Here we have made use of the expansion

$$z^2 - \gamma z + 1 = (\sinh\theta)^{-1} \sum_{l=1}^{\infty} \sinh l\theta\, z^{l-1} .$$

The second boundary condition is that, since for a surfaceless periodic lattice the Green's functions have the property $\lim G_{lm}^{\alpha\epsilon}(\lambda) = 0$ as $|l-m| \to \infty$, then we shall require that $\lim X_{lm}^{f}(\alpha, \beta) = 0$ as $|l-m| \to \infty$. The solutions in (17) follow.

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Logarithmic Corrections to the Molecular-Field Behavior of Critical and Tricritical Systems

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The asymptotic critical form of thermodynamic functions is analyzed by means of renormalization-group techniques. If certain exponent relations are satisfied, then the critical behavior is not described by a simple power law, but a power law multiplied by a fractional power of a logarithm. The approach is applied to two special systems whose critical exponents are molecular-field-like. (i) For ordinary critical transitions in four dimensions we find the same logarithmic factors previously computed by Larkin and Khmel'nitskii. (ii) For tricritical transitions in three dimensions we compute the logarithmic corrections to the molecular-field tricritical behavior discussed in an earlier publication.

I. INTRODUCTION

Renormalization-group techniques yield power laws for the expectation values of different operators and susceptibilities near criticality,[1,2] which can be characterized by sets of critical exponents. If these exponents satisfy certain relations, then the power laws are modified by logarithmic *factors*.[3] If in particular an operator has a vanishing scaling index y (for a definition of y see Sec. II of the present paper and Ref. 3), then this gives rise to factors of fractional powers of logarithms. In terms of the renormalization-group procedure this is due to the very slow decay of the field of this operator. Examples where this situation occurs are (i) the critical behavior of four-dimensional systems[2] and (ii) the tricritical behavior of three-dimensional systems.[4] For both cases[5] the Gaussian fixed point of the renormalization-group equations leads to respective sets of molecular-field values for the critical exponents.[2,4,6] In this paper we show that the molecular-field results for the two systems are modified by logarithmic factors: The asymptotic critical forms of thermodynamic functions have the form of a power law *times* a fractional power of a logarithm.

We calculate these logarithmic corrections and determine the powers of the logarithms from the Wilson recursion relations.[2] In Sec. II we review those features of the renormalization-group procedure we need for our calculations. In Sec. III the general scheme for determining logarithmic corrections is derived. The second-order critical behavior in four dimensions is discussed in Sec. IV, and the tricritical behavior in three dimensions is investigated in Sec. V. Our results for the logarithmic corrections to the molecular-field behavior in four dimensions agree with previous results by Larkin and Khmel'nitskii.[7] The predictions for the tricritical behavior in three dimensions can be tested experimentally, for example at the He^3-He^4 tricritical point.

We note that besides the critical and the tricritical behavior described here, other critical and tricritical fixed points might exist which yield different sets of critical exponents. That question will not be investigated further in this paper.

II. RENORMALIZATION-GROUP PROCEDURE

In this section we review the renormalization-group procedure that has been outlined by Wilson[1] and one of the authors.[3] We derive the homogeneity relation for the free energy and show that near critical points it is sufficient (in most cases) to consider only the relevant scaling fields.

The state of a thermodynamic system can be changed by varying certain experimental fields which we denote by $\tilde{\mu}_i^{\text{expt}}$. The subscript i numbers the fields, whereas the superscript expt indicates that the particular field is experimentally available. Examples of such fields are the inverse temperature β, a magnetic field multiplied by β, a chemical potential multiplied by β, etc. We introduce the fields such that they enter linearly into $H=\beta\mathcal{H}$, where $\mathcal{H}$ is the Hamiltonian of the system. For conceptual reasons one often introduces additional "theoretical fields" which cannot be experimentally realized. In the following we denote the set of experimental *and* theoretical fields by $\{\tilde{\mu}_i\}$.

The renormalization group connects the free energy of systems in different states. For states infinitesimally close to each other this relation can be written

$$F=d^{-1}\sum_i \tilde{c}_i' \frac{\partial F}{\partial \tilde{\mu}_i} , \qquad (2.1)$$

where d is the dimensionality of the system, and where the coefficients $\tilde{c}_i'$ are functions of the fields $\tilde{\mu}_i$. By F we denote always the free energy multiplied by β. The relation (2.1) can be easily derived from Eqs. (2.8) and (2.12) of Ref. 3; a simplified version of it can be obtained from Eqs. (9)–(11) of Ref. 1. A point $\{\tilde{\mu}_i^*\}$ at which $\tilde{c}_i'=0$ for all i is called a *fixed point* of the renormalization-group equation (2.1). Note that the free energy F has been normalized such that $F\{\tilde{\mu}_i^*\}=0$.

We discuss the relation (2.1) first in a linearized form. We may expand the coefficients $\tilde{c}_i'$ to linear order about the fixed point:

$$\tilde{c}_i'=\sum_j \tilde{a}_{ij}'(\tilde{\mu}_j-\tilde{\mu}_j{}^*)+O(\tilde{\mu}-\tilde{\mu}^*)^2 . \qquad (2.2)$$

By taking appropriate linear combinations of the deviating fields $(\tilde{\mu}-\tilde{\mu}^*)$, we can define a new set of fields μ_i in terms of which an expansion (2.2) leads to a diagonal matrix a_{ij}'. We denote the diagonal elements (eigenvalues) of this matrix by y_i. In terms of the new fields μ_i, one obtains

$$F=d^{-1}\sum_i c_i' \frac{\partial F}{\partial \mu_i} , \qquad (2.3)$$

and in linear approximation

$$c_i'=y_i\,\mu_i+O(\mu^2) . \qquad (2.4)$$

Retaining only this linear term in Eq. (2.3), we obtain the homogeneous differential equation

$$F=d^{-1}\sum_i y_i\,\mu_i \frac{\partial F}{\partial \mu_i} . \qquad (2.5)$$

Therefore, in this approximation F is a homogeneous function of the fields μ_i,

$$F\{\mu_i\}=e^{-dl}F\{\mu_i\,e^{y_i l}\} , \qquad (2.6)$$

where e^l is an arbitrary scale factor. This factor has the meaning of a momentum cutoff factor[1,3] in the renormalization-group equations.

