs. (64) into Eqs. (65), we obtain C_1 and C_2. In the SU-TF case,

$$C_1=(1/\Delta_1)\,(b\underline{\mu}^0k_2\sin k_2d) , \quad C_2=-(1/\Delta_1)\,(b\underline{\mu}^0k_1\sin k_1d) , \quad (66)$$

where

$$\Delta_1=(ic\underline{\mathrm{w}}^1k_1+b)k_2\sin k_2d\cos k_1d-(ic\underline{\mathrm{w}}^2k_2+b)k_1\sin k_1d\cos k_2d .$$

In the SP-TF case,

$$C_1=-\frac{\underline{\mathrm{w}}^2k_2\mu^0}{(\underline{\mathrm{w}}^1k_1-\underline{\mathrm{w}}^2k_2)\cos k_1d} , \quad C_2=\frac{\underline{\mathrm{w}}^2k_1\mu^0}{(\underline{\mathrm{w}}^1k_1-\underline{\mathrm{w}}^2k_2)\cos k_2d} . \quad (67)$$

In the SP-DF case,

$$C_1=-(1/\Delta_2)\,(\underline{\mathrm{w}}^2\underline{\mu}^0\sin k_2d) , \quad C_2=(1/\Delta_2)\,(\mathrm{w}^1\,\underline{\mu}^0\sin k_1d) , \quad (68)$$

where

$$\Delta_2=\underline{\mathrm{w}}^1\sin k_1d\cos k_2d-\underline{\mathrm{w}}^2\cos k_1d\sin k_2d .$$

To calculate the power absorption from Eq. (20), we express m_1 and m_2 in terms of μ:

$$\vec{\mathrm{m}}_{\parallel}=\begin{pmatrix}m_1\\ m_2\end{pmatrix}=\begin{pmatrix}U_{11}\cos\theta\\ U_{21}\end{pmatrix}\vec{\mu} . \quad (69)$$

Since $\underline{\mu}'$ is proportional to $\underline{\mu}^0$ as can be seen from the expressions for C_s and since $\underline{\mu}^0$ is related to $\underline{\mathrm{h}}_1^0$ and $\underline{\mathrm{h}}_2^0$ by (38a), we can write (69) as

$$\underline{\vec{\mathrm{m}}}_{\parallel}=\overleftrightarrow{\chi}\cdot\underline{\vec{\mathrm{h}}}^0 , \quad (70)$$

where

$$\overleftrightarrow{\chi}=\left(1-\sum_{s=1}^{2}C'_s\cos k_sx_3\right)\overleftrightarrow{\chi}^0 \quad (71)$$

and

$$\overleftrightarrow{\chi}^0=-\frac{M_0}{A'_{11}}\begin{pmatrix}U_{11}^2\cos2\theta & U_{11}U_{21}^*\cos\theta\\ U_{11}U_{21}\cos\theta & |U_{21}|^2\end{pmatrix} , \quad C'_s=C_s/\underline{\mu}^0 . \quad (72)$$

Substituting Eq. (70) into Eq. (20) we obtain

$$P = -\tfrac{1}{2}\omega \,\mathrm{Im}\{\underline{\vec{h}}^{0*}\cdot \overleftrightarrow{\chi}^{A}\cdot \underline{\vec{h}}^{0}\},$$

where $\overleftrightarrow{\chi}^{A}$ is the average susceptibility and is given by

$$\overleftrightarrow{\chi}^{A} = \left[1+\sum_{s=1}^{2} C_s'\left(\frac{\sin k_s d}{k_s d}\right)\right]\overleftrightarrow{\chi}^{0}.$$

Following the same diagonalization procedure discussed in Sec. II C, we obtain the eigenvalues of $\overleftrightarrow{\chi}^{A}$ as

$$\chi^{\pm} = \left[1+\sum_{s=1}^{2} C_s'\left(\frac{\sin k_s d}{k_s d}\right)\right]\chi_{\pm}^{0}, \tag{73}$$

where $\overleftrightarrow{\chi}_{\pm}^{0}$ are the eigenvalues of the uniform-precession susceptibility, $\overleftrightarrow{\chi}^{0}$, given in (72), and are

$$\chi_{+}^{0} = -(M_0/A_{11}')(U_{11}^{2}\cos 2\theta + |U_{21}|^{2}),$$
$$\chi_{-}^{0} = 0. \tag{74}$$

Clearly, χ_{+}^{0} and χ_{-}^{0} correspond to the resonant and antiresonant responses, respectively. Of course, the antiresonant response is identically zero in the approximate calculation due to the fact that the antiresonant mode μ_2 has been neglected. The eigenvectors $\underline{\vec{h}}^{\pm}$ corresponding to the eigenvalues are obtained from Eqs. (24) with $\overleftrightarrow{\chi}^{0}$ in place of $\overleftrightarrow{\chi}^{A}$. The resonance frequency and linewidth are calculated from the resonant absorption P^{+} in the same manner as discussed in Sec. II C.

