PHYSICAL REVIEW B VOLUME 7, NUMBER 5 1 MARCH 1973

Diagrammatic Analysis of the Dynamics of Localized Moments in Metals*

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(Received 28 June 1972)

Using the conventional exchange Hamiltonian $\mathcal{H}_{\text{es}}$, together with an additional model Hamiltonian which provides a source of lattice relaxation for the conduction-electron magnetization, the coupled local-moment–conduction-electron transverse dynamic susceptibility is calculated. Feynman temperature-ordered Green's functions are used and are "dressed" with self-energies which are correct to second order in interaction parameters. A pair of coupled vertex equations are constructed which includes all diagrams correct to second order in the interaction parameters. In order to obtain the desired two-particle characteristics of the response function, some new manipulative and mathematical methods are introduced. These enable the vertex equations to be reduced to a coupled pair of linear equations which determine the coupled susceptibility. At sufficiently high temperatures the Kondo "g shifts" and the weak frequency dependence of the self-energies can be ignored. These linear equations are then equivalent to the linearized version of the following Bloch equations: $d/dt\ \vec{\text{M}}_s = g_s[\vec{\text{M}}_s \times (\vec{\text{H}}_{\text{ext}} + \lambda \vec{\text{M}}_e)] - (1/T_{\text{se}}) [\vec{\text{M}}_s - \chi_s^0 (\vec{\text{H}}_{\text{ext}} + \lambda \vec{\text{M}}_e)] + (g_s/g_e T_{\text{es}}) [\vec{\text{M}}_e - \chi_e^0 (\vec{\text{H}}_{\text{ext}} + \lambda \vec{\text{M}}_s)]$, $d/dt\ \vec{\text{M}}_e = g_e[\vec{\text{M}}_e \times (\vec{\text{H}}_{\text{ext}} + \lambda \vec{\text{M}}_s)] - (1/T_{\text{es}} + 1/T_{\text{el}}) [\vec{\text{M}}_e - \chi_e^0 (\vec{\text{H}}_{\text{ext}} + \lambda \vec{\text{M}}_s)] + (g_e/g_s T_{\text{se}}) [\vec{\text{M}}_s - \chi_s^0 (\vec{\text{H}}_{\text{ext}} + \lambda \vec{\text{M}}_e)] + D\, \nabla^2 [\vec{\text{M}}_e - \chi_e^0 (\vec{\text{H}}_{\text{ext}} + \lambda \vec{\text{M}}_s)]$, where T_{se}, T_{es}, and T_{el} are the relaxation times for the local-moment–conduction-electron-spin, the conduction-electron-spin–local-moment, and the conduction-electron-spin–lattice systems. The diffusion constant is $D = \frac{1}{3} v_F^2 T_i$, where T_i is the nonmagnetic-impurity scattering time, and $\vec{\text{M}}_e$ and $\vec{\text{M}}_s$ are, respectively, the conduction-electron and local-moment magnetizations. The spin-orbit scattering in the presence of large-potential-impurity scattering results in relaxation effects identical to those obtained above with the model electron-lattice Hamiltonian. $\vec{\text{H}}_{\text{ext}}$ is the external static and rf field, χ_e^0 and χ_s^0 are the conduction-electron and local-moment-unenhanced (by the s-d exchange) susceptibilities, and g_e and g_s are the respective g factors ($\hbar$ and μ_B have been set equal to unity). These Bloch equations show that the relaxation destination vectors are those appropriate to the total internal field. The presence of the g-factor ratios are consistent with the fact that $\mathcal{H}_{\text{es}}$ transfers spin, rather than magnetization, from the conduction electrons to local moments and vice versa. For temperatures $kT < \omega_s$, where ω_s is the local-moment Zeeman energy, these equations fail, the self-energies become frequency dependent, and there are modifications to χ_s^0. Most important, the electronic self-energy becomes dependent on the magnitude of the electronic momentum, so that a single equation for the *total* electron magnetization cannot be written. A microscopic derivation of the detailed-balance condition is given.

I. INTRODUCTION

On a phenomenological basis, Hasegawa[1] proposed a pair of coupled Bloch equations which describe the motion of the magnetic moment of the local-impurity and conduction-electron system in disordered dilute magnetic alloy. A similar pair of equations is displayed in Eqs. (4.10)–(4.11). Hasegawa discussed two possibilities for the "destination vectors" which appear in relaxation terms of these equations: relaxation to the thermal equilibrium value of the magnetization appropriate to first, the average internal field, and second, the instantaneous internal field. Cottet *et al.*[2] later provided arguments in support of the latter form for the destination vectors. These authors also showed that for positive absorption, there must exist a so-called "detailed-balance" condition relating the conduction-electron and the local-moment relaxation rates and their unenhanced susceptibilities.

There have been several microscopic calculations of the various quantities involved in these Bloch equations, and some calculations of the coupled dynamic sysceptibility in attempts to validate their general form.

Starting with the s-d Hamiltonian, and introducing the electron-lattice and local-moment–lattice relaxation in a phenomenological way, Orbach and Spencer[3] used the decoupling method and obtained a denominator correct to the second order in the exchange interaction constant J, which is consistent with either of the destination fields considered by Hasegawa. Very recently, using the method of Kadanoff and Baym, Langreth and Wilkins[4] synthesized the Bloch equations in the form proposed by Cottet *et al.*[2] They limited themselves to equal g factors and high temperatures and ignored the frequency dependence of the relaxation times. Götze and Wölfe[5] used a method which *assumes* that the susceptibility is of the form of a certain regular analytic anzatz, and obtained results which are also consistent with either type of the destination vectors. Sasada and Hasegawa[6] begin with the

Anderson model[7] and used diagrammatic methods. They conclude that the unenhanced average equilibrium magnetizations are the correct destination vectors.

The present calculation uses the *s-d* Hamiltonian and spin-orbit scattering to induce electron-lattice relaxation. Feynman diagram techniques[8] are used, and a fermion representation[9] of the spin operators is introduced in order that the fermion Wick theorem may be used. The mathematical and manipulative methods developed in this paper enable the analysis to be performed for arbitrary temperatures. The susceptibility (a two-particle propagator) is calculated by solving a pair of coupled vertex equations which are written in terms of dressed single-particle propagators. The results are obtained in a straightforward way and display the characteristics expected from the two particle quantities. For example, the arguments of the single-particle self-energies which appear in the susceptibility can be simply interpreted in terms of the energy conservation requirement for the two particle processes to which they correspond. Inhomogeneous terms appear, which are appropriate to the propagators associated with the intermediate states involved in the self-energies. With this method, there is an exact correspondence between the spin-flip scattering amplitudes calculated by Izuyama *et al.*[10] and the corresponding spin-flip local-moment self-energies calculated by the present methods.

All relevant two-particle quantities appropriate to the evaluation of the dynamic susceptibility are calculated to the second order in the interaction parameters. The cross-magnetization relaxation terms in Hasegawa equations[1] arise from the "scattering-in" terms in the coupled vertex equation. They are treated on equal footing with the "scattering-out" relaxation terms, arising from the true self-energies. The detailed-balance condition is discussed and shown to be dynamic in nature. It is weighted by the g factor ratio squared, which reflects the fact that it is the spin rather than magnetization which is conserved. The results for the dynamic susceptibility are reduced to, and compared with, the Bloch equations, and also with the existing microscopic theories. Since the present methods of solving the coupled vertex equations are believed to be new, they will be described in some detail in the text and appendices.

Besides the known frequency dependence of the relaxation rates, a frequency modification of the susceptibilities in the destination vectors is found, and a momentum dependence of the two-particle electronic self-energies arises. The magnetizations are found to relax towards their thermal equilibrium values in the internal instantaneous local field. At high temperatures ($kT < \omega_s$, ω_0, the resonant and external frequencies), or, roughly speaking, for the large spin, the momentum dependence of the conduction-electron self-energies is weak, and the two coupled equations which determine the dynamic susceptibility are obtained. Apart from the Kondo logarithmic terms and the frequency dependence mentioned above, these equations are equivalent to the phenomenological Bloch equations in the form displayed by Cottet *et al.*,[2] except that the ratios of g factors should appear in front of the cross magnetization relaxation rates, demonstrating that it is the spin rather than magnetization which is involved in this transfer. This is why Cottet *et al.* obtain susceptibilities $\chi_e^{+-}(0)$ and $\chi_s^{+-}(0)$ which do not reduce to their real static values, as they should.

At low temperatures ($kT \ll \omega_s$, ω_0) and for spins of small magnitude, it is not found possible to write a single equation for the *total* electronic susceptibility because of the strong momentum dependence of the conduction-electron relaxation, although an equation may be written which involves that part of the susceptibility which comes from the conduction electrons with a given momentum. In the present calculation where the interaction parameter in the *s-d* Hamiltonian is assumed to be momentum independent, this momentum dependence arises solely through the arguments of the single-particle self-energies (two-particle self-energies are of the form of the single-particle self-energies with definite arguments), and for this reason it *is* possible to write a closed form for the total susceptibility in terms of appropriate integrals.

Although it would be possible, using the methods presented in this paper, to evaluate exactly the single-particle self-energies and similar quantities involved in the evaluation of the vertices to any order in perturbation theory by using the dressed propagators, the authors make the approximation of using undressed propagators. This is justified for sufficiently high temperatures (the quasiparticle regime), where kT is much greater than the single-particle widths. In the opposite limit, the self-energies should be evaluated self-consistently, since the self-energies must be written in terms of propagators which contain the self-energy itself (a formidable task, especially when the self-energies are frequency dependent).

The plan of the paper is as follows. In Sec. II the model Hamiltonian, the method of representing the local-moment spin operators in terms of fermions and the method of performing the configuration averages are described, and then the resulting additional rules for evaluating Feynman diagrams are introduced. Section III contains the calculation of the dynamic transverse susceptibility, and in Sec. IV the approximation that the momentum dependence of the conduction-electron self-energies is weak is made, and a pair of coupled linear equations are obtained, which determine the transverse susceptibility. These equations are then related to the Bloch equations. In Sec. V the so-called detailed-balance relationship is discussed and a new relationship between the real second-order self-energies is obtained. Finally, in Sec. VI some additional discussion is given and the present results are compared with those of previous calculations.

II. MODEL AND FORMULATION

The s-d Hamiltonian is defined by

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{es}\ , \tag{2.1}$$

where $\mathcal{H}_0$ represents the noninteracting Hamiltonian and $\mathcal{H}_{es}$ the electron–localized-spin exchange coupling. The Hamiltonian $\mathcal{H}_0$ is given by

$$\mathcal{H}_0=\omega_s\sum_j S_j^z+\sum_{\vec{p},\sigma}\epsilon_{\vec{p},\sigma}a^\dagger_{\vec{p},\sigma}a_{\vec{p},\sigma}\ , \tag{2.2}$$

where $\omega_s=g_s\mu_BH_0$, the $\epsilon_{\vec{p},\sigma}$ are the one-electron self-energies,

$$\epsilon_{\vec{p},\sigma}=\epsilon_{\vec{p}}+\tfrac{1}{2}\sigma\omega_e\ ,\quad \omega_e=g_e\mu_BH_0\ ,$$

and where H_0 is a static magnetic field applied in the z direction. $a^\dagger_{\vec{p},\sigma}$ and $a_{\vec{p},\sigma}$ are the usual electron creation and destruction operators for electron with momentum $\vec{p}$ and spin σ (we have not included band indices, σ may be the arrows $\uparrow$ or $\downarrow$ or numerically ± 1 for the electron spin "up" or "down"). S_j^z is the usual spin matrix for the spin moment at the jth site. The electrons will be assumed to be "free-electron-like," except when numerically evaluating self-energies and thermal averages. Then we shall assume that the electrons occupy a half-filled square band of the total width $2D$.

The exchange Hamiltonian is given by

$$\mathcal{H}_{es}=-(J/N)\sum_{\vec{q},j}e^{-i\vec{q}\cdot\vec{R}_j}\vec{S}_j\cdot\vec{\sigma}_{\vec{q}}\ , \tag{2.3a}$$

with

$$\vec{\sigma}_{\vec{q}}=\sum_{p,\sigma,\sigma'}a^\dagger_{\vec{p}+\vec{q},\sigma}\vec{\sigma}_{\sigma,\sigma'}a_{\vec{q},\sigma'}\ , \tag{2.3b}$$

where the $\vec{\sigma}_{\sigma\sigma'}$ are the Pauli spin matrices for $s=\frac{1}{2}$ and N is the number of lattice sites.

The interaction has been taken to be a δ function in coordinate space. First, because the interaction is believed to be short range in nature, and second because it considerably simplifies the calculations. It will be indicated where a $\vec{q}$ dependence of J would be important.

It is also desired to investigate in what way the electron-lattice coupling enters our results. This consists of a spin-independent interaction, the so-called impurity potential scattering, and the spin-dependent interaction. It is well known, that the potential scattering does not directly contribute to the relaxation rates for the magnetization. However, it can have certain interference effects. It is currently popular to take the spin-orbit scattering as the source of the electron-lattice relaxation. The exact form of this coupling brings mathematical complications, especially if the magnitudes of the potential and the spin-orbit scattering were comparable. Therefore, a simple rotationally invariant Hamiltonian will be included here as a model Hamiltonian for the spin-dependent part of the electron-lattice interaction.