Next we include the higher-order contributions to c_i' in Eq. (2.4). In general, one obtains an expansion

$$c_i'=y_i\,\mu_i+\tfrac{1}{2}\sum_{jk} a_{ijk}'\,\mu_j\,\mu_k+\cdots . \qquad (2.7)$$

Now Eq. (2.6) no longer holds. But it has been shown[3] that *scaling fields* g_i can be introduced such that an analogous homogeneity relation for F holds exactly in these new variables:

$$F\{g_i\}=e^{-dl}F\{g_i\,e^{y_i l}\} . \qquad (2.8)$$

The Hamiltonian H of the system does not depend linearly on the scaling fields g_i. Usually the fields μ_i can be expanded in terms of the scaling fields g_i:

$$\mu_i=g_i+\tfrac{1}{2}\sum_{jk} b_{ijk}\,g_j g_k+\cdots . \qquad (2.9)$$

Based on the homogeneity assertion for the free energy, which has been very successful for the explanation of critical phenomena, one might expect that at critical points the scaling fields g_i

vanish. However, one can reach criticality by adjusting only a few experimental fields. Therefore one has the suspicion that only a small number of scaling fields g_i must vanish at criticality. From the theory this can be seen as follows: The scaling laws describe the behavior of a system very close to the critical point. Let us assume, for example, that g_1 is approximately given by the relative temperature difference $(T-T_c)/T_c$, where T_c is the critical temperature. Then for the scaling laws to be valid g_1 has to be very small. Let us apply Eq. (2.8) and choose l such that

$$|g_1|\, e^{y_1 l}=1 \; . \tag{2.10}$$

Then we obtain

$$F\{g_i\}=|g_1|^{d/y_1} F\{g_i|g_1|^{-y_i/y_1}\} \; . \tag{2.11}$$

Microscopic calculations yield $y_1>0$.[2,6] (Note that the scaling index y_1 and the quantity λ used in Ref. 6 are related via $\lambda=2^{y_1}$.) Therefore $g_i|g_1|^{-y_i/y_1}$ vanishes for all g_i with negative y_i in the limit $g_1\to 0$. These terms can be neglected and give only corrections of the relative order $|g_1|^{-y_i/y_1}$. (A restriction is discussed in the next paragraph.) Hence it is sufficient to consider only the scaling fields g_i with $y_i\geq 0$. The other scaling fields are irrelevant. Criticality is thus determined by the condition that the scaling fields g_i with $y_i>0$ vanish. We call these fields relevant fields and denote them by g_i^{rel}. The third case, namely, the contribution of marginal fields g_i with $y_i=0$, will be discussed in Sec. III. We mention that the operator 1 with the conjugate field μ_0 has an exponent $y_0=d$. Nevertheless, the condition $g_0=0$ at criticality need not be satisfied since an additive constant to the Hamiltonian does not change its critical properties. Therefore g_0 is not considered to be a relevant field.

The argument that the irrelevant fields can be neglected holds only if the free energy exists in the limit of vanishing irrelevant fields. If, however, the fixed point Hamiltonian, which is defined by $\mu_i=0$, plus the "relevant" contributions lead to a Hamiltonian without a lower bound, then the free energy does not exist in the limit of vanishing irrelevant fields. In this case at least one irrelevant scaling field has to be taken into account.[8]

Now let us consider the condition for criticality. The theoretical fields for a given system are certain constants. Therefore, the fields μ_i, which are linear functions of the $\tilde{\mu}_i$, depend explicitly only on the experimental fields $\tilde{\mu}_i^{\text{expt}}$,

$$\mu_i=\mu_i\{\tilde{\mu}^{\text{expt}}\} \; . \tag{2.12}$$

We can express the scaling fields g_i in terms of the fields $\tilde{\mu}_i$ by inverting Eq. (2.9) and substituting Eq. (2.12). As a result, we obtain the scaling fields as nonlinear functions of the experimental fields $\tilde{\mu}_i^{\text{expt}}$:

$$g_i=g_i\{\tilde{\mu}^{\text{expt}}\} \; . \tag{2.13}$$

Criticality is now determined by the condition that all relevant scaling fields vanish,

$$g_i^{\text{rel}}\{\tilde{\mu}^{\text{expt}}\}=0 \; . \tag{2.14}$$

This yields a finite set of equations for a finite number of experimentally available parameters. We assume that the fields g_i are analytic functions of the fields $\tilde{\mu}^{\text{expt}}$.

III. LOGARITHMIC ANOMALIES

In this section we discuss how logarithmic anomalies in the asymptotic critical behavior arise. First, we describe how simple logarithms are obtained within the renormalization scheme. Second, we show that a field with a scaling index $y=0$ may even lead to fractional powers of logarithms.