IV. CONCLUSIONS

In an attempt to provide a foundation for the evaluation of anisotropic broadening of FMR lines due to magnetoelasticity, we have presented here a formal treatment of the solution of the coupled magnetoelastic equations of motion for an obliquely magnetized thin film. We have considered two methods. The first, based on a formalism developed by Tiersten,[9–11] is exact within the framework of the quasistatic approximation. The second is an approximate method by which the mathematical treatment of the boundary-value problem has been greatly simplified. This method is based on the fact that a strong coupling exists only between the resonant spin wave and the elastic wave which un-

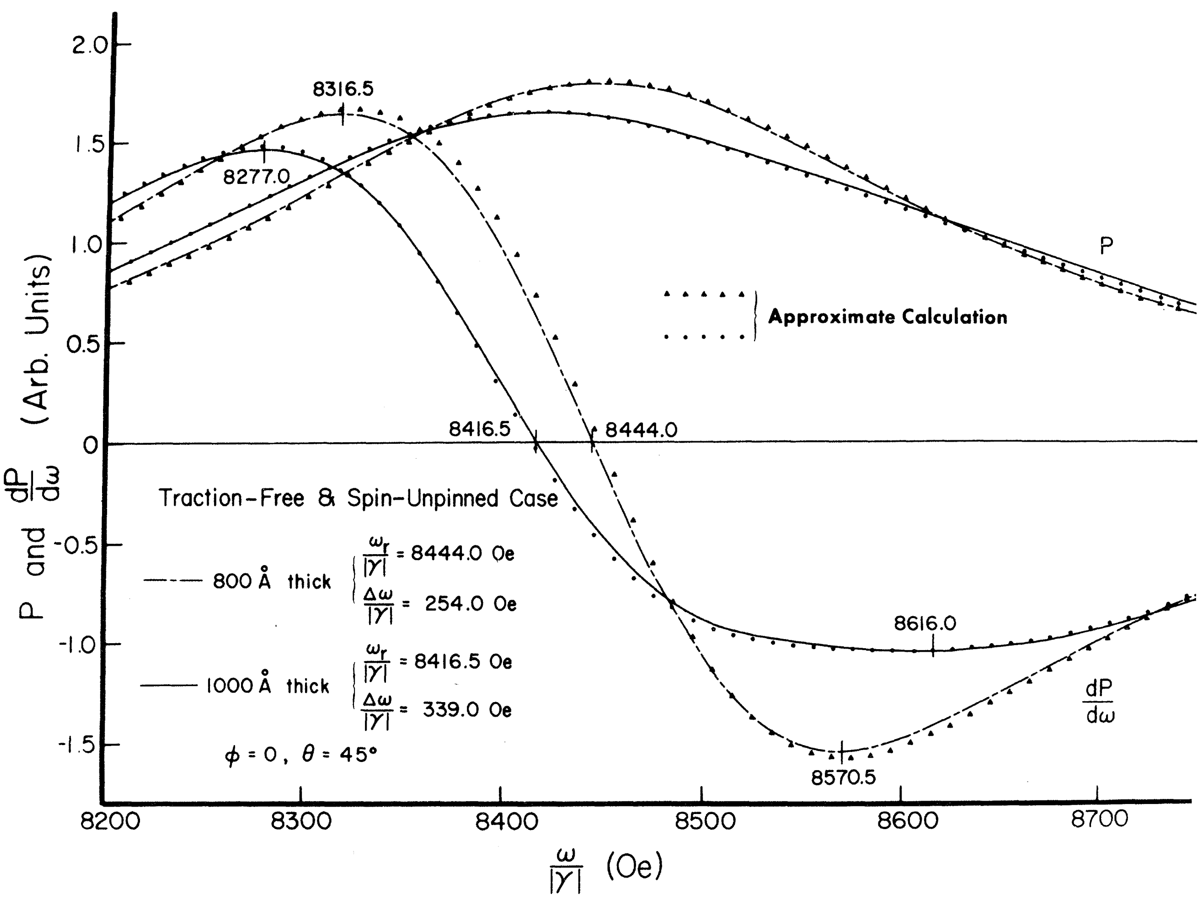

FIG. 3. Ferromagnetic resonance line shapes as calculated by the exact and approximate methods. The approximate solutions, represented by the points, depart from the exact solutions, represented by the lines, by 1% or less everywhere.

dergoes a thickness resonance near the FMR frequency. Thus, the simplification was made possible without sacrificing the physical model of the exact method.

We now quantitatively justify the validity of the approximate method by a discussion of some examples. Figure 3 shows the resonance absorption lines and their frequency derivatives for nickel films of 800- and 1000-Å thickness, calculated for $\omega_0/|\gamma| = 8432$ Oe by both the exact and the approximate methods. They represent the SU-TF case with $\vec{M}_0$ along the $\langle 101 \rangle$ axis. 1000 Å corresponds to the half-wavelength of the L elastic wave at $\omega/|\gamma| = 8432$ Oe. We see that the agreement between the two calculations is excellent for this case. The accuracy of the approximate calculation is within 1% of the exact one. For the thickness of 800 Å neither the L nor the T elastic wave undergoes a thickness resonance near the FMR frequency, since the thickness is considerably larger than the half-wavelength of the T elastic wave (~700 Å) and far smaller than that of the L elastic wave. We have used the transverse-thickness-resonance equations for this case. Even in this case both calculations agree quite well as far as the linewidth is concerned. This is not unexpected because neither of the elastic waves has an appreciable response. There is, however, a slight discrepancy in the resonance frequency. This will be discussed elsewhere in conjunction with the resonance-frequency shift due to thickness resonance.[17] Under the FMER conditions, both calculations agree just as well for the SP-TF and SP-DF cases as they do for the SU-TF case.

It is possible that for some materials the elastic damping is small enough and at the same time the magnetoelastic coupling is large enough for the off-thickness resonant elastic wave to have an appreciable response.[27] In this case, both the L and T elastic waves (v_1 for the T elastic wave) have to be included in the approximate equations of motion. Although this results in a cubic secular equation, it can still be solved analytically.

Finally, the boundary conditions can easily be generalized to arbitrary spin pinning and elastic conditions at each surface of the plate.[31] By doing so, we can evaluate the effect of substrate acoustic impedance on FMR[18] and also calculate the phonon power transmitted into the substrate.[18,19] For this problem, it is frequently necessary to employ the cubic secular equation.[19]

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