The electron-lattice Hamiltonian is then written

$$\mathcal{H}_i+\mathcal{H}_{el}=-(A/N)\sum_{i,\vec{q},\sigma}a^\dagger_{\vec{k}+\vec{q},\sigma}a_{\vec{k},\sigma}e^{-i\vec{q}\cdot\vec{R}_i}-(H/N)\sum_{i,\vec{q}}\vec{h}_i\cdot\vec{\sigma}_{\vec{q}}e^{-i\vec{q}\cdot\vec{R}_i}\ , \tag{2.4}$$

where A is the magnitude of the potential scattering, and the vector $\vec{h}_i$ is a dimensionless random variable associated with a scattering center i, which is assumed to be in a thermal equilibrium with the lattice. The averages $\langle(h^\alpha)^n\rangle$ are defined by

$$\langle(h^\alpha)^n\rangle=\int_{-\infty}^{+\infty}(h^\alpha)^n p(h^\alpha)\,dh^\alpha\ , \tag{2.5}$$

where the distribution function $p(h^\alpha)$ is normalized to unity. The averages $\langle h^\alpha\rangle=0$ for $\alpha=x,\ y,\ z$. Because of its rotational symmetry, the vectorial field $\vec{h}_i$ adequately represents the s-wave component of the scattering that is due to the spin-orbit interaction in how it invokes the relaxation effects in the present problem. Furthermore, large-impurity-potential scattering redistributes vectors of the electronic momentum over the Fermi sphere, surpressing the directionally dependent components of the spin-orbit scattering. This is a common assumption in most relaxation theories.[11] For the present purposes, the two scattering Hamiltonians, i.e., the $\mathcal{H}_{el}$ in Eq. (2.4) and the true spin-orbit interaction Hamiltonian, are then fully equivalent, as is shown in Appendix D. The form of the model Hamiltonian [Eq. (2.4)] distinguishes longitudinal and spin-flip contributions to the relaxation rate, and furthermore is mathematically simple.

The field-theoretical methods[8] involving Feynman diagrams will be used. Such methods are dependent on the availablity of a Wick-type theorem for spin operators. A fermion representation of the localized moment operators[9] is used here and the fermion Wick theorem. In this paper only $S=\frac{1}{2}$ will be considered. An extension for $S>\frac{1}{2}$ is straightforward and will be presented elsewhere.[12]

The spin operators are defined in terms of fermions by

$$\vec{S}_i=\sum_{s,s'}c^\dagger_{is}\vec{S}_{ss'}c_{is'}\ , \tag{2.6}$$

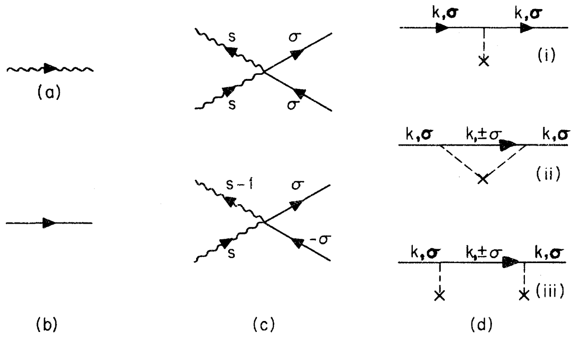

FIG. 1. (a) Propagator $C_s^0\ (i\nu)$; (b) propagator $G^0_{p,\sigma}\ (i\nu)$; (c) e-s vertices; (d) conduction-electron–lattice vertices.

where $\vec{S}_{ss'}$ are the Pauli spin matrices.

The fermion operators obey the usual anticommutation relations. We shall refer to these fermions as spin fermions to distinguish them from the conduction electrons. This representation of the spin operators obeys the spin commutation rules between the states $|n_\uparrow = 1, n_\downarrow = 0\rangle$ and $|n_\uparrow = 0, n_\downarrow = 1\rangle$, corresponding to the spin states $|S_z = \frac{1}{2}\rangle$ and $|S_z = -\frac{1}{2}\rangle$, respectively. The unphysical states $|n_\uparrow = 0, n_\downarrow = 0\rangle$ and $|n_\uparrow = 1, n_\downarrow = 1\rangle$ have no spin equivalents. However, only thermal averages of combinations of the spin operators $[\langle S_i S_j \cdots\rangle = (Z_s)^{-1}\,\mathrm{Tr}(e^{-\beta\mathcal{H}} S_i S_j \cdots)$, where Z_s is the true spin partition function] will be calculated. Spin operators have no matrix elements between these unphysical states. Thus, the only mistake in taking the thermal average with respect to the spin-fermion states, as compared to that one taken in the spin space is in that the partition functions are different, which is easily corrected for by multiplying by the ratio of the partition functions.

The two unperturbed propagators associated with the spin fermions are $\langle T c_s^\dagger(\tau) c_s\rangle$; $s(s = \pm 1)$ is the spin index obeying the same convention as the electron-spin indices. In complex frequency space these are

$$C_s^0(i\nu) = (\tfrac{1}{2}s\omega_s - i\nu)^{-1}, \quad S = \pm 1. \qquad (2.7)$$

The superscript indicates that this is the unperturbed propagator. $i\nu$ is the conventional imaginary fermionlike complex frequency. Diagrammatically, they are represented by directed wiggly lines shown in Fig. 1(a).

The corresponding electron propagators $\langle T a^\dagger_{\vec{p}\sigma}(\tau) a_{\vec{p}\sigma}\rangle$, $\sigma = \pm 1$ in frequency space are

$$G^0_{\vec{p},\sigma}(i\nu) = (\epsilon_{p,\sigma} - i\nu)^{-1}, \quad \sigma = \pm 1, \qquad (2.8)$$

and diagrammatically represented by a straight directed line in Fig. 1(b).

There are two types of electron-spin vertices [Fig. 1(c)] implied by the $\mathcal{H}_{es}$. The longitudinal one has a factor $\frac{1}{2}\sigma s(J/N)$ connected with it, whereas the transversal one has a factor (J/N). The electron-impurity vertices arising from $\mathcal{H}_{el}$ are illustrated in Fig. 1(d).

The representation of the spin operators as formulated so far is not a convenient one. Consider a system which contains a single local moment; then the usual definitions of a self-energy apply. Consider the spin self-energy shown in Fig. 2(a). Evaluating the propagator $C_s(i\nu)$ using just this self-energy, we obtain

$$C_s(i\nu) = \{(\tfrac{1}{2}s\omega_s - i\nu) - [I^2/(\tfrac{1}{2}s\omega_s - i\nu)] \times \sum_{s'} f^+(\tfrac{1}{2}s'\omega_s) f^-(\tfrac{1}{2}s'\omega_s)\}^{-1} \qquad (2.9)$$

where

$$I = (J/N)^2\, \mathcal{P} \sum_{p,p',\sigma} \frac{f^+_{p,\sigma} - f^+_{p',\sigma}}{\epsilon_{p,\sigma} - \epsilon_{p',\sigma}},$$

the Fermi factors $f^+_{p,\sigma} = (1 + e^{\beta\epsilon_{p,\sigma}})^{-1}$, $f^+(x) = (1 + e^{\pm\beta x})^{-1}$, and $\mathcal{P}$ means the principal part.

Thus $C_s(i\nu)$ has *two* poles, and therefore so does the transverse susceptibility $\chi^{+-}(i\nu)$. On physical grounds, this seems unreasonable for $S = \frac{1}{2}$. This and similar terms arise from the presence of the unphysical states with the total fermion occupation number at a single site bigger than one as intermediate states, as is implied by inspection of relevant diagrams. These exist because we choose to work with the one-particle propagators, and, as a consequence of the diagrammatic methods, at intermediate steps of the calculations the expectation values of operators which are not spin operators can be present, and hence contributions from the unphysical states can arise.

Some other problems may also arise. The local-moment susceptibility for a single site is corrected by only a *single* factor of the ratio of the unphysical to physical partition function and this appears in the susceptibility numerator. Consider the self-energy diagram of Fig. 2(b). This and similar types of diagrams, involving only a single site, yield a contribution proportional to $\sum_{s'} s' f^+(\frac{1}{2}s'\omega_s)$. For a system with only a single site this quantity repre-

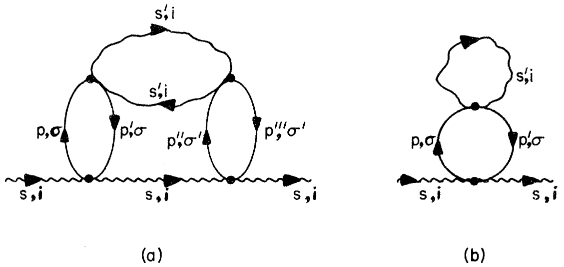

FIG. 2. Unphysical contributions (a) a fourth order, (b) a second order.

sents a self-energy contribution, and since it will then appear in the denominator of the susceptibility, it must represent a physical spin expectation value. However, it does not represent any physical spin expectation value. Therefore, it is not possible to write the total susceptibility in terms of only spin expectation values as one must be able to do for obvious physical reasons.

One can remove the above mentioned difficulties by making the contributions to the thermal averages involving the unphysical states indefinitely small. To do this, one adds to the $\mathcal{H}_0$ a projection Hamiltonian $\mathcal{H}_p$,

$$\mathcal{H}_p = \lambda \sum_i (c^\dagger_{i\uparrow} c_{i\uparrow} + c^\dagger_{i\downarrow} c_{i\downarrow}) , \tag{2.10}$$

where λ is a constant much greater than kT. With λ positive, a factor $e^{-n\beta\lambda}$ will appear in that contribution to the trace of the spin-fermion operators which involves a state with a fermion occupation number n. The unphysical partition function then becomes unity. The propagator $C_s(i\nu)$ is modified:

$$C^0_s(i\nu) = (\lambda + \tfrac{1}{2} s\omega_s - i\nu)^{-1} , \quad s = \pm 1 . \tag{2.11}$$

Both of the unphysical "self-energy-like" contributions of Fig. 2 are now proportional to $e^{-\beta\lambda}$ and therefore do not contribute. By inspection one finds that any diagram with more than one spin-fermion loop at the same site does not contribute. Hence, in what follows, these types of diagrams will be discarded. It is believed that such troublesome self-energies are removed to all orders, typical topologies up to J^4 order have been explicitly examined.

In the usual diagrammatic methods for evaluation of correlation functions, the "vacuum-polarization diagrams" cancel the partition function (appropriate to the full Hamiltonian) which appears in the thermal weighting. Owing to the above restriction upon the number of spin-fermion loops, this complete cancellation no longer is obtained. In order to correct the thermal averages, to the first order in the concentration, the following rule is found: include a factor $e^{\beta\lambda}/Z_s$ for each distinct lattice site, where Z_s is the full many-body partition function for a single local-moment site, divided by the unperturbed conduction-electron partition function [the unperturbed $Z_s = Z^0_s \equiv (e^{-\beta\omega_s/2} + e^{\beta\omega_s/2})^{-1}$]. The Hamiltonian $\mathcal{H}_p$ thus projects out the unphysical states with the fermion occupation number n greater than one.

At finite concentrations one must sum the Feynman diagrams over all possible local-moment-site labels. To facilitate this, label each spin-fermion loop with a site index, discard all diagrams with two or more identical site labels. Multiply each diagram by an additional factor

$$\eta_1 \eta_2 \eta_3 \cdots \eta_{n_s} (e^{\lambda\beta}/Z_s)^{n_s} , \tag{2.12}$$

where the η_i are unity if the ith lattice site is occupied by a local moment and zero otherwise, and where n_s is the number of spin-fermion loops involved in the diagram.

One now performs the sum over the site indices. Consider, as an example Fig. 3(a)(i), where i and j are the local-moment-site indices. The value of the diagram is

$$\sum_{i, j \neq i} \sum_{k'} \langle S_z \rangle^2 \eta_i \eta_j (J/N)^2 e^{-i(\vec{k}-\vec{k}')\cdot(\vec{R}_i - \vec{R}_j)} (\epsilon_{k',\sigma} - i\nu)^{-1} , \tag{2.13}$$

where i, j run over all lattice sites. Even in the absence of the restriction $i \neq j$, the sum over i and j *would not* produce a δ function, since η_i and η_j are random variables. The easiest way to bring about the necessary translational invariance is to perform a "configuration average." This method is well known in the theory of the electronic structure of highly disordered alloys and involves an averaging of a given diagram over all possible arrangements of the occupied local-moment sites. It can be shown that this average equals the sum over all i and j in the limit of infinite number of the occupied sites (that is for an infinite system of finite concentration). This averaging makes the sum over lattice sites much easier to perform.

The only quantity in Eq. (2.13) which is dependent on the configuration is the product $\eta_i \eta_j$. Their configuration average is

$$\langle \eta_i \eta_j \rangle = \langle \eta_i \rangle \langle \eta_j \rangle = c^2 , \quad i \neq j \tag{2.14a}$$

$$\langle \eta_i \eta_j \rangle = \langle \eta_i \rangle = c , \quad i = j . \tag{2.14b}$$

Using the Eq. (2.14a), Eq. (2.13) becomes

$$\sum_{i, j \neq i} \sum_{k'} c^2 \langle S_z \rangle^2 (J/N)^2 e^{-i(\vec{k}-\vec{k}')\cdot(\vec{R}_i - \vec{R}_j)} (\epsilon_{k',\sigma} - i\nu)^{-1} . \tag{2.15}$$

Now one can use explicit form of the δ function. Adding and subtracting the restriction ($i \neq j$), one

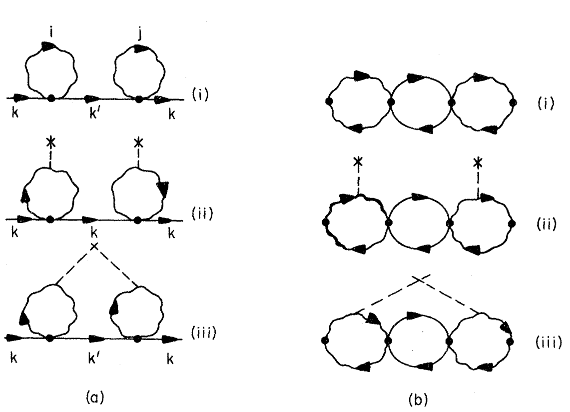

FIG. 3. Configuration averages: (a) (i) A second-order self-energy; (a) (ii) a second-order term due to the first-order self-energy; (a) (iii) second-order self-energy; (b) off-diagonal coupling.