We return to Eq. (2.3) and consider the fields μ_i now as functions of the parameter l. With

$$\frac{\partial\mu_i(l)}{\partial l}=c_i' \; , \tag{3.1}$$

we obtain for the free energy $F=F\{\mu_i(l)\}$ a relation analogous to the result (2.3),

$$F=d^{-1}\sum_i \frac{\partial\mu_i}{\partial l}\,\frac{\partial F}{\partial\mu_i} \; , \tag{3.2}$$

with the solution

$$F\{\mu_i(0)\}=e^{-dl}F\{\mu_i(l)\} \; . \tag{3.3}$$

[In the linearized approximation c_i' is given by $c_i'=y_i\mu_i$, and Eq. (3.1) yields $\mu_i(l)=\mu_i(0)\,e^{y_i l}$. On substituting this result into Eq. (3.3), we obtain again Eq. (2.6).] The exact homogeneity relation (2.8) suggests the definition

$$g_i(l)=g_i(0)\,e^{y_i l} \tag{3.4}$$

for the scaling fields g_i, which leads to

$$\frac{\partial g_i(l)}{\partial l}=y_i g_i \; . \tag{3.5}$$

In linear order, g_i and μ_i are identical as seen from Eq. (2.9).

Next we calculate the derivative $\partial\mu_i/\partial l$ to second order in g_i. Using Eq. (3.5), we obtain from Eq. (2.9)

$$\frac{\partial\mu_i}{\partial l}=y_i g_i+\frac{1}{2}\sum_{jk}(y_j+y_k)b_{ijk}g_j g_k+O(g^3) \; . \tag{3.6}$$

On the other hand, by substituting Eqs. (3.1) and (2.9) into Eq. (2.7), we find

$$\frac{\partial\mu_i}{\partial l}=y_i g_i+\frac{1}{2}\sum_{jk}(y_i b_{ijk}+a_{ijk}')g_j g_k+O(g^3) \; . \tag{3.7}$$

Equating the coefficients of $g_j g_k$ in Eqs. (3.6) and (3.7), we obtain an equation for the coefficients b_{ijk}:

$$(y_j + y_k - y_i)\, b_{ijk} = a'_{ijk} \ . \qquad (3.8)$$

Provided that $y_i \neq y_j + y_k$, this equation yields the coefficients b_{ijk} in terms of the expansion coefficients a'_{ijk}. However, if $y_i = y_j + y_k$ holds and if a'_{ijk} is nonzero, then no constant b_{ijk} satisfies Eq. (3.8). One can repair this[3] by letting b_{ijk} depend on l. Then Eq. (3.6) reads

$$\frac{\partial \mu_i}{\partial l} = y_i g_i + \frac{1}{2} \sum_{jk} \left((y_j + y_k)\, b_{ijk} + \frac{\partial b_{ijk}}{\partial l} \right) g_j g_k + O(g^3) \ . \qquad (3.9)$$

For $y_i = y_j + y_k$ this equation yields

$$\frac{\partial b_{ijk}}{\partial l} = a'_{ijk}, \qquad b_{ijk} = a'_{ijk}(l + l_0) \ . \qquad (3.10)$$

Therefore $\mu_i(l)$ contains in that case a contribution proportional to $e^{y_i l}(l + l_0)$. Together with Eq. (2.10), this leads to a logarithmic correction term proportional to $|g_1|^{-y_i/y_1} \ln |g_1|$.

A similar situation occurs for higher-order terms $b_{ij_1 \cdots j_n}$ if the condition

$$y_i = y_{j_1} + \cdots + y_{j_n} \qquad (3.11)$$

is satisfied. As long as we are interested only in the contributions from relevant operators and if no marginal field with $y = 0$ has to be taken into account,[9] then there are only a finite number of sets $(y_i, y_{j_1}, \ldots, y_{j_n})$ which satisfy the condition (3.11). These sets determine the logarithmic corrections.

If, however, a field μ_u with $y_u = 0$ has to be considered,[9] then Eq. (3.11) is satisfied for an infinite number of sets. This might lead to an anomalous behavior. Here we discuss only how the most singular contributions arise. A more complete discussion is given in the Appendix A. Let us consider the following approximation for the differential equation for μ_u:

$$\frac{\partial \mu_u}{\partial l} = \tfrac{1}{2} a'_{uuu}\, \mu_u^2 \ . \qquad (3.12)$$

From this equation one obtains the solution[2,4,10]

$$\mu_u = s\,(l + l_0)^{-1}, \quad s = -2/a'_{uuu} \ . \qquad (3.13)$$

According to our definition (3.4), g_u would be a constant. Therefore we do not use g_u to define the state of the system for $l = 0$, but the parameter l_0. Then the state of the system is completely determined by the scaling fields g_i (without g_u) and l_0. Again we assume that the fields g_i and l_0 are analytic functions of the fields $\tilde{\mu}^{\text{expt}}$ within a certain region of l_0.

Now we consider the effect of the coupling of μ_u to μ_i. We use the approximation

$$\frac{\partial \mu_i}{\partial l} = y_i\, \mu_i + a'_{iui}\, \mu_i\, \mu_u = y_i\, \mu_i + s a'_{iui} (l + l_0)^{-1} \mu_i \ , \qquad (3.14)$$

and try the ansatz

$$\mu_i(l) = g_i(l)(l + l_0)^{p_i} \ . \qquad (3.15)$$

On substituting Eq. (3.15) into Eq. (3.14) and using Eq. (3.5), we obtain

$$p_i = s a'_{iui} \ . \qquad (3.16)$$

Therefore, μ_i contains, in general, factors of fractional powers of $\ln |g_1|$ as corrections to the fields $g_i(l)$. The discussion in Appendix A shows that the fields μ_i can be expressed as a polynomial of $g_i(l)(l + l_0)^{p_i}$ in leading order (that is, for large l) provided that Eq. (3.11) is not satisfied for the exponents y (except y_u). If Eq. (3.11) is satisfied for a set of exponents y which does not include y_u, then we obtain additional powers of $(l + l_0)$ as described in the first part of this section.