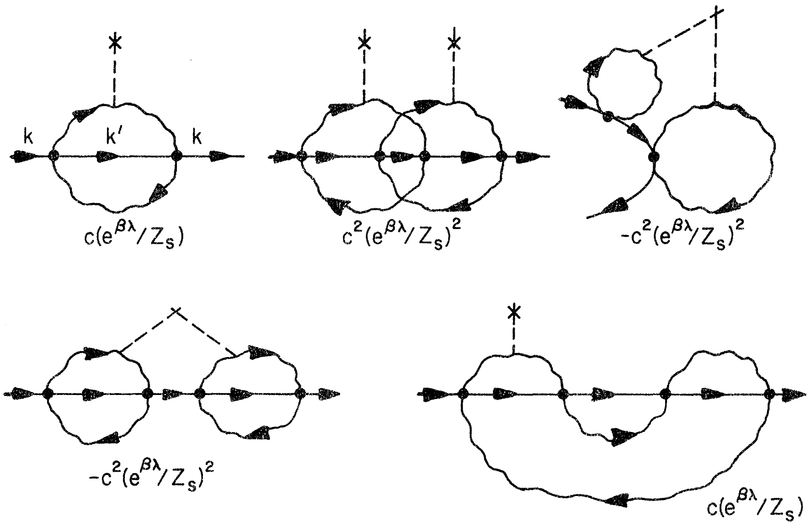

FIG. 4. Some examples of configuration averages.

obtains finally a value of the diagram 3(a)(i):

$$c^2J^2\langle S_z\rangle^2(\epsilon_{k,\sigma}-i\nu)^{-1}-c^2J^2\langle S_z\rangle^2\frac{1}{N}\sum_{k'}(\epsilon_{k',\sigma}-i\nu)^{-1}\ . \tag{2.16}$$

These two types of contribution will be represented by two separate diagrams. The first is illustrated in Fig. 3 (a)(ii) and the second in Fig. 3 (a)(iii). Notice that now they are not labeled by the local-moment-site indices. This convention for diagrams allows the usual definition of proper self-energy's part to be re-established: A diagram may not be reducable by breaking a dotted line. Thus the first term in Eq. (2.16) is second-order contribution appropriate to the first-order self-energy $cJ\langle S_z\rangle$ in the Dyson equation. The second term *does* represent a self-energy contribution.

This configuration average must be performed upon each diagram and not only self-energy diagrams. Thus Fig. 3(b)(i) yields the contributions denoted by Figs. 3(b)(ii) and 3(b)(iii). One can now formulate the additional final rule necessary to evaluate diagrams involving spin-fermion loops.

The configuration-average rule. With each diagram with n_s spin-fermion loops, associate a number of crosses (less or equal to n_s). Join each loop with a single dotted line to a single cross, sum over all possible numbers of crosses and arrangements. Do not label them by site indices. Discard arrangements in which any cross has no dashed line attached to it. Associate an additional factor $C_{n_i}(-1)^{n_i+1}(cNe^{\beta\lambda}/Z_s)^{n_i}$ with each cross, where n_i is the number of the fermion loops attached to the cross i, $n_s=\sum_i n_i$, and C_{n_i} is a constant, $C_1=1$, $C_2=1$. The total momentum change at the vertices of the loops attached to each cross must be zero. Allow λ to become indefinitely large. Typical types of diagrams and the additional factors are illustrated in Fig. 4.

This configuration average has a restriction that no two site indices can be the same. There is another type of configuration average which enters the problem, associated with the vertices that are due to the $\mathcal{H}_i$ and $\mathcal{H}_{\text{el}}$. It is of the type encountered in the electronic-band-structure problem, and has no such restriction on the site indices. In our convention, the configuration average yields the value

$$(c'\langle h_z^2\rangle-c'^2\langle h_z\rangle^2)(H^2/N)\sum_{k'}(\epsilon_{k',\sigma}-i\nu)^{-1}$$

for the longitudinal contribution of Fig. 1(d)(ii) and $(\sigma c')^2\langle h_z\rangle^2(\epsilon_{k\uparrow}-i\nu)^{-1}$ will obtain for the longitudinal contribution of Fig. 1(d)(iii). The last contribution is again a second-order contribution appropriate to the first-order term in Fig. 1(d)(i); the 1(d)(ii) is a new self-energy contribution. The additional term here, $c'\langle h_z^2\rangle$, arises because of the absence of the restriction mentioned above. The factors introduced by the configuration averages for diagrams which appear in the vertex equations will be similar to the factors obtained for these self-energy-type diagrams.

III. CALCULATION OF DYNAMIC RESPONSE

The transverse susceptibility $\chi^{+-}(t)$ is equal to a retarded response function $i\theta(t)(1/N)\langle M^+(t), M^-(0)\rangle$, where $M^\pm=g_s\mu_B\sum_i S_i^\pm+\frac{1}{2}g_e\mu_B\sigma^\pm_{\vec{q}=0}$, $M^\pm(t)$ is the Heisenberg operator. The Fourier transform $\chi^{+-}(\omega_0)$ of this response function is equal to the analytic continuation of the imaginary frequency transform of the corresponding temperature-ordered function $(1/N)\langle TM^+(u)M^-(0)\rangle$.

The present diagrammatic calculation will make use of the one-fermion propagators. The degree of accuracy will be $(J)^2$.

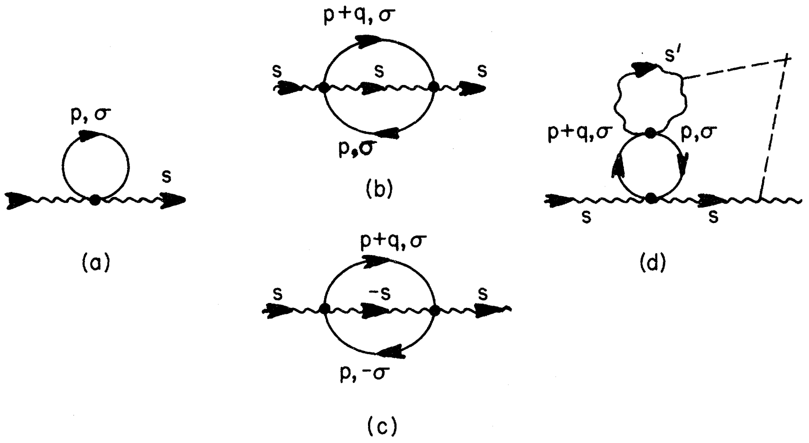

FIG. 5. Spin self-energies: (a) first order; (b) second-order longitudinal; (c) second-order spin flip; (d) concentration-dependent second order.

A. Single-Particle Propagators

The appropriate self energies are evaluated up to the second order in J and at least to the lowest order in the concentrations of both localized moments and the lattice scattering sites for the electrons. The local spin-fermion self-energies are illustrated in Figs. 5(a)–5(d). The corresponding algebraic expressions are, using "bare" propagators,

$$\Sigma^{(1)}_{\mathrm{se},\alpha}=\alpha(J/2N)\sum_{\sigma,p}\sigma\,\langle n_{p,\sigma}\rangle=\alpha(\tfrac{1}{2}J)\sum_{\sigma}\sigma\langle n_{e,\sigma}\rangle\,, \tag{3.1a}$$

$$\Sigma^{(2l)}_{\mathrm{se},\alpha}(i\nu)=\left(\frac{J}{2N}\right)^2\sum_{p,q,\sigma}\frac{f^+(\lambda+\frac{1}{2}\alpha\omega_s)f^+_{p+q,\sigma}f^-_{p,\sigma}+f^-(\lambda+\frac{1}{2}\alpha\omega_s)f^-_{p+q,\sigma}f^+_{p,\sigma}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\alpha\frac{1}{2}\omega_s-i\nu}\quad, \tag{3.1b}$$

$$\Sigma^{(2t)}_{\mathrm{se},\alpha}(i\nu)=\left(\frac{J}{N}\right)^2\sum_{p,q}\frac{f^+(\lambda-\frac{1}{2}\alpha\omega_s)f^+_{p+q,\alpha}f^-_{p,-\alpha}+f^-(\lambda-\frac{1}{2}\alpha\omega_s)f^-_{p+q,\alpha}f^+_{p,-\alpha}}{\epsilon_{p+q,\alpha}-\epsilon_{p,-\alpha}+\lambda-\frac{1}{2}\alpha\omega_s-i\nu}\quad, \tag{3.1c}$$

$$\Sigma^{(2l,c)}_{\mathrm{se},\alpha}=c\left(\frac{J}{N}\right)^2\left(-\frac{\alpha}{2}\right)\langle S_z\rangle\sum_{p,q,\sigma}\frac{f^+_{p+q,\sigma}-f^+_{p,\sigma}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}}\quad. \tag{3.1d}$$

In Eq. (3.1), $\alpha=+,-$ for the local spin ↑, ↓, respectively. The superscripts 1, 2 designate the power of J, l and t are the longitudinal and transverse character of the contributions, and c is the local-spin's concentration. All other symbols were defined in Sec. II.

The algebraic expressions for the electronic

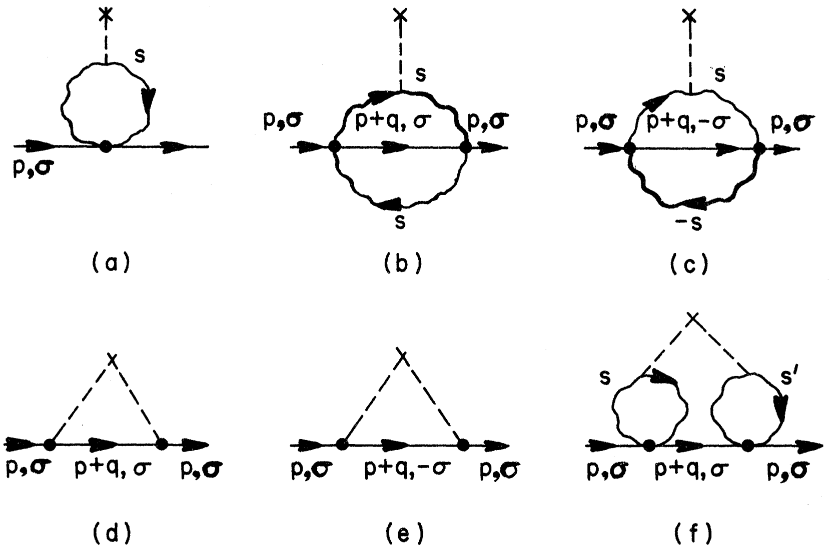

FIG. 6. Electronic self-energies: (a) first order; (b) second-order longitudinal; (c) second-order spin flip; (d) potential scattering; (e) spin lattice; (f) $(c)^2$ dependent.

self-energies illustrated in Fig. 6 are

$$\Sigma^{(1)}_{\mathrm{es},\sigma} = c(\tfrac{1}{2}\sigma)\, J(e^{\beta\lambda}/Z_s) \sum_{\nu,\alpha} \alpha\, C_\alpha(i\nu) = c\sigma(\tfrac{1}{2}J)\langle n_{s\uparrow} - n_{s\downarrow}\rangle \,, \tag{3.2a}$$

$$\Sigma^{(2l)}_{\mathrm{es},\sigma}(i\nu) = c(J^2/N)\,\langle S_z^2\rangle \sum_q (\epsilon_{p+q,\sigma} - i\nu)^{-1} \,, \tag{3.2b}$$

$$\Sigma^{(2t)}_{\mathrm{es},\sigma}(i\nu) = c\left(\frac{J^2}{N}\right)\left(\frac{1}{2}\sum_q \frac{1}{\epsilon_{p+q,-\sigma} + \sigma\omega_s - i\nu} + \sigma\,\langle S_z\rangle \sum_q \frac{f^{+}_{p+q,-\sigma} - f^{-}_{p+q,-\sigma}}{\epsilon_{p+q,-\sigma} + \sigma\omega_s - i\nu}\right) \,, \tag{3.2c}$$

$$\Sigma_{\mathrm{ei},\sigma}(i\nu) = c'(A^2/N) \sum_q (\epsilon_{p+q,\sigma} - i\nu)^{-1} \,, \tag{3.2d}$$

$$\Sigma^{(2l)}_{\mathrm{el},\sigma}(i\nu) = c'(H^2/N)\,(\langle h_z^2\rangle - c'\,\langle h_z\rangle^2) \sum_q (\epsilon_{p+q,\sigma} - i\nu)^{-1} \,, \tag{3.2e}$$

$$\Sigma^{(2t)}_{\mathrm{el},\sigma}(i\nu) = c'(H^2/N)\,(\langle h_x^2\rangle + \langle h_y^2\rangle) \sum_q (\epsilon_{p+q,-\sigma} - i\nu)^{-1} \,, \tag{3.2f}$$

$$\Sigma^{(2l,c)}_{\mathrm{es},\sigma}(i\nu) = c^2(J/N)^2\,\langle S_z\rangle^2 \sum_q (\epsilon_{p+q,\sigma} - i\nu)^{-1} \,. \tag{3.2g}$$

The self-energies (3.1) are as yet unprojected, but the appropriate projection factors in the sense of Sec. II have been used in Eq. (3.2). Using one of several methods[3,13] it is possible to show that the factor in front of the first term in (3.2c) is in fact the spin $S=\frac{1}{2}$ value for $(\langle S_x^2\rangle + \langle S_y^2\rangle)$. It has been assumed that J is not $\vec{q}$ dependent and hence the electronic self-energies are not momentum dependent. If this is not the case, J must be replaced by $J(\vec{q})$ and $\vec{p}$ will label the now momentum-dependent self-energies.

In what follows, $\Sigma^{(2)}_{\mathrm{se},\alpha}(i\nu)$ will be defined as the sum of the longitudinal and the spin-flip contributions (second order in J) to the self-energy of the spin fermion with the spin label α. $\Sigma_{\mathrm{se},\alpha}(i\nu)$ sums both first and second order in J contributions. The self-energies (3.1d) and (3.2g) are not of the lowest order in the concentration, and hence will be omitted until the end of this section.

B. Dynamical Response

For $S=\frac{1}{2}$, there are two types of fully dressed coupled vertices appropriate to the evaluation of the transverse susceptibility, the localized-spin and conduction-electron-like vertices, illustrated in Figs. 7(a) and 7(b), respectively. In Fig. 7(a), the full vertex on the left-hand side represents the sum of all diagrams which begin with a pair of incoming spin-fermion lines, and end with unspecified external-field vertex. Closing the vertex, and multiplying by the factor $(g_s/\sqrt{2})(1/N)$ appropriate for the incoming spin-fermion external-field vertex, and then summing over the internal frequency $i\nu$, yields the spin susceptibility $\chi_s^{+-}(i\nu_0) = \chi_{ss}^{+-}(i\nu_0) + \chi_{se}^{+-}(i\nu_0)$. For $S>\frac{1}{2}$, the factors would also contain the matrix elements appropriate to the coupling to the external rf field. The first term $\chi_{ss}^{+-}(i\nu_0)$ represents sum of all diagrams coupled to the external rf field by the local-spin–external-field vertices on both ends; the latter has the last external vertex of the electronic type. The $\chi_e^{+-}(i\nu_0) = \chi_{ee}^{+-}(i\nu_0) + \chi_{ee}^{+-}(i\nu_0)$ is obtained in a similar way from the full vertex of Fig. 7(b), replacing g_s by g_e and summing over the additional electronic momentum label $\vec{p}$.

On the right-hand side of Fig. 7(a), the first term is the simple outgoing vertex. The second term represents vertex corrections to the full vertex. Up to the second order in J and the lowest order in the concentration, there are only longitudinal-like vertex corrections, none of the spin-flip type. The third and the last two terms describe off-diagonal couplings (i.e., coupling of the local-spin–conduction-electron bubbles in the vertex equations), which transfer spin excitation in the coupled system. The third term represents a direct, spin-flip coupling of the first order in J and the last two terms represent a transfer via intermediate states in the second order in J. The latter coupling enters the vertex equation through a pair of exchange vertices, one of them always longitudinal, the other one transverse.

The terms in the electronic vertex in Fig. 7(b) are interpreted in a similar way; the vertex-correction diagram represents the effect of correlated scattering of the particle-hole pair off the same local-moment site; and the last term in

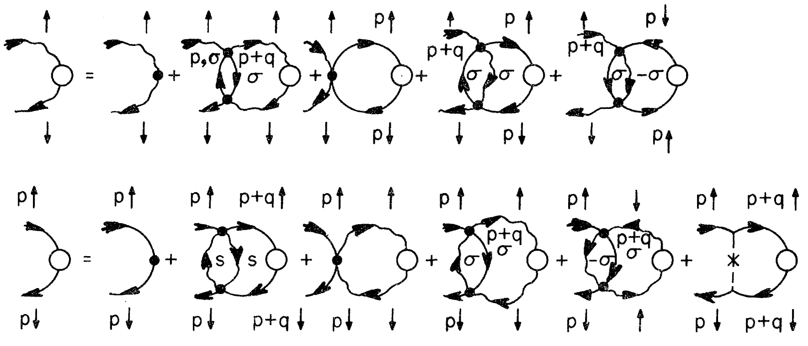

FIG. 7. Vertex equation for the coupled susceptibilities.

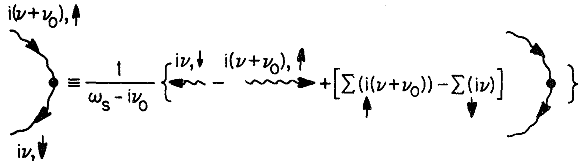

FIG. 8. Identity of the partial fraction.

Fig. 7(b) a similar vertex correction for both the impurity scattering, and the scattering from the inhomogeneous vector field describing electron-spin–lattice interaction that is due to the Hamiltonian $\mathcal{H}_{\mathrm{el}}$. In Figs. 7(a) and 7(b) all propagators are dressed by their self-energies, and all diagrams have been included up to the second order in the interaction parameters and to the lowest order in the concentration. Formally, to this degree of accuracy, any vertex correction or off-diagonal coupling diagram can be associated with a corresponding one-particle self-energy by comparing the propagators, cut by intersecting the central part of the diagram.

The dynamic susceptibility is obtained by solving these coupled vertex equations. The susceptibility is a two-particle response function and should be given in terms of quantities which display certain two-particle properties. For example, two-particle character of the self-energies associated with the physical processes involved can be inferred by the examination of the denominators of these quantities.[14] It is also desirable to treat the contributions which arise from the vertices and the self-energies upon an equal footing.

To perform the sum over the internal frequencies and momenta, a few algebraic rearrangements will be used which transform terms of Figs. 7(a) and 7(b) into a more convenient form. The identity illustrated in Fig. 8,

$$C_\uparrow(i(\nu+\nu_0))\, C_\downarrow(i\nu) = (\omega_s - i\nu_0)^{-1}\{C_\downarrow(i\nu) - C_\uparrow(i(\nu+\nu_0)) + (\Sigma_{se\uparrow}(i(\nu+\nu_0)) - \Sigma_{se\downarrow}(i\nu))C_\uparrow(i(\nu+\nu_0))C_\downarrow(i\nu)\} \tag{3.3}$$

is used for the product of all incoming pairs of the dressed Green functions (both fermion-spin or electronic) in all terms of the diagrammatic equations [Figs. 7(a) and 7(b)]. Using the algebraic rearrangement of the various terms, as shown diagrammatically in Fig. 9 and algebraically in Appendix A, it is possible to rearrange all terms in such a way that they may be written exactly solely in terms of the self-energy-like quantities. The following equations are then obtained:

$$\begin{aligned}(\omega_s - i\nu_0)\,\chi_s^{+-}(i\nu_0) &= (\omega_s - i\nu_0)\sum_\nu c(e^{\beta\lambda}/Z_s)\,(g_s/\sqrt{2}\,)\,\Lambda_s(i\nu,\, i(\nu+\nu_0))\\ &= \sum_\nu \Big((\tfrac{1}{2}g_s^2)\, c(e^{\beta\lambda}/Z_s)\,[C_\downarrow(i\nu) - C_\uparrow(i(\nu+\nu_0))] + (e^{\beta\lambda}/Z_s)[\Sigma_{se\uparrow}(i(\nu+\nu_0)) - \Sigma_{se\downarrow}(i\nu)](\tfrac{1}{2}cg_s)\Lambda_s(i\nu,\, i(\nu+\nu_0))\\ &\quad + (e^{\beta\lambda}/Z_s)\,[\Sigma_{se\uparrow}^{(2l)}(i(\nu+\nu_0)) - \Sigma_{se\downarrow}^{(2l)}(i\nu)]\,(\tfrac{1}{2}cg_s)\Lambda_s(i\nu,\, i(\nu+\nu_0))\\ &\quad - (g_s/g_e)\sum_p [\Sigma_{es\uparrow}(i(\nu+\nu_0)) + \Sigma_{es\uparrow}^{(2l)}(i(\nu+\nu_0)) - \Sigma_{es\downarrow}(i\nu) - \Sigma_{es\downarrow}^{(2l)}(i\nu)]\,(g_e/N\sqrt{2}\,)\Lambda_p(i\nu,\, i(\nu+\nu_0))\Big)\ ,\end{aligned} \tag{3.4}$$

where $\Lambda_s(i\nu,\ i(\nu+\nu_0))$ is the full vertex on the left-hand side of Fig. 7(a), and similarly, $\Lambda_p(i\nu,\ i(\nu+\nu_0))$ is an electron contribution to the full vertex of Fig. 7(b), labeled with the momentum p. Use has been made of the fact that the internal frequency ν involved in the summation is a dummy variable. For the electronic vertex one obtains

$$\begin{aligned}(\omega_e - i\nu_0)\chi_p^{+-}(i\nu_0) &= (\omega_e - i\nu_0)\sum_\nu (g_e/\sqrt{2}\,)\,\Lambda_p(i\nu,\, i(\nu+\nu_0)) = \sum_\nu \Big[\frac{g_e^2}{2}[G_{p\downarrow}(i\nu) - G_{p\uparrow}(i(\nu+\nu_0))] + [\Sigma_{es\uparrow}(i(\nu+\nu_0))\\ &\quad + \Sigma_{el\uparrow}(i(\nu+\nu_0)) + \Sigma_{ei\uparrow}(i(\nu+\nu_0)) - \Sigma_{es\downarrow}(i\nu) - \Sigma_{el\downarrow}(i\nu) - \Sigma_{ei\downarrow}(i\nu)]\frac{g_e}{\sqrt{2}}\,\Lambda_p(i\nu,\, i(\nu+\nu_0))\\ &\quad + \frac{1}{N}\,(cJ^2\langle S_z^2\rangle - c'A^2 + c'H^2\langle h_z^2\rangle)\left(\frac{1}{\epsilon_{p\uparrow} - i(\nu+\nu_0)} - \frac{1}{\epsilon_{p\downarrow} - i\nu}\right)\sum_{p'}\frac{g_e}{\sqrt{2}}\,\Lambda_{p'}(i\nu,\, i(\nu+\nu_0))\\ &\quad - \frac{g_e}{g_s}\,\frac{J}{N}\sum_\sigma \sigma\,\langle n_{p\sigma}\rangle\frac{cNg_s}{\sqrt{2}}\,\frac{e^{\beta\lambda}}{Z_s}\Lambda_s(i\nu,\, i(\nu+\nu_0)) - \frac{g_e}{g_s}\,\frac{e^{\beta\lambda}}{Z_s}\left(\frac{J}{N}\right)^2\frac{1}{2}\sum_q\left(\frac{f^+(\lambda+\frac{1}{2}\omega_s)f_{p\uparrow}^+ f_{q\uparrow}^- + f^-(\lambda+\frac{1}{2}\omega_s)f_{p\uparrow}^- f_{q\uparrow}^+}{\epsilon_{p\uparrow} - \epsilon_{q\uparrow} + \lambda + \frac{1}{2}\omega_s - i(\nu+\nu_0)}\right.\end{aligned}$$

$$+\frac{f^{+}(\lambda+\frac{1}{2}\omega_s)f^{+}_{q\downarrow}f^{-}_{p\downarrow}+f^{-}(\lambda+\frac{1}{2}\omega_s)f^{-}_{q\downarrow}f^{+}_{p\downarrow}}{\epsilon_{q\downarrow}-\epsilon_{p\downarrow}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}+\frac{f^{+}(\lambda-\frac{1}{2}\omega_s)f^{+}_{p\uparrow}f^{-}_{q\downarrow}+f^{-}(\lambda-\frac{1}{2}\omega_s)f^{-}_{p\uparrow}f^{+}_{q\downarrow}}{\epsilon_{p\uparrow}-\epsilon_{q\downarrow}+\lambda-\frac{1}{2}\omega_s-i(\nu+\nu_0)}$$

$$+\frac{f^{+}(\lambda-\frac{1}{2}\omega_s)f^{+}_{q\uparrow}f^{-}_{p\downarrow}+f^{-}(\lambda-\frac{1}{2}\omega_s)f^{-}_{q\uparrow}f^{+}_{p\downarrow}}{\epsilon_{q\uparrow}-\epsilon_{p\downarrow}+\lambda-\frac{1}{2}\omega_s-i(\nu+\nu_0)}+\frac{f^{+}(\lambda-\frac{1}{2}\omega_s)f^{+}_{p\downarrow}f^{-}_{q\downarrow}+f^{-}(\lambda-\frac{1}{2}\omega_s)f^{-}_{p\downarrow}f^{+}_{q\downarrow}}{\epsilon_{p\downarrow}-\epsilon_{q\downarrow}+\lambda-\frac{1}{2}\omega_s-i\nu}$$

$$-\frac{f^{+}(\lambda-\frac{1}{2}\omega_s)f^{+}_{q\uparrow}f^{-}_{p\uparrow}+f^{-}(\lambda-\frac{1}{2}\omega_s)f^{-}_{q\uparrow}f^{+}_{p\uparrow}}{\epsilon_{q\uparrow}-\epsilon_{p\uparrow}+\lambda-\frac{1}{2}\omega_s-i\nu}-\frac{f^{+}(\lambda+\frac{1}{2}\omega_s)f^{+}_{p\downarrow}f^{-}_{q\uparrow}+f^{-}(\lambda+\frac{1}{2}\omega_s)f^{-}_{p\downarrow}f^{+}_{q\uparrow}}{\epsilon_{p\downarrow}-\epsilon_{q\uparrow}+\lambda+\frac{1}{2}\omega_s-i\nu}$$

$$-\frac{f^{+}(\lambda+\frac{1}{2}\omega_s)f^{+}_{q\downarrow}f^{-}_{p\uparrow}+f^{-}(\lambda+\frac{1}{2}\omega_s)f^{-}_{q\downarrow}f^{+}_{p\uparrow}}{\epsilon_{q\downarrow}-\epsilon_{p\uparrow}+\lambda+\frac{1}{2}\omega_s-i\nu}\Bigg)\frac{cNg_s}{\sqrt{2}}\Lambda_s(i\nu,\,i(\nu+\nu_0))\Bigg]\ , \tag{3.5a}$$

$$\begin{aligned}(\omega_e-i\nu_0)\chi_e^{+-}(i\nu_0)&=(\omega_e-i\nu_0)\sum_{p,\nu}(g_e/N\sqrt{2})\Lambda_p(i\nu,\,i(\nu+\nu_0))=\sum_{\nu}\Bigg((g_e^2/2N)\sum_p[G_{p\downarrow}(i\nu)-G_{p\uparrow}(i(\nu+\nu_0))]\\&+\sum_p[\Sigma_{es\uparrow}(i(\nu+\nu_0))+\Sigma_{el\uparrow}(i(\nu+\nu_0))+\Sigma^{(2l)}_{es\uparrow}(i(\nu+\nu_0))+\Sigma^{(2l)}_{el\uparrow}(i(\nu+\nu_0))\\&-\Sigma_{es\downarrow}(i\nu)-\Sigma_{el\downarrow}(i\nu)-\Sigma^{(2l)}_{es\downarrow}(i\nu)-\Sigma^{(2l)}_{el\downarrow}(i\nu)](g_e/N\sqrt{2})\Lambda_p(i\nu,\,i(\nu+\nu_0))\\&-(g_e/g_s)(e^{\beta\lambda}/Z_s)[\Sigma_{se\uparrow}(i(\nu+\nu_0))+\Sigma^{(2l)}_{se\uparrow}(i(\nu+\nu_0))-\Sigma_{se\downarrow}(i\nu)-\Sigma^{(2l)}_{se\downarrow}(i\nu)](cg_s/\sqrt{2})\Lambda_s(i\nu,\,i(\nu+\nu_0))\Bigg)\ .\end{aligned} \tag{3.5b}$$