In Secs. IV and V we will apply these ideas to discuss the critical behavior of four-dimensional systems and the tricritical behavior of three-dimensional systems.

IV. CRITICAL BEHAVIOR IN FOUR DIMENSIONS

We consider now the critical behavior of the isotropic n-vector model[11] (compare Refs. 6, 12, and 13). A fixed point of this model is the Gaussian fixed point[2,4,6] which is described by the Hamiltonian

$$H^* = \tfrac{1}{2} \sum_q q^2 z_q^\alpha z_{-q}^\alpha \ , \qquad (4.1)$$

where z_q^α is the Fourier transform with wave vector q of the α component of the vector $\vec{z}(r)$. As shown in Appendix B, the local and rotationally invariant operators δQ_m (which were denoted by δQ_{ms} in Refs. 4 and 13) have the exponents

$$y_m = d - m(d - 2) \ . \qquad (4.2)$$

Since $m = 0$ corresponds to the operator 1, only those operators with $m > 0$ and $y_m \geq 0$ are relevant. We note that the rotationally invariant operators which correspond to short-range interactions have exponents $y \leq 0$. The only such operator with $y = 0$ arises from the scale transformation[9] $z \to cz$ applied to H^*, which reproduces H^* since it is homogeneous in z. For that reason, we can restrict ourselves to the local operators δQ_m.

For $d = 4$ we find from Eq. (4.2) the exponents $y_1 = 2$ and $y_2 = 0$. Thus, only one rotationally invariant operator δQ_1, given by Eq. (B11), is relevant. The corresponding scaling field g_1 has to vanish at criticality and determines the critical temperature [compare Eq. (2.14)]. For small g_1 we may assume that g_1 is proportional to $(T - T_c)/T_c$:

$$g_1 \propto (T - T_c)/T_c = \tau \ . \qquad (4.3)$$

The coefficients a' which determine the logarithmic singularities are calculated in Appendix B. Wilson's recursion relation[2] is used for this calculation. It yields

$$a'_{222} = -(n+8)(1-b^{-2})^2/\ln b \ , \tag{4.4}$$

$$a'_{121} = -(n+2)(1-b^{-2})^2/(2\ln b) \ , \tag{4.5}$$

$$a'_{011} = -n(1-b^{-2})^2/(2\ln b) \ , \tag{4.6}$$

where b is the momentum cutoff factor. Using Eq. (3.16), we obtain the exponent

$$p_1 = -(n+2)/(n+8) \ . \tag{4.7}$$

It is independent of the cutoff parameter b. Although it is encouraging that p_1 is independent of b, it is not known whether Wilson's recursion relation gives the exact result for this case. Since $2y_1 = y_0$ and $a'_{011} \neq 0$, we find that there is an additional factor $(l+l_0)$ associated with the g_1^2 term in μ_0

$$\mu_0(l) = g_0 e^{4l} + \tfrac{1}{2} a'_{011} g_1^2 e^{4l} (l+l_0)^{2p_1+1}/(2p_1+1) + \delta\mu_0 (g_1 e^{2l} (l+l_0)^{p_1}) \ . \tag{4.8}$$

We now use Eq. (3.3) and

$$F\{\mu_i\} = \mu_0 + F(\mu_0 = 0, \{\mu_{i\neq 0}\}) \tag{4.9}$$

to obtain

$$F\{\mu_i(0)\} = e^{-4l}\mu_0(l) + e^{-4l} F(\mu_0 = 0, \{\mu_{i\neq 0}(l)\}) \ . \tag{4.10}$$

We discuss both terms of Eq. (4.10) as a function of τ and an external magnetic field h. The magnetic field h enters F via

$$g_h = \beta h, \quad y_h = 3 \ . \tag{4.11}$$

The exponent y_h follows from Eq. (B9) with $m=0$, $k=1$. Since the magnetic field couples to the $q=0$ component of z, it is not eliminated by the renormalization procedure. Therefore, only μ_h depends on g_h and

$$a'_{h2h} = 0, \quad p_h = 0 \ . \tag{4.12}$$

Similarly the field g_0 is never eliminated and couples only to μ_0. Therefore, the additional factor $(l+l_0)$ multiplying g_1^2 appears only in μ_0.

Defining l by

$$|g_1| \, e^{2l} (l+l_0)^{p_1} = 1 \ , \tag{4.13}$$

we obtain the free energy

$$F = g_0 + \tfrac{1}{2} a'_{011} g_1^2 (l+l_0)^{2p_1+1}/(2p_1+1) + e^{-4l} f_\pm (g_h e^{3l}) \ , \tag{4.14}$$

with

$$f_\pm(g_h(l)) = \delta\mu_0(\mathrm{sgn} g_1) + F(\mu_0 = 0, \mu_1 1 \,\mathrm{sgn} g_1, g_h(l)) \ . \tag{4.15}$$

Since g_1 is proportional to τ, one finds from Eq. (4.13) in the limit $l \gg l_0$, i.e., $|\tau| \ll 1$,

$$l + l_0 \propto |\ln|\tau|| \ . \tag{4.16}$$

The most singular contribution to the specific heat c at constant $g_h = 0$ for $|\tau| \ll 1$ follows from the second term of Eq. (4.14),

$$c \propto |\ln|\tau||^{2p_1+1} = |\ln|\tau||^{(4-n)/(n+8)} \ . \tag{4.17}$$

We obtain the singular behavior of the susceptibility χ from the last term in Eq. (4.14),[14]

$$\chi \propto e^{2l} \propto |\tau|^{-1} |\ln|\tau||^{-p_1} = |\tau|^{-1} |\ln|\tau||^{(n+2)/(n+8)} \ . \tag{4.18}$$