Equation (3.5b) is the sum over the p labels of Eq. (3.5a). The self-energies Σ_{es} or Σ_{se} will be assumed to contain both first- and second-order-in-J contributions, unless specified otherwise. $\chi_e^{+-}(i\nu_0)=(1/N)\sum_p\chi_p^{+-}(i\nu_0)$ is the electron susceptibility per lattice site.

In Eq. (3.4), the first inhomogeneous term is a contribution from the first simple outgoing vertex in the vertex equation in Fig. 7(a). It will give rise to a static magnetization in the numerator of the dynamic susceptibility. The second term has contributions from all of the vertices. The third term has arisen from the vertex correction diagram, and it adds,[15] for $S=\frac{1}{2}$, to the longitudinal one-fermion true self-energy which appears in the second terms because the exchange interaction is spin dependent. The last term is the result of the off-diagonal coupling. Its first part of the first order in J will give rise to the local-field terms in the Bloch equations. Note that the off-diagonal coupling of the second order in J contains the longitudinal self-energy doubled. Apart from the momentum dependence, the same description holds for the analogical terms in the electronic equations (3.5) plus the direct electron-lattice coupling terms. The self-energies that are due to the spin-independent potential scattering were cancelled by their corresponding vertex correction.

Thus, apart from the g-factor ratio, terms of the second order in J which are equal in magnitude appear in both the electron and local-moment equations. In the local-spin equation, the terms Σ_{se}'s (the one-fermion true self-energies and vertex correction) and the terms $(g_s/g_e)\Sigma_{es}$'s (the off-diagonal spin transfer) represent scattering-out and scattering-in processes for the local-moment magnetization. In the electronic equation, the reverse is true. However, it should be noted that the terms Σ_{es}'s in Eq. (3.4) are proportional to the product of two *different* matrix elements, for example to $\langle\frac{1}{2}|S^{+}|-\frac{1}{2}\rangle\langle\frac{1}{2}|S^{z}|\frac{1}{2}\rangle$, while the true electron self-energies in Eq. (3.5) are composed from terms proportional to the matrix elements squared, for example $|\langle\frac{1}{2}|S_z|\frac{1}{2}\rangle|^2$ as is expected from the "golden rule." This general algebraic equality is a consequence of the mathematical symmetry (rotational invariance) of the interaction Hamiltonian $\mathcal{H}_{es}$. Had not it been the case (e.g., $J_\perp\neq J_\parallel$) this

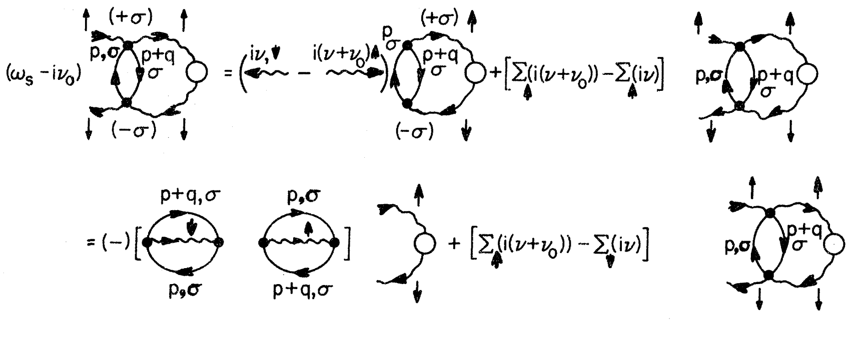

FIG. 9. Algebraic rearrangement.

equality would not have obtained.

The equations (3.4) and (3.5) are exact rearrangements of the original vertex equations in Figs. 7(a) and 7(b), provided that dressed propagators (by both the exchange self-energies and the lattice self-energies) are used for evaluation of the self-energy-like quantities, obtained by their rearrangements. This would require the self-consistent evaluation of these self-energies. Only the first-order self-energies, used in this paper, will be dressed. Evaluation of the second-order self-energies can be done to finite order in perturbation theory, by the methods presented in this section. However, this is very complicated and will not be considered here. In what follows, their undressed values given in Eqs. (3.1) and (3.2) will be used. This simplification is justified in the high-temperature limit where $kT > \omega_0$, ω_s, ω_e and the imaginary parts of the self-energies themselves; since the range of variations involved in their evaluation is kT, the self-energies must also be slowly varying over the thermal width kT. Generally, provided that the total one-fermion widths are smaller than kT, their frequency dependence remains the same as for the "bare ones." At low temperatures, the self-energies become frequency dependent and the complications connected with the self-consistent evaluation are beyond the scope of the present study.

The next step is to evaluate the terms in Eqs. (3.4) and (3.5). They are of the form of the sums:

$$\sum_{\nu}\left[\Sigma_{se\uparrow}(i(\nu+\nu_0))\Lambda_s(i\nu,\, i(\nu+\nu_0))\right].$$

The self-energies of the first order in J are frequency independent, so that they can be factored out of the summation. It is obvious from Eqs. (3.1) and (3.2) that the general form of the self-energies second order in J or H is such that they have frequency dependent denominator, in a very similar manner to a propagator. The identity (3.3) will then be used with advantage for the product of this denominator with that Green's function contained in Λ with the opposite spin label to that of the self-energy.

The reader is referred for more detailed derivation to Appendix B, all of the summations are of the same general form. The following is a typical term:

$$\begin{aligned}(cg_s/\sqrt{2})\sum_{\nu}\left[(e^{\beta\lambda}/Z_s)\Sigma^{(2l)}_{se\uparrow}(i(\nu+\nu_0))\Lambda_s(i\nu,\, i(\nu+\nu_0))\right] = {}&\Sigma^{(2l)}_{se\uparrow}(\lambda-\tfrac{1}{2}\omega_s-\Sigma_{se\downarrow}(\epsilon_{p+q}-\epsilon_p+\lambda+\tfrac{1}{2}\omega_s-i\nu_0)+i\nu_0)\chi^{+-}_s(i\nu_0)\\ &-\Xi^{(2l)}_{se\uparrow}(\lambda\tfrac{1}{2}\omega_s-\Sigma_{se\downarrow}(\epsilon_{p+q}-\epsilon_p+\lambda+\tfrac{1}{2}\omega_s-i\nu_0)+i\nu_0)\,(1+\lambda\chi^{+-}_e(i\nu_0))\\ &+c(g_s/\sqrt{2})\sum_{\nu}\left[\Sigma^{(4)}_{se\uparrow}(i(\nu+\nu_0))-\Sigma^{(4)}_{se\downarrow}(i\nu)\right]\Lambda_s(i\nu,\, i(\nu+\nu_0))+(g_s/g_e)(g_e/\sqrt{2})\\ &\times\sum_{\nu,p}\left\{\left[\Sigma^{(4)}_{es\uparrow}(i(\nu+\nu_0))-\Sigma^{(4)}_{es\downarrow}(i\nu)\right]\Lambda_p(i\nu,\, i(\nu+\nu_0))\right\}\,, \qquad (3.6)\end{aligned}$$

where $\lambda = 2J/g_e\, g_s$, and

$$\begin{aligned}\Xi^{(2l)}_{se\uparrow}\left(\lambda-\frac{\omega_s}{2}+i\nu_0\right)=\frac{ce^{\beta\lambda}g_s^2}{2Z_s}\sum_{p,q,\sigma}\Bigg[(J/2N)^2\sum_{\nu}\left(\frac{f^{+}(\lambda+\frac{1}{2}\omega_s)f^{+}_{p+q,\sigma}f^{-}_{p,\sigma}+f^{-}(\lambda-\frac{1}{2}\omega_s)f^{-}_{p+q,\sigma}f^{+}_{p,\sigma}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}\right)C_{\uparrow}(i(\nu+\nu_0))&\\ \times(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0)^{-1}\Bigg]\,.&\qquad (3.7)\end{aligned}$$

In Eq. (3.6), the first term on the right-hand side is a product of the self-energy-like quantity with the fixed arguments times the spin susceptibility $\chi^{+-}_s(i\nu_0)$. A comparison of Eq. (3.6) with Eq. (3.4) tells the meaning of this quantity; it is a two-particle self-energy, a contribution to the relaxation rates in the denominator of the local-moment susceptibility. The second term in Eq. (3.6) represents both the inhomogeneous and the off-diagonal term in the linear equation for $\chi^{+-}_s(i\nu_0)$. They involve the quantity designated Ξ; its value is given by Eq. (3.7). The off-diagonal term is the product of the inhomogeneous term times the direct spin-flip coupling of the spin and conduction-electron "bubbles." It produces the local-exchange-field term in the Bloch equations.

The superscript (4) on quantities such as $\Sigma^{(4)}_{se\uparrow}$ means that they are of the fourth order in J. Some of these terms are explicitly displayed in Appendix B. Their form shows that they can be interpreted as fourth-order processes which arise from the overlap in time of the scattering processes occurring to each of the two particles involved in the two-particle propagator, from the overlap of the vertices with the self-energies of the particles, and also from the overlap of the vertices with themselves. The interest of the present paper is to evaluate the *two-particle* quantities up to the J^2. It is argued in Appendix B that it is consistent with the declared degree of accuracy to omit these fourth-order terms.

Again, the lowest-order approximation when evaluating quantities $\Sigma^{(2)}$ and $\Xi^{(2)}$ is to ignore Σ_{se} in their arguments and to replace propagators $C_{\uparrow}(i(\nu+\nu_0))$ by $C_{\uparrow}^{(0)}(i(\nu+\nu_0))$ when evaluating $\Xi^{(2)}$. Also, at this point, it is convenient to perform the final projection by letting $\lambda\to\infty$. Notice that as a result of the algebraic rearrangements above, the Σ and Ξ contain λ as an argument, which cancels the λ which appears in the single-particle self-energy, so that the only dependence on λ which remains is the one in the fermi functions in the numerators of Σ's and Ξ's. The spin characteristics are obtained from the one-fermion quantities. With the above approximation, the following expressions are obtained:

$$\lim_{\lambda\to\infty}\Sigma_{se\uparrow}^{(2l)}\left(\lambda-\frac{\omega_s}{2}+i\nu_0\right)=\left(\frac{J}{2N}\right)^2\sum_{p,q,\sigma}\frac{f^-_{p+q,\sigma}f^+_{p,\sigma}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0}\ , \tag{3.8}$$

$$\lim_{\lambda\to\infty}\Xi_{se\uparrow}^{(2l)}\left(\lambda-\frac{\omega_s}{2}+i\nu_0\right)=\frac{g_s^2}{2}\sum_{p,q,\sigma}\frac{c(J/2N)^2[n^0_{s\uparrow}(f^+_{p+q,\sigma}-f^+_{p,\sigma})/(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma})]}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0}\ , \tag{3.9}$$

where $n^0_{s\uparrow}=e^{-\beta\omega_s/2}/Z_s$. An examination of both of these quantities and especially of the denominators of the expressions (3.8) and (3.9) justifies our efforts. They have two-particle-like character associated with the corresponding excitations of the spin system and they are of a form to obey the energy conservation requirements for the process.

The numerator of the term $\Xi^{(2l)}$ resulted from the frequency summation in the numerator of Eq. (3.7). It is the first-order contribution to the expectation value $\langle[a^\dagger_{p+q,\sigma}a_{p,\sigma}S^+,\ S^-]\rangle$ and the whole term $\Xi^{(2l)}$ is then a contribution to the inhomogeneous term in the equation of motion[16] for the propagator $\langle Ta^\dagger_{p+q,\sigma}a_{p,\sigma}S^+(\tau)S^-\rangle$. In turn, the last propagator appears as an intermediate state via which the transition in the local spin system proceeds in the equation of motion for the propagator $\langle TS^+(\tau)S^-\rangle$ with the factor of proportionality J. The form of the inhomogeneous term $\Xi_{se}^{(2l)}$ is thus seen to be appropriate to the propagator $\langle Ta^\dagger_{p+q,\sigma}a_{p,\sigma}S^+(\tau)S^-\rangle$ which is associated with the intermediate state involved in the local-spin self-energy $\Sigma_{se}^{(2l)}$. In this particular case, the transition (towards equilibrium) is due to the exchange interaction; the electronic excitation are longitudinal-like. The self-energy $\Sigma_{se}^{(2l)}$ is expressed in terms of the one-fermion self-energy with argument appropriate to the energy conservation of this process.