The Hamiltonian has no lower bound for vanishing irrelevant fields below the critical temperature. Therefore we cannot deduce the spontaneous magnetization m from Eq. (4.14). However, we may use

$$m = \frac{\partial F\{\mu_i(0)\}}{\partial g_h} = e^{-4l} \frac{\partial F\{\mu_i(l)\}}{\partial g_h} = e^{-l} \frac{\partial F\{\mu_i(l)\}}{\partial g_h(l)} = e^{-l} m(l) \ , \tag{4.19}$$

where $m(l)$ denotes the magnetization for the fields $\mu_i(l)$. Within Wilson's approximation, the Hamiltonian reads

$$H = H^* + \int d^4 r \sum_k v_k z^{2k}(r) \ . \tag{4.20}$$

Defining l by Eq. (4.13), it turns out that the leading terms for v_k are proportional to $(l+l_0)^{-k}$ for $k > 2$, whereas $0 > v_1 \propto (l+l_0)^0$ and $0 < v_2 \propto (l+l_0)^{-1}$. Since for this value of l the system in the state $\{\mu_i(l)\}$ is far away from criticality, one obtains $m(l)$ by determining the value of z that minimizes the Hamiltonian (4.20). This yields

$$m(l) \propto (l+l_0)^{1/2} \ . \tag{4.21}$$

For $n > 1$ this is an estimation only, since the persistence of an infinite correlation length associated with breaking a continuous symmetry might result in some violations of Landau's mean-field theory even for $|g_1| e^{2l} > 1$ below T_c. From Eqs. (4.19) and (4.21) we obtain

$$m \propto e^{-l}(l+l_0)^{1/2} \propto |\tau|^{1/2} |\ln|\tau||^{(p_1+1)/2} = |\tau|^{1/2} |\ln|\tau||^{3/(n+8)} \ . \tag{4.22}$$

Similarly one finds, for $T = T_c$ in an external magnetic field with l defined by $|g_h|^{3l} = 1$, the magnetization

$$m \propto e^{-l}(l+l_0)^{1/3} \propto |h|^{1/3} |\ln|h||^{1/3} \ . \tag{4.23}$$

The results in Eqs. (4.17), (4.18), and (4.22) agree with the results obtained by Larkin and Khmel'nitskii[7] from diagram techniques. In the limit $n \to \infty$ one obtains the same results as for the spherical model[15] as expected from Stanley's proof.[16]

V. TRICRITICAL BEHAVIOR IN THREE DIMENSIONS

In this section we discuss the behavior of an n-vector model near the Gaussian fixed point $Q_t^* = 0$

in three dimensions. We have shown that such a system exhibits a tricritical behavior with molecular-field-type tricritical exponents.[4]

From Eq. (4.2) we find the exponents

$$y_0=3,\quad y_1=2,\quad y_2=1,\quad y_3=0\ . \tag{5.1}$$

Hence, there are two relevant operators δQ_1 and δQ_2 given by Eqs. (B11) and (B12) with $d=3$. According to Eq. (2.14) two scaling fields, g_1 and g_2, have to vanish at criticality. Since both operators are rotationally invariant this situation does not correspond to a normal critical point but to a tricritical point.[4,17] In a tricritical system *two* fields (in addition to the ordering field) have to be adjusted to approach the tricritical point. For example, in a He^3-He^4 mixture one has to adjust both the temperature and the difference between the chemical potentials of the He^3 and He^4 components to reach the superfluid–phase-separation tricritical point. The scaling fields g_1 and g_2 are, in general, functions of these two experimental fields. For He^3-He^4 mixtures the ordering field h is the field conjugate to the superfluid order parameter. This field is experimentally not accessible, and the physical field space is defined by $h=0$.

The exponents $y_1=2$, $y_2=1$, and the exponent $y_h=\frac{5}{2}$ for the ordering field lead to molecular-field-like tricritical exponents[18,19] as shown previously.[4] [The exponent y_h can be obtained from Eq. (B9) with $m=0$, $k=1$.] Since there is also a marginal operator δQ_3, given by Eq. (B13), with a vanishing exponent y_3, we expect logarithmic corrections to the molecular-field behavior.

Again we calculate the coefficients a' using Wilson's recursion relation in Appendix B. We obtain

$$a'_{333}=6(3n+22)(1-b^{-1})^2/(b\ln b)\ , \tag{5.2}$$

$$a'_{232}=6(n+4)(1-b^{-1})^2/(b\ln b)\ , \tag{5.3}$$

$$a'_{131}=0\ , \tag{5.4}$$

$$a'_{122}=3(n+2)(1-b^{-1})^2/(b\ln b)\ , \tag{5.5}$$

$$a'_{012}=0\ . \tag{5.6}$$

This yields the exponents

$$p_1=0,\qquad p_2=-2(n+4)/(3n+22)\ , \tag{5.7}$$

which are again independent of the cutoff parameter b.

Since $2y_2=y_1$ and $a'_{122}\neq 0$, we obtain a contribution

$$\tfrac{1}{2}a'_{122}\,g_2{}^2\,e^{2l}(l+l_0)^{2p_2+1}/(2p_2+1) \tag{5.8}$$

to μ_1. Since μ_1 couples back to the other fields μ_i, the leading contribution from g_2 is proportional to

$$g_2\,e^l(l+l_0)^{p_2+1/2}\quad . \tag{5.9}$$

Since a'_{012} vanishes, we do not obtain an extra logarithmic factor to μ_0. Therefore, we obtain for the free energy F as a function of the scaling fields g_i in leading order

$$F(g_0,g_1,g_2\,l_0^{p_2+1/2},g_h)$$
$$=g_0+e^{-3l}\,F(0,g_1e^{2l},g_2e^l(l+l_0)^{p_2+1/2},g_he^{5l/2})\ . \tag{5.10}$$

With the choice $|g_1|\,e^{2l}=1$, this leads to

$$F=g_0+|g_1|^{3/2}$$
$$\times f_\pm(g_2|g_1|^{-1/2}\,|\ln|g_1||^{p_2+1/2},\,g_h|g_1|^{-5/4})\ , \tag{5.11}$$

where $f_\pm$ is $F(g_1=\pm1)-g_0$ and g_0 is the regular part of the free energy. The result (5.11) differs from the prediction

$$F^{\text{sing}}=|g_1|^{(2-\dot{\alpha}_t)}f_\pm(g_2|g_1|^{-\dot{\varphi}_t},g_h|g_1|^{-\dot{\Delta}_t})$$

of the phenomenological tricritical scaling approach [compare Eq. (6) of Ref. 17] by the logarithmic correction factor in the first argument. The tricritical exponents can be identified with $\dot{\alpha}_t=\frac{1}{2}$, $\dot{\varphi}_t=\frac{1}{2}$, and $\dot{\Delta}_t=\frac{5}{4}$.[4]

First we discuss the consequences of Eq. (5.11) if we approach the tricritical point from the disordered phase along a line $g_1/g_2=\text{const}\neq 0$, that is, a path not parallel to the second-order critical line [compare Eq. (5.15)] or the first-order transition line.[17] Then we obtain for the singular contribution of the entropy S^{sing}, specific heat c (for constant $g_h=0$), and susceptibility χ_h with respect to the ordering field,

$$S^{\text{sing}}\propto|g_1|^{1/2}\ , \tag{5.12}$$

$$c\propto|g_1|^{-1/2}\ , \tag{5.13}$$

$$\chi_h\propto|g_1|^{-1}\ . \tag{5.14}$$

These results are in agreement with the molecular-field theory.[4] Equations (5.12) and (5.13) describe also the tricritical behavior of the "nonordering density" and its conjugate-field derivative.[4,17] Throughout our tricritical calculations we have neglected terms of the relative order $(l+l_0)^{-1/2}$. Therefore we expect additive corrections to our results which are smaller by a factor $|\ln|g_1||^{-1/2}$.

In the ordered regime the Hamiltonian has no lower bound. In analogy to Eq. (4.19) we find

$$m=e^{-l/2}m(l)\ . \tag{5.15}$$

Moreover, with $|g_1|e^{2l}=1$ and $g_2^2\,|\ln|g_2||^{2p_1+1}<|g_1|$, we obtain for the leading terms of the coefficients v_k in the Hamiltonian (4.20) now $v_k\propto(l+l_0)^{\iota_k}$ with $\iota_{2k}=\iota_{2k-1}=-k$ for integer k except $\iota_1=0$ and $\iota_3=-1$. This yields the estimation

$$m(l)\propto(l+l_0)^{1/4}\ , \tag{5.16}$$

and hence for the "ordering density"

$$m\propto|g_1|^{1/4}\,|\ln|g_1||^{1/4}\ . \tag{5.17}$$

Again the exponent $\dot{\beta}_t = \frac{1}{4}$ agrees with the molecular-field result, but the power law is modified by an (here) n-independent fractional power of a logarithm. The appearance of logarithmic correction factors makes a determination of tricritical exponents by series expansion techniques difficult.

The critical line in the $g_h = 0$ plane is determined by a nonanalyticity of the function $f_\pm(\tilde{q}_2, \tilde{q}_h = 0)$ in Eq. (5.11) in the variable $\tilde{q}_2$. From this we deduce that the shape of the critical line is given by

$$g_{1,c} \propto g_{2,c}^2 \left|\ln|g_{2,c}|\right|^{2p_2+1} = g_{2,c}^2 \left|\ln|g_{2,c}|\right|^{(6-n)/(3n+22)} . \quad (5.18)$$

This result reproduces the tricritical crossover exponent $\dot{\varphi}_t = \frac{1}{2}$, but exhibits an additional logarithmic correction factor in the equation for the critical line.

The tricritical scaling fields g_1 and g_2 are nonlinear functions of two experimental fields, the temperature, and a nonordering field.[4,17,20] Contrary to a phenomenological approach, the renormalization theory defines uniquely the directions of the g axes (relative to Cartesian experimental field axes, for example). To discuss the critical line as a function of the experimental fields $\tilde{\mu}_i^{\text{expt}}$, we start from the assumption that these fields can be expanded in terms of the scaling fields g_i,

$$\tilde{\mu}_1^{\text{expt}} = d_1 + d_2 g_1 + d_3 g_2 + d_4 g_2^2 + \cdots , \quad (5.19)$$

$$\tilde{\mu}_2^{\text{expt}} = e_1 + e_2 g_1 + e_3 g_2 + e_4 g_2^2 + \cdots . \quad (5.20)$$

In the expression $(e_3 \tilde{\mu}_1^{\text{expt}} - d_3 \tilde{\mu}_2^{\text{expt}})$ the term linear in g_2 is eliminated. Therefore, on using Eq. (5.18) and $(2p_2 + 1) > 0$ for $n < 6$, we obtain

$$e_3(\tilde{\mu}_{1,c}^{\text{expt}} - d_1) - d_3(\tilde{\mu}_{2,c}^{\text{expt}} - e_1) \propto (\tilde{\mu}_{2,c}^{\text{expt}} - e_1)^2 \left|\ln|\tilde{\mu}_{2,c}^{\text{expt}} - e_1|\right|^{2p_2+1} . \quad (5.21)$$

This result for the critical line can be tested experimentally.[4]

We mention that Migdal[21] has considered the correlation functions at a tricritical point by using diagram techniques. Migdal interpreted his calculations in terms of a normal critical point since he assumed implicitly $g_1 = 0$ and hence was left with only one parameter, g_2, that corresponds to his V_{eff}. It is interesting to calculate the free energy near the tricritical point by diagram methods.