The quantity $\Xi^{(2l)}$ is then continued into a complex plane by letting $i\nu_0\to\omega_0+is$ and then is separated into its real and imaginary parts with the result

$$\begin{aligned}\Xi_{se\uparrow}^{(2l)}\left(\lambda-\frac{\omega_s}{2}+\omega_0+is\right)&=\mathrm{Re}\,\Xi_{se\uparrow}^{(2l)}\left(\lambda-\frac{\omega_s}{2}+\omega_0\right)\\&\quad+i\pi\left(\frac{J}{2N}\right)^2\sum_{p,q,\sigma}\delta(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-\omega_0)\,\frac{cn^0_{s\uparrow}(f^+_{p+q,\sigma}-f^+_{p,\sigma})}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}}\\&=\mathrm{Re}\,\Xi_{se\uparrow}^{(2l)}\ (\lambda-\tfrac{1}{2}\omega_s+\omega_0)+i\,\mathrm{Im}[\Sigma_{se\uparrow}^{(2l)}\ (\lambda-\tfrac{1}{2}\omega_s+\omega_0+is)]\,\chi^0_{s\uparrow}\ ,\end{aligned} \tag{3.10a}$$

where

$$\chi^0_{s\uparrow}=(cg_s^2/2Z_s)[(e^{[(\omega_s/2)-\omega_0]\beta}-e^{-(\omega_s/2)\beta})/(\omega_s-\omega_0)]\ , \tag{3.10b}$$

and where the δ function has been used in the last step.

In exactly the same way the rest of the terms entering the spin equation, and all terms in the electronic vertex equation, are derived. Their general form as well is the interpretation and identification of contributing processes and is similar to that one of the term exemplified above. For derivation see Appendix B.

In the spin equation, there will be two resulting susceptibilitylike quantities $\chi^0_{s\uparrow}$, $\chi^0_{s\downarrow}$ connected with the self-energies $\Sigma_{se\uparrow}$ or $\Sigma_{se\downarrow}$, respectively. The same situation arises in the electronic equation; there, the additional index p must be added to label contributions of a single electronic momentum p.

In what follows, the separation into the real and imaginary parts of the Ξ quantities will always have been made and hence ReΞ will be written simply as Ξ. The superscript and subscript convention of Ξ's will be exactly the same as that one for Σ terms. The net result of the above approximations and rearrangments are the following replacements for the sums involving products of self-energies and vertex functions:

$$\sum_{\nu}\left[c(e^{\beta\lambda}/Z_s)\Sigma_{\mathrm{se}\uparrow}(i(\nu+\nu_0))\,(g_s/\sqrt{2})\,\Lambda_s(i\nu,\,i(\nu+\nu_0))\right]\to i\,\mathrm{Im}\Sigma_{\mathrm{se}\uparrow}(\lambda-\tfrac{1}{2}\omega_s+\omega_0+is)$$
$$\times\{\chi_s^{+-}(\omega_0+is)-\chi_{s\uparrow}^{0}[1+\lambda\chi_e^{+-}(\omega_0+is)]\}+\mathrm{Re}\Sigma_{\mathrm{se}\uparrow}(\lambda-\tfrac{1}{2}\omega_s+\omega_0)\,\chi_s^{+-}(\omega_0+is)-\Xi_{\mathrm{se}\uparrow}(\lambda-\tfrac{1}{2}\omega_s+\omega_0)[1+\lambda\chi_e^{+-}(\omega_0+is)]\,, \tag{3.10c}$$

$$\sum_{\nu}\left[c(e^{\beta\lambda}/Z_s)\Sigma_{\mathrm{se}\downarrow}(i\nu)\,(g_s/\sqrt{2})\,\Lambda_s(i\nu,\,i(\nu+\nu_0))\right]\to i\,\mathrm{Im}\Sigma_{\mathrm{se}\downarrow}(\lambda+\tfrac{1}{2}\omega_s-\omega_0-is)\{\chi_s^{+-}(\omega_0+is)-\chi_{s\downarrow}^{0}[1+\lambda\chi_e^{+-}(\omega_0+is)]\}$$
$$+\mathrm{Re}\Sigma_{\mathrm{se}\downarrow}(\lambda+\tfrac{1}{2}\omega_s-\omega_0)\chi_s^{+-}(\omega_0+is)-\Xi_{\mathrm{se}\downarrow}(\lambda+\tfrac{1}{2}\omega_s-\omega_0)[1+\lambda\chi_e^{+-}(\omega_0+is)]\,, \tag{3.11a}$$

where

$$\chi_{s\downarrow}^{0}=(cg_s^2/2Z_s)[(e^{(\omega_s/2)\beta}-e^{-[(\omega_s/2)-\omega_0]\beta})/(\omega_s-\omega_0)], \tag{3.11b}$$

and for the electronic vertex functions

$$\sum_{\nu}\left[\Sigma_{\mathrm{es}\uparrow}(i(\nu+\nu_0))\,(g_e/\sqrt{2})\,\Lambda_p(i\nu,\,i(\nu+\nu_0))\right]\to i\,\mathrm{Im}\Sigma_{\mathrm{es}\uparrow}(\epsilon_{p\downarrow}+\omega_0+is)\{\chi_p^{+-}(\omega_0+is)-\chi_{p\uparrow}^{0}[1+\lambda\chi_s^{+-}(\omega_0+is)]\}$$
$$+\mathrm{Re}\Sigma_{\mathrm{es}\uparrow}(\epsilon_{p\downarrow}+\omega_0)\,\chi_p^{+-}(\omega_0+is)-\Xi_{\mathrm{es},p\uparrow}(\epsilon_{p\downarrow}+\omega_0)[1+\lambda\chi_s^{+-}(\omega_0+is)]\,, \tag{3.12a}$$

where

$$\chi_{p\uparrow}^{0}=\tfrac{1}{2}g_e^2\{[f^+(\epsilon_{p\downarrow}+\omega_0)-f^+(\epsilon_{p\uparrow})]/(\omega_e-\omega_0)\}\,, \tag{3.12b}$$

and

$$\sum_{\nu}\left[\Sigma_{\mathrm{es}\downarrow}(i\nu)\,(\tfrac{1}{2}g_e)\Lambda_p(i\nu,\,i(\nu+\nu_0))\right]\to i\,\mathrm{Im}\Sigma_{\mathrm{es}\downarrow}(\epsilon_{p\uparrow}-\omega_0-is)\{\chi_p^{+-}(\omega_0+is)-\chi_{p\downarrow}^{0}[1+\lambda\chi_s^{+-}(\omega_0+is)]\}$$
$$+\mathrm{Re}\Sigma_{\mathrm{es}\downarrow}(\epsilon_{p\uparrow}-\omega_0)\chi_p^{+-}(\omega_0+is)-\Xi_{\mathrm{es},p\downarrow}(\epsilon_{p\uparrow}-\omega_0)[1+\lambda\chi_s^{+-}(\omega_0+is)]\,, \tag{3.13a}$$

where

$$\chi_{p\downarrow}^{0}=\tfrac{1}{2}g_e^2\{[f^+(\epsilon_{p\downarrow})-f^+(\epsilon_{p\uparrow}-\omega_0)]/(\omega_e-\omega_0)\}\,, \tag{3.13b}$$

and where the various Ξ terms are listed in Appendix B. The $\Sigma_{\mathrm{el}}(i\nu)$ self-energies in the electronic equation describe the electron-lattice relaxation and their rearrangements yields equations which are exact analogies of Eqs. (3.12) and (3.13).

From Eq. (3.4) and (3.5) it is obvious that the vertex correction contribution will always add to the self-energies. Also, the arguments of all terms have now been fixed, and so some new definitions will be made. Using the above results, Eqs. (3.4) and (3.5) become, after the final projection and continuation ($i\nu_0\to\omega_0+is$),

$$(\omega_s-\omega_0-is)\,\chi_s^{+-}(\omega_0+is)=\tfrac{1}{2}cg_s^2\,\langle n_{s\downarrow}-n_{s\uparrow}\rangle+R(\omega_0+is)\,, \tag{3.14}$$

$$(\omega_e-\omega_0-is)\chi_e^{+-}(\omega_0+is)=\tfrac{1}{2}g_e^2\,\langle n_{e\downarrow}-n_{e\uparrow}\rangle-(g_e/g_s)R(\omega_0+is)$$
$$+(1/N)\sum_{p}i\,\mathrm{Im}\Sigma_{\mathrm{el}\uparrow}(p,\,\omega_0+is)\{\chi_p^{+-}(\omega_0+is)-\chi_{p\uparrow}^{0}[1+\lambda\chi_s^{+-}(\omega_0+is)]\}$$
$$+(1/N)\sum_{p}i\,\mathrm{Im}\Sigma_{\mathrm{el}\downarrow}(p,\,\omega_0+is)\{\chi_p^{+-}(\omega_0+is)-\chi_{p\downarrow}^{0}[1+\lambda\chi_s^{+-}(\omega_0+is)]\}$$
$$+(1/N)\sum_{p}\mathrm{Re}[\Sigma_{\mathrm{el}\uparrow}(p,\,\omega_0)+\Sigma_{\mathrm{el}\downarrow}(p,\,\omega_0)]\,\chi_p^{+-}(\omega_0+is)-\Xi_{\mathrm{el}}(\omega_0)[1+\lambda\chi_s^{+-}(\omega_0+is)]\,, \tag{3.15}$$

where

$$R(\omega_0+is)=i\,\mathrm{Im}\Sigma_{\mathrm{se}\uparrow}(\omega_0+is)\{\chi_s^{+-}(\omega_0+is)-\chi_{s\uparrow}^{0}[1+\lambda\chi_e^{+-}(\omega_0+is)]\}+i\,\mathrm{Im}\Sigma_{\mathrm{se}\downarrow}(\omega_0+is)$$
$$\times\{\chi_s^{+-}(\omega_0+is)-\chi_{s\downarrow}^{0}[1+\lambda\chi_e^{+-}(\omega_0+is)]\}+\mathrm{Re}[\Sigma_{\mathrm{se}\uparrow}(\omega_0)+\Sigma_{\mathrm{se}\downarrow}(\omega_0)]\,\chi_s^{+-}(\omega_0+is)-\Xi_{\mathrm{se}}(\omega_0)[1+\lambda\chi_e^{+-}(\omega_0+is)]$$
$$-(g_s/g_eN)\sum_{p}i\,\mathrm{Im}\Sigma_{\mathrm{es}\uparrow}(p,\,\omega_0+is)\{\chi_p^{+-}(\omega_0+is)-\chi_{p\uparrow}^{0}[1+\lambda\chi_s^{+-}(\omega_0+is)]\}$$
$$-(g_s/g_eN)\sum_{p}i\,\mathrm{Im}\Sigma_{\mathrm{es}\downarrow}(p,\,\omega_0+is)\{\chi_p^{+-}(\omega_0+is)-\chi_{p\downarrow}^{0}[1+\lambda\chi_s^{+-}(\omega_0+is)]\}$$

$$-(g_s/g_eN)\sum_p \mathrm{Re}[\Sigma_{\mathrm{es}\uparrow}(p,\omega_0)+\Sigma_{\mathrm{es}\downarrow}(p,\omega_0)]\,\chi_p^{+-}(\omega_0+is)+(g_s/g_e)\,\Xi_{\mathrm{es}}(\omega_0)\,[1+\lambda\chi_s^{+-}(\omega_0+is)]\,, \quad (3.16)$$

and where the newly defined quantities are

$$\Sigma_{\mathrm{se}\uparrow}(\omega_0+is)=\lim_{\lambda\to\infty}\left[\Sigma_{\mathrm{se}\uparrow}\left(\lambda-\frac{\omega_s}{2}+\omega_0+is\right)+\Sigma_{\mathrm{se}\uparrow}^{(2l)}\left(\lambda-\frac{\omega_s}{2}+\omega_0+is\right)\right]$$

$$=\frac{J}{2}\langle n_{e\uparrow}-n_{e\downarrow}\rangle+\left(\frac{J}{N}\right)^2\sum_{p,q}\left(\frac{1}{2}\sum_\sigma\frac{f^{-}_{p+q,\sigma}f^{+}_{p,\sigma}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-\omega_0-is}+\frac{f^{-}_{p+q\uparrow}f^{+}_{p\downarrow}}{\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_0-is}\right)\,, \quad (3.17\mathrm{a})$$

$$\Sigma_{\mathrm{se}\downarrow}(\omega_0+is)=\lim_{\lambda\to\infty}\left[-\Sigma_{\mathrm{se}\downarrow}\left(\lambda+\frac{\omega_s}{2}-\omega_0-is\right)-\Sigma_{\mathrm{se}\downarrow}^{(2l)}\left(\lambda+\frac{\omega_s}{2}-\omega_0-is\right)\right]$$

$$=\frac{J}{2}\langle n_{e\uparrow}-n_{e\downarrow}\rangle+\left(\frac{J}{N}\right)^2\sum_{p,q}\left(\frac{1}{2}\sum_q\frac{f^{+}_{p+q,\sigma}f^{-}_{p,\sigma}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-\omega_0-is}+\frac{f^{+}_{p+q\uparrow}f^{-}_{p\downarrow}}{\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_0-is}\right)\,, \quad (3.17\mathrm{b})$$

$$\Xi_{\mathrm{se}}(\omega_0)=\lim_{\lambda\to\infty}\left[\Xi_{\mathrm{se}\uparrow}\left(\lambda-\frac{\omega_s}{2}+\omega_0-is\right)+\Xi_{\mathrm{se}\uparrow}^{(2l)}\left(\lambda-\frac{\omega_s}{2}+\omega_0+is\right)-\Xi_{\mathrm{se}\downarrow}\left(\lambda+\frac{\omega_s}{2}-\omega_0-is\right)-\Xi_{\mathrm{se}\downarrow}^{(2l)}\left(\lambda+\frac{\omega_s}{2}-\omega_0-is\right)\right]$$

$$=\frac{cg_s^2}{2}\left(\frac{J}{N}\right)^2\sum_{p,q}\left(\frac{1}{2}\sum_\sigma\frac{f^{+}_{p+q,\sigma}-f^{+}_{p,\sigma}}{(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-\omega_0-is)(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma})}+\frac{2(n^0_{s\uparrow}f^{-}_{p+q\uparrow}f^{+}_{p\downarrow}-n^0_{s\downarrow}f^{+}_{p+q\uparrow}f^{-}_{p\downarrow})}{(\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_0-is)(\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_s)}\right)\,, \quad (3.17\mathrm{c})$$

$$\Sigma_{\mathrm{es}\uparrow}(p,\omega_0+is)=\Sigma_{\mathrm{es}\uparrow}(\epsilon_{p\downarrow}+\omega_0+is)+\Sigma_{\mathrm{es}\uparrow}^{(2l)}(\epsilon_{p\downarrow}+\omega_0+is)=\frac{cJ}{2}\langle n_{s\uparrow}-n_{s\downarrow}\rangle$$

$$+\frac{cJ^2}{N}\left(2\langle S_z^2\rangle\sum_q\frac{1}{\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_0-is}+\langle S_x^2+S_y^2\rangle\sum_q\frac{1}{\epsilon_{p+q\downarrow}-\epsilon_{p\downarrow}+\omega_s-\omega_0-is}+\frac{\langle S_z\rangle}{2}\sum_q\frac{f^{+}_{p+q\downarrow}-f^{-}_{p+q\downarrow}}{\epsilon_{p+q\downarrow}-\epsilon_{p\downarrow}+\omega_s-\omega_0-is}\right), \quad (3.18\mathrm{a})$$

$$\Sigma_{\mathrm{es}\downarrow}(p,\omega_0+is)=-\Sigma_{\mathrm{es}\downarrow}(\epsilon_{p\uparrow}-\omega_0-is)-\Sigma_{\mathrm{es}\downarrow}^{(2l)}(\epsilon_{p\uparrow}-\omega_0-is)=\frac{cJ}{2}\langle n_{s\uparrow}-n_{s\downarrow}\rangle$$

$$+\frac{cJ^2}{N}\left(2\langle S_z^2\rangle\sum_q\frac{1}{\epsilon_{p\uparrow}-\epsilon_{p+q\downarrow}-\omega_0-is}+\langle S_x^2+S_y^2\rangle\sum_q\frac{1}{\epsilon_{p\uparrow}-\epsilon_{p+q\uparrow}+\omega_s-\omega_0-is}-\frac{\langle S_z\rangle}{2}\sum_q\frac{f^{+}_{p+q\uparrow}-f^{-}_{p+q\uparrow}}{\epsilon_{p\uparrow}-\epsilon_{p+q\uparrow}+\omega_s-\omega_0-is}\right), \quad (3.18\mathrm{b})$$

$$\Xi_{\mathrm{es}}(\omega_0)=\frac{1}{N}\sum_p\Xi_{\mathrm{es}}(p,\omega_0)=\frac{1}{N}\sum_p[\Xi_{\mathrm{es}\uparrow}(\epsilon_{p\downarrow}+\omega_0)+\Xi_{\mathrm{es}\uparrow}^{(2l)}(\epsilon_{p\downarrow}-\omega_0)-\Xi_{\mathrm{es}\downarrow}(\epsilon_{p\uparrow}-\omega_0)-\Xi_{\mathrm{es}\downarrow}^{(2l)}(\epsilon_{p\uparrow}-\omega_0)]$$

$$=\frac{cg_e^2}{2}\left(\frac{J}{N}\right)^2 2\sum_{p,q}\left(\langle S_z^2\rangle\sum_\sigma\frac{f^{+}_{p+q,\sigma}-f^{+}_{p,\sigma}}{(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_e-\omega_0)(\epsilon_{p,\sigma}-\epsilon_{p+q,\sigma})}\right.$$

$$\left.-\langle S_x^2+S_y^2\rangle\frac{f^{+}_{p+q\downarrow}-f^{+}_{p\uparrow}}{(\epsilon_{p+q}-\epsilon_p+\omega_s-\omega_0)(\epsilon_{p+q\downarrow}-\epsilon_{p\uparrow}+\omega_s)}-\langle S_z\rangle\frac{f^{+}_{p+q\downarrow}f^{-}_{p\uparrow}+f^{-}_{p+q\downarrow}f^{+}_{p\uparrow}}{(\epsilon_{p+q}-\epsilon_p+\omega_s-\omega_0)(\epsilon_{p+q\downarrow}-\epsilon_{p\uparrow}+\omega_s)}\right)\,, \quad (3.18\mathrm{c})$$

$$\Sigma_{\mathrm{el}\uparrow}(p,\omega_0+is)=\Sigma_{\mathrm{el}}(\epsilon_{p\downarrow}+\omega_0+is)+\Sigma_{\mathrm{el}}^{(2l)}(\epsilon_{p\downarrow}+\omega_0+is)=\frac{H^2}{N}\sum_q\left(\frac{2c'(\langle h_z^2\rangle-c'\langle h_z\rangle^2)}{\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_0-is}+\frac{c'\langle h_x^2+h_y^2\rangle}{\epsilon_{p+q\downarrow}-\epsilon_{p\downarrow}-\omega_0-is}\right)\,, \quad (3.19\mathrm{a})$$

$$\Sigma_{\mathrm{el}\downarrow}(p,\omega_0+is)=-\Sigma_{\mathrm{el}}(\epsilon_{p\uparrow}-\omega_0-is)-\Sigma_{\mathrm{el}}^{(2l)}(\epsilon_{p\uparrow}-\omega_0-is)=\frac{H^2}{N}\sum_q\left(\frac{2c'(\langle h_z^2\rangle-c'\langle h_z\rangle^2)}{\epsilon_{p\uparrow}-\epsilon_{p+q\downarrow}-\omega_0-is}+\frac{c'\langle h_x^2+h_y^2\rangle}{\epsilon_{p\uparrow}-\epsilon_{p+q\uparrow}-\omega_0-is}\right)\,, \quad (3.19\mathrm{b})$$

$$\Xi_{\mathrm{el}}(\omega_0)=\frac{1}{N}\sum_p\Xi_{\mathrm{el}}(p,\omega_0)=\frac{1}{N}\sum_p[\Xi_{\mathrm{el}\uparrow}(\epsilon_{p\downarrow}+\omega_0)+\Xi_{\mathrm{el}\uparrow}^{(2l)}(\epsilon_{p\downarrow}+\omega_0)-\Xi_{\mathrm{el}\downarrow}(\epsilon_{p\uparrow}-\omega_0)-\Xi_{\mathrm{el}\downarrow}(\epsilon_{p\uparrow}-\omega_0)]$$

$$=c'g_e^2(H/N)^2\sum_{p,q}\left((\langle h_z^2\rangle-c'\langle h_z\rangle^2)\sum_\sigma\frac{f^{+}_{p+q,\sigma}-f^{+}_{p,\sigma}}{(\epsilon_{p+q}-\epsilon_p-\omega_0+\omega_e)(\epsilon_p-\epsilon_{p+q})}-\langle h_x^2+h_y^2\rangle\frac{f^{+}_{p+q,\sigma}-f^{+}_{p,\sigma}}{(\epsilon_{p+q}-\epsilon_p-\omega_0)(\epsilon_{p+q\downarrow}-\epsilon_{p\uparrow})}\right)\,. \quad (3.19\mathrm{c})$$

Equations (3.14) and (3.15) are a set of linear coupled equations which describe dynamical behavior of the local-spin–conduction-electron system at all temperatures. The form of the relaxation terms in Eqs. (3.15) and (3.16) is of the form of the relaxation towards the local instantaneous internal field, which was evaluated up to the first order in J.

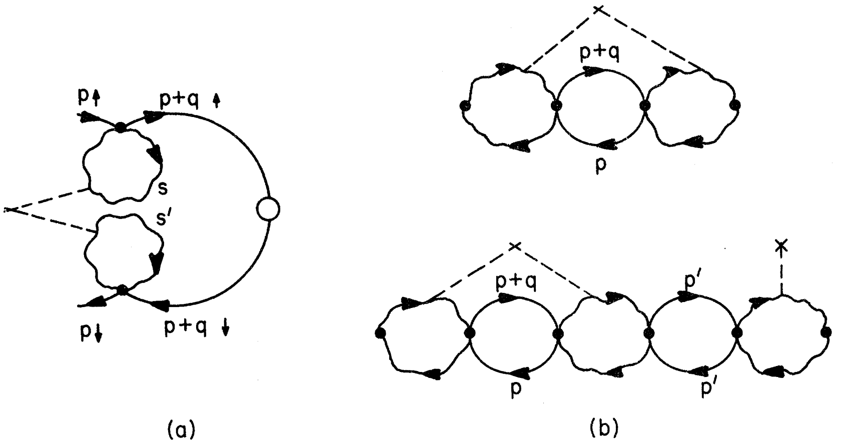

FIG. 10. (a) The (c) order electronic vertex correction; (b) the anomalous $(c)^2$ diagrams.

All quantities in Eqs. (3.17)–(3.19) are of the same general form, and their interpretation goes along the same lines as for the exemplified term $\Sigma_{\mathrm{se}\uparrow}^{(21)}$. They are written in such a way that their first part represents a term of longitudinal character, and the remaining part a term of transversal (spin-flip) character; each contains relevant contributions from both up and down fermions. The electron-lattice terms, the Σ_{el}'s and Ξ_{el}'s, are due to the model Hamiltonian used, and are exact analogies of the electron-local spin terms.

The evaluation of all of these quantities is straightforward. For a square-band model of the width $2D$, in the high-temperature limit ($kT \gg \omega_e$, ω_s, ω_0), the following explicit formulas are obtained[16]:

$$\Sigma_{\mathrm{se},\sigma}(\omega_0+is)=(\tfrac{1}{2}J)\langle n_{e\uparrow}-n_{e\downarrow}\rangle+2(\rho J)^2 \times[(\omega_0-\tfrac{1}{2}\omega_s)\ln(\gamma D/\pi kT)+\tfrac{1}{2}\omega_e\ln 2 -2\sigma D\ln 2+i\pi kT]\ , \tag{3.20a}$$

$$\Xi_{\mathrm{se}}(\omega_0)=(\tfrac{1}{2}g_s^2)\,4c(\rho J)^2\,\langle S_z\rangle\ln(\gamma D/\pi kT)\ , \tag{3.20b}$$

$$\Sigma_{\mathrm{es}\uparrow}(p,\,\omega_0+is)=(\tfrac{1}{2}cJ)\langle n_{s\uparrow}-n_{s\downarrow}\rangle +c\rho J^2\,\langle S_z\rangle[2\ln(\pi kT/2\gamma D) +i\pi\tanh((\omega_s-\omega_0-\epsilon_{p\downarrow})/2kT)] +i\pi c\rho J^2\,(2\langle S_z^2\rangle+\langle S_x^2\rangle+\langle S_y^2\rangle)\ , \tag{3.21a}$$

$$\Sigma_{\mathrm{es}\downarrow}(p,\,\omega_0+is)=(\tfrac{1}{2}cJ)\langle n_{s\uparrow}-n_{s\downarrow}\rangle +c\rho J^2\langle S_z\rangle[2\ln(\pi kT/2\gamma D) +i\pi\tanh((\omega_s-\omega_0+\epsilon_{p\uparrow})/2kT)] +i\pi c\rho J^2\,(2\langle S_z^2\rangle+\langle S_x^2\rangle+\langle S_y^2\rangle)\ , \tag{3.21b}$$

$$\Xi_{\mathrm{es}}(\omega_0)=-(\tfrac{1}{2}g_e^2)\,4c(\rho J)^2\,\langle S_z\rangle\ln(\gamma D/\pi kT)\ , \tag{3.21c}$$

$$\Sigma_{\mathrm{el}\uparrow}(p,\,\omega_0+is)=\Sigma_{\mathrm{el}\downarrow}(p,\,\omega_0+is)$$

$$=i\pi c'\rho H^2[2(\langle h_z^2\rangle-c'\langle h_z\rangle^2)+\langle h_x^2+h_y^2\rangle]\ , \tag{3.22a}$$

$$\Xi_{\mathrm{el}}(\omega_0)=0\ , \tag{3.22b}$$

where $\rho=1/\epsilon_F$ is the constant one-spin density of states close to the Fermi surface, and $\gamma\cong 0.58$ is Euler's constant. The imaginary parts of the expressions $\Sigma_{\mathrm{es}\uparrow}$ and $\Sigma_{\mathrm{es}\downarrow}$ are in fact valid at all temperatures. Their momentum dependence, which is due to a part of the transverse contribution, is explicitly displayed. At high temperatures, the ratio of the momentum-dependent part to the momentum-independent contributions is ω_s/kT. In the low-temperature regime and for general spin, this ratio can be roughly considered to be $1/2S$.[12] The low-temperature evaluations are done in the same way and are listed in Appendix C.