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APPENDIX A

Here we give a more complete discussion of the calculation of $\mu_i(l)$ and show that in Sec. III we have obtained the leading terms for large l. We expand the fields μ in powers of the scaling fields (except g_u)

$$\mu_j = t_j(l) + \sum t_{ji}(l) g_i(l) + \tfrac{1}{2}\sum t_{ji_1i_2}(l) g_{i_1}(l) g_{i_2}(l) + \cdots . \quad (A1)$$

(Throughout this appendix we use the convention that the summation runs over all indices i.) We substitute Eq. (A1) into

$$\frac{\partial \mu_j}{\partial l} = y_j \mu_j + \sum_{n=2} \frac{1}{n!} \sum a'_{jI} \mu_I , \quad (A2)$$

in which I denotes the set $i_1, i_2, \ldots, i_n$ and μ_I stands for $\mu_I = \mu_{i_1} \cdots \mu_{i_n}$. Equating equal powers of g_i, we obtain the Eqs. (A3), (A8), and (A13) below for the coefficients t.

The zeroth-order terms t_j obey the equations

$$\frac{\partial t_j}{\partial l} = y_j t_j + \tfrac{1}{2}\sum a'_{ji_1i_2} t_{i_1} t_{i_2} + \cdots . \quad (A3)$$

Suppose that $t_u \propto l^{-1}$ and all other coefficients $t_j \propto l^{-2}$; then we obtain for $j = u$

$$\frac{\partial t_u}{\partial l} = \tfrac{1}{2} a'_{uuu} t_u^2 + O(l^{-3}) , \quad (A4)$$

which leads to

$$t_u = s(l + l_0)^{-1} + O(l^{-2}), \quad s = -2/a'_{uuu} \quad (A5)$$

[compare Eq. (3.13)]. For $j \neq u$ we obtain (we assume $y_j \neq 0$)

$$y_j t_j + \tfrac{1}{2} a'_{juu} t_u{}^2 + O(l^{-3}) = 0 , \quad (A6)$$

from which it follows that

$$t_j = -(a'_{juu}/2y_j)(l + l_0)^{-2} + O(l^{-3}) . \quad (A7)$$

Next we consider the first-order terms t_{jk}. They obey

$$\frac{\partial t_{jk}}{\partial l} = (y_j - y_k) t_{jk} + \sum a'_{ji_1i_2} t_{i_1} t_{i_2k} + \cdots . \quad (A8)$$

Suppose that $t_{kk} \propto l^{p_k}$ and $t_{jk} \propto l^{p_k - 1}$ for $j \neq k$; then we obtain from Eq. (A8)

$$\frac{\partial t_{kk}}{\partial l} = a'_{kuk} t_{kk} s(l + l_0)^{-1} + O(l^{p_k - 2}) , \quad (A9)$$

which yields

$$t_{kk} = (l + l_0)^{p_k} + O(l^{p_k - 1}), \quad p_k = s a'_{kuk} . \quad (A10)$$

For $j \neq k$ provided that $y_j \neq y_k$, we obtain from Eq. (A8)

$$(y_j - y_k) t_{jk} + a'_{juk} s(l + l_0)^{p_k - 1} + O(l^{p_k - 2}) = 0 , \quad (A11)$$

which leads to

$$t_{jk}=\frac{a'_{juk}}{y_k-y_j}\; s(l+l_0)^{p_k-1}+O(l^{p_k-2}) \,. \tag{A12}$$

We omit a discussion of the case $y_j=y_k$.

At last we consider the terms t in general order. From Eq. (A1) and (A2) we obtain

$$\frac{\partial t_{jK}}{\partial l}+(y_K-y_j)t_{jK}=\tfrac{1}{2}\sum a'_{ji_1i_2}t_{i_1K_1}t_{i_2K_2}+\cdots \,. \tag{A13}$$

Here K stands for $k_1,\ldots,k_n$ and $y_K=y_{k_1}+\cdots+y_{k_n}$. The summation runs also over all decompositions of the set K into two sets K_1 and K_2. If $y_K\neq y_j$ then the leading contribution to t_{jK} comes from the leading term of the right-hand side of Eq. (A13). If $y_j=y_K$ then we obtain one more power in $(l+l_0)$ for t_{jK}. Therefore we obtain

$$t_{jK}\propto(l+l_0)^{p_K+\Delta_{jK}}\,,\quad p_K=p_{k_1}+\cdots+p_{k_n}\,, \tag{A14}$$

where Δ_{jK} gives the number of extra powers of $(l+l_0)$ which arise because of $y_j=y_k$. It might be that the leading power of t_{jK} is less than that given in Eq. (A14) because of vanishing coefficients a' or because of contributions canceling each other. We note that we have neglected factors $\ln(l+l_0)$ in this discussion.

APPENDIX B

We sketch the calculation of the coefficients a' from Wilson's recursion relations [Eqs. (3.41) and (3.43) of Ref. 2]. With general momentum cutoff b and without the constant term $I_l(0)$, these relations are

$$I_l(z)=\int d^ny\,\exp[-y^2-\tilde{Q}_l(z,y)]\,, \tag{B1}$$

$$\tilde{Q}_l(z,y)=\tfrac{1}{2}Q_l(z+y)+\tfrac{1}{2}Q_l(z-y)\,, \tag{B2}$$

$$Q_{l+1}(z)=-b^d\ln I_l(b^{1-d/2}z)\,. \tag{B3}$$

The fixed point under consideration is given by $Q^*=0$. Then we obtain from Eq. (B1) for the deviations from the fixed point

$$\delta\ln I(z)=-\langle\tilde{Q}\rangle+\tfrac{1}{2}(\langle\tilde{Q}^2\rangle-\langle\tilde{Q}\rangle^2)+\cdots\,, \tag{B4}$$

with

$$\langle A\rangle=\int d^ny\,A(z,y)\,e^{-y^2}\Big/\int d^ny\,e^{-y^2}\,. \tag{B5}$$

To obtain the eigenfunctions δQ and the eigenvalues $\lambda=b^y$, we use the method of Ref. 13. Since $r^*=u^*=0$, Eq. (12) of Ref. 13 yields

$$b^y\,\delta Q(b^{d/2-1}z)=b^d e^{\Delta/4}\,\delta Q(z)\,. \tag{B6}$$

The eigenfunctions δQ are products of the Laguerre polynomials L_m with the harmonic polynomials H_k. Since

$$e^{\Delta/4}\left[L_m^{(n/2+k-1)}\left(\frac{z^2}{a}\right)H_k(z)\right]=\left(\frac{a-1}{a}\right)^m L_m^{(n/2+k-1)}\left(\frac{z^2}{a-1}\right)H_k(z)\,, \tag{B7}$$

we obtain the eigenfunctions

$$\delta Q(z)=L_m^{(n/2+k-1)}((1-b^{2-d})z^2)H_k(z) \tag{B8}$$

and

$$y=d-(2m+k)(\tfrac{1}{2}d-1)\,. \tag{B9}$$

In particular, for the rotationally invariant solutions ($k=0$) we obtain Eq. (4.2) and

$$\delta Q_0=1\,, \tag{B10}$$

$$\delta Q_1=\tfrac{1}{2}n-\hat{z}^2\,, \tag{B11}$$

$$\delta Q_2=\tfrac{1}{2}n(\tfrac{1}{2}n+1)-2(\tfrac{1}{2}n+1)\hat{z}^2+\hat{z}^4\,, \tag{B12}$$

$$\delta Q_3=\tfrac{1}{2}n(\tfrac{1}{2}n+1)(\tfrac{1}{2}n+2)-3(\tfrac{1}{2}n+1)(\tfrac{1}{2}n+2)\hat{z}^2+3(\tfrac{1}{2}n+2)\hat{z}^4-\hat{z}^6\,, \tag{B13}$$

with

$$\hat{z}^2=(1-b^{2-d})z^2\,. \tag{B14}$$

To obtain the coefficients a' we calculate first the contributions to $\delta\ln I(z)$ [Eq. (B4)] by using Eq. (C1), and expanding the polynomials

$$2[\delta\ln I(z)]_{jk}=\langle\tilde{Q}_j\tilde{Q}_k\rangle-\langle\tilde{Q}_j\rangle\langle\tilde{Q}_k\rangle=-b^d\sum_i a_{ijk}L_i^{(n/2-1)}((1-b^{2-d})z^2)\,. \tag{B15}$$

The last step is the calculation of a'_{ijk} from a_{ijk}. From

$$\mu_i(l+l')=e^{y_il}\mu_i(l')+\tfrac{1}{2}\sum_{jk}a_{ijk}(l)\,\mu_j(l')\,\mu_k(l')+\cdots \tag{B16}$$

and

$$\frac{\partial\mu_i(l+l')}{\partial l}=\frac{\partial\mu_i(l+l')}{\partial l'}\,, \tag{B17}$$

we obtain in the limit $l'=0$

$$\frac{\partial a_{ijk}(l)}{\partial l}=a'_{ijk}e^{y_il}+(y_j+y_k)\,a_{ijk}(l)\,. \tag{B18}$$

In particular, for $i=k$ and $j=u$ Eq. (B13) yields

$$a_{iui}(l)=a'_{iui}\,le^{y_il}=a'_{iui}\,b^{y_i}\ln b\,. \tag{B19}$$

Although Wilson's recursion relation gives b-dependent coefficients a', it turns out that the exponents p do not depend on b.

APPENDIX C

The expectation values $\langle A\rangle$ [Eq. (B5)] can be calculated from

$$\langle y^{2\mu}(\vec{y}\,\vec{z})^{2\nu}\rangle=\frac{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2}n+\nu+\mu)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\nu)}\,z^{2\nu}\,. \tag{C1}$$

One obtains Eq. (C1) from evaluating the generating function

$$\int d^ny\,\exp[-(1-\rho)y^2+\beta\vec{y}\,\vec{z}]$$

in two ways: (a) by first expanding in powers of ρ and β and expressing the integrals by expectation values and (b) by first evaluating the integral and then expanding.

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[9]Due to scale transformations there are always some fields with $y=0$. It is not necessary to consider them, since they can be transformed away by the scale transformations.

[10]If a'_{uuu} vanishes but $a'_{uuuu}\neq 0$, then one obtains $\mu_u \propto l^{-1/2}$ which might lead to anomalous results for μ_1.

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