C. Higher-Order Concentration Effects

The diagrams which would have yielded higher-order concentration effects, were dropped from the beginning of this calculation. However, they are useful in the interpretation of the longitudinal electronic self-energies. The diagrams in question include electronic self-energies proportional to c^2 in Fig. 6(f); their corresponding vertex corrections are illustrated in Fig. 10(a) and localized moment self energies in Fig. 5(d). The localized-moment self-energies do not have any corresponding vertex corrections, but rather the series of diagrams illustrated in Fig. 10(b). Again, as expected for regular self-energies, the electronic self-energies 6(f) have corresponding off-diagonal coupling illustrated in Fig. 11(a) and the spin self-energies 5(d) are associated with diagrams in Fig. 11(b).

Inclusion of the conduction electron self-energies 6(f), their corresponding vertex corrections, and off-diagonal coupling diagrams results in only one algebraic correction in the expressions for the electronic self-energies; namely, the factor $c\langle S_z^2\rangle$

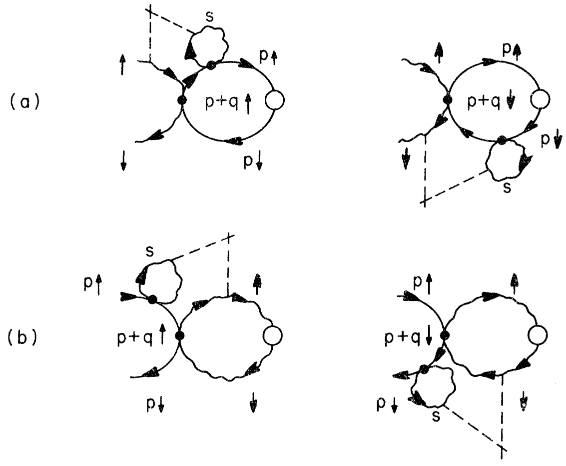

FIG. 11. (a) Off-diagonal coupling associated with diagrams 6(f) and 10(a); (b) the off-diagonal coupling associated with diagrams 5(d) and 10(b).

in Eqs. (3.2f), (3.18), (3.21), and elsewhere is to be replaced by the factor $c(\langle S_z^2\rangle - c\langle S_z\rangle^2)$. Comparing Eqs. (3.18) and (3.19), it can be seen that the longitudinal electronic self-energies may be interpreted as the result of electron scattering off the microscopic inhomogeneous fluctuating magnetic field produced by the local moments, giving rise to the so-called "frequency modulation" contribution $1/T_1'$ to the relaxation rate. Note that for concentrations $c=1$, the above contribution is identical to the usual scattering contribution by a thermally fluctuating magnetic field of the local-moment system.

The changes corresponding to the diagrams associated with these higher-order spin self-energies consist of adding the following term containing two self-energy-like quantities to Eq. (3.14):

$$[\Sigma_{\text{se}}^{(2l,c)}(\omega_0+is)+\Sigma_{\text{se}}^{(2t,c)}(\omega_0+is)]\,\chi_e^{+-}(\omega_0+is)\ , \tag{3.23a}$$

where

$$\Sigma_{\text{se}}^{(2l,c)}(\omega_0+is)=\Sigma_{\text{se}\uparrow}^{(2l,c)}\,(\lambda-\tfrac{1}{2}\omega_s+\omega_0+is)$$

$$-\Sigma_{\text{se}\downarrow}^{(2l,c)}(\lambda+\tfrac{1}{2}\omega_s-\omega_0-is)$$

$$=-c\langle S_z\rangle\left(\frac{J}{N}\right)^2\sum_{p,q,\sigma}\frac{f_{p+q,\sigma}^{+}-f_{p,\sigma}^{+}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}} \tag{3.23b}$$

and

$$\Sigma_{\text{se}}^{(2t,c)}(\omega_0+is)=2c\langle S_z\rangle\left(\frac{J}{N}\right)^2\sum_{p,q}\frac{f_{p+q\uparrow}^{+}-f_{p\downarrow}^{+}}{\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_0-is}\ . \tag{3.23c}$$

The self-energy $\Sigma_{\text{se}}^{(2l,c)}(\omega_0+is)$ is real, and hence enters the equations in the same way as the other real terms. However, the quantity $\Sigma_{\text{se}}^{(2l,c)}(\omega_0+is)$, which is not real at finite frequencies, does not enter the equations in the same way as the other self-energies which represent the effects linear in c. The meaning of this quantity is not understood.

IV. TRANSVERSE SUSCEPTIBILITY

A. Susceptibility

In order that Eqs. (3.14) and (3.15) may be written in terms of $\chi_s^{+-}(\omega_0+is)$ and $\chi_e^{+-}(\omega_0+is)$ only, and to identify explicitly expressions for the relaxation rates, the sums over momenta p must be moved to the right of the electronic self-energies. This is justifiable in such a regime where over a region of the width kT around the Fermi surface, the dependence of the electronic self-energies on the argument ϵ_p is weak as is the case, e.g., for high temperatures and/or for large local-moment-spin magnitude. At very low temperatures and small spin, Eq. (3.5a) rather than (3.5b) would have to be summed over the internal frequency $i\nu$. By comparing Eqs. (3.5a) and (3.5b), the result of this sum can be inferred from Eqs. (3.14) and (3.15). The result, though formidable, is quite straightforward, and would represent the solution of the given Hamiltonian to the declared degree of accuracy.

In the high-temperature or large-spin limit, the momentum dependence of the electronic self-energies can be considered weak. There is no such approximation necessary for the spin self-energies. In this limit, Eqs. (3.14) and (3.15) will reduce to

$$\begin{aligned}(\omega_s-\omega_0-is)\chi_s^{+-}(\omega_0+is)=&\,g_sM_s^z-\Xi_{\text{se}}(\omega_0)[1+\lambda\chi_e^{+-}(\omega_0+is)]+(g_s/g_e)\,\Xi_{\text{es}}(\omega_0)[1+\lambda\chi_s^{+-}(\omega_0+is)]\\&+(g_s/g_e)\{\lambda g_eM_s^z-\text{Re}[\Sigma_{\text{es}\uparrow}^{(2)}(p_F,\omega_0)+\Sigma_{\text{es}\downarrow}^{(2)}(p_F,\omega_0)]\}\,\chi_e^{+-}(\omega_0+is)\\&-\{\lambda g_sM_e^z-\text{Re}[\Sigma_{\text{se}\uparrow}^{(2)}(\omega_0)+\Sigma_{\text{se}\downarrow}^{(2)}(\omega_0)]\}\,\chi_s^{+-}(\omega_0+is)+i\,\text{Im}\Sigma_{\text{se}\uparrow}(\omega_0+is)\{\chi_s^{+-}(\omega_0+is)-\chi_{s\uparrow}^0[1+\lambda\chi_e^{+-}(\omega_0+is)]\}\\&+i\,\text{Im}\Sigma_{\text{se}\downarrow}(\omega_0+is)\{\chi_s^{+-}(\omega_0+is)-\chi_{s\downarrow}^0[1+\lambda\chi_e^{+-}(\omega_0+is)]\}\\&-i(g_s/g_e)\,\text{Im}[\Sigma_{\text{es}\uparrow}(\omega_0+is)+\Sigma_{\text{es}\downarrow}(\omega_0+is)]\{\chi_e^{+-}(\omega_0+is)-\chi_e^0[1+\lambda\chi_s^{+-}(\omega_0+is)]\}\ ,\end{aligned} \tag{4.1}$$

$$\begin{aligned}(\omega_e-\omega_0-is)\,\chi_e^{+-}(\omega_0+is)=&\,g_eM_e^z-\Xi_{\text{es}}(\omega_0)[1+\lambda\chi_s^{+-}(\omega_0+is)]+(g_e/g_s)\Xi_{\text{se}}(\omega_0)[1+\lambda\chi_e^{+-}(\omega_0+is)]\\&+(g_e/g_s)\{\lambda g_sM_e^z-\text{Re}[\Sigma_{\text{se}\uparrow}^{(2)}(\omega_0)+\Sigma_{\text{se}\downarrow}^{(2)}(\omega_0)]\}\,\chi_s^{+-}(\omega_0+is)-\{\lambda g_eM_s^z-\text{Re}[\Sigma_{\text{es}\uparrow}^{(2)}(p_F,\omega_0)+\Sigma_{\text{es}\downarrow}^{(2)}(p_F,\omega_0)]\}\,\chi_e^{+-}(\omega_0+is)\end{aligned}$$

$$+i\,\mathrm{Im}[\Sigma_{\mathrm{es}\uparrow}(p_F,\omega_0+is)+\Sigma_{\mathrm{es}\downarrow}(p_F,\omega_0+is)+\Sigma_{\mathrm{el}\uparrow}(p_F,\omega_0+is)+\Sigma_{\mathrm{el}\downarrow}(p_F,\omega_0+is)]\{\chi_e^{+-}(\omega_0+is)$$
$$-\chi_e^0[1+\lambda\chi_s^{+-}(\omega_0+is)]\}-(g_e/g_s)\,i\,\mathrm{Im}\Sigma_{\mathrm{se}\uparrow}(\omega_0+is)\{\chi_s^{+-}(\omega_0+is)-\chi_{s\uparrow}^0[1+\lambda\chi_e^{+-}(\omega_0+is)]\}$$
$$-(g_e/g_s)\,i\,\mathrm{Im}\Sigma_{\mathrm{se}\downarrow}(\omega_0+is)\{\chi_s^{+-}(\omega_0+is)-\chi_{s\downarrow}^0[1+\lambda\chi_e^{+-}(\omega_0+is)]\}\,, \tag{4.2}$$

where $\lambda=2J/g_eg_s$ is the molecular-field constant and where the two-particle self-energies have been decomposed into their first- and second-order contributions, the superscript (2) designating the latter quantity. They are expressed by Eqs. (3.17)–(3.19) in terms of the one-particle self-energies. The approximations $\chi_{e\uparrow}^0\simeq\chi_{e\downarrow}^0\simeq\chi_e^0$, the conventional Pauli susceptibility, and $\Xi_{\mathrm{el}}=0$, $\mathrm{Re}\Sigma_{\mathrm{el}}=0$ have been made, all of which are valid for $[kT,\ \omega_0,\ \omega_s,\ \omega_e,\ \mathrm{Im}(\Sigma_{\mathrm{es}}+\Sigma_{\mathrm{el}})]\ll\epsilon_F$. $\chi_{s\uparrow}^0$ and $\chi_{s\downarrow}^0$ as given by Eqs. (3.10b) and (3.11b) represent modifications of the static spin susceptibility appearing in the relaxation to the internal field terms at temperatures which are not large compared to the external frequency ω_0 and the resonant frequency ω_s. $M_e^z=\frac{1}{2}g_e\langle n_{e\downarrow}-n_{e\uparrow}\rangle$ and $M_s^z=\frac{1}{2}cg_s\langle n_{s\downarrow}-n_{s\uparrow}\rangle$ are, respectively, the electron and local-spin magnetizations per lattice site in the z direction.

Before going to further approximations, the diffusion term in the presence of a large impurity potential scattering will be introduced. Its derivation requires rederivation of the equations of Sec. III for the $\vec{q}\neq0$ susceptibility $\chi^{+-}(\vec{q},\omega_0)$. For $|\vec{q}|$ small, the present mathematical framework is used with the substitution $\epsilon_{p\uparrow}\rightarrow\epsilon_{p+q\uparrow}$ and the additional replacement $\sum_p(f_{p\downarrow}-f_{p+q\uparrow})\rightarrow\langle n_{e\downarrow}-n_{e\uparrow}\rangle-\rho(\vec{p}\cdot\vec{q}/m)$ everywhere it is relevant. Spin diffusion is associated with the electronic density transport represented by the vertex correctionlike diagrams for finite $\vec{q}$ and thus determined by the largest scattering cross section for the process. A more detailed derivation is contained in Appendix D. If the usual approximation (well satisfied in practice) that the single-particle impurity-scattering cross section is big, namely, that $|\mathrm{Im}\Sigma_i|\gg\Sigma_{\mathrm{el}},\ \Sigma_{\mathrm{es}},\ |\omega_e-\omega_0|,\ |\vec{q}\cdot\vec{k}_F|$, is made, it is seen that Eq. (D15) for the susceptibility averaged over the directions $\hat{p}$, is the same as Eq. (4.2) except that the diffusion term iDq^2 has been added to the right-hand side of Eq. (4.2) everywhere the self-energy Σ_{el} appears and where the diffusion constant $D=[1/3\Sigma_i(\omega_0+is)]v_F^2$, $v_F=p_F/m$. Owing to its large cross section, the impurity-scattering time appears in the diffusion constant, though it does not directly contribute any relaxation rate for the magnetization. The diffusion term as it appears in Eq. (D16), and consequently in Eq. (4.2) for $\vec{q}\neq0$ susceptibility, is again of the form appropriate to the instantaneous local field.

Collecting then the terms in Eqs. (4.1), modified by the diffusion term Eq. (4.2), and making the high-temperature approximation that the susceptibility $\chi_{s\uparrow}^0\simeq\chi_{s\downarrow}^0\simeq\chi_s^0$, the local-moment static susceptibility, the following two equations for the coupled dynamic susceptibility are obtained:

$$(\epsilon_s-\omega_0)\chi_s^{+-}(\omega_0+is)-\zeta_s\chi_e^{+-}(\omega_0+is)=\eta_s\,, \tag{4.3}$$

$$(\epsilon_e-\omega_0)\chi_e^{+-}(\omega_0+is)-\zeta_e\chi_s^{+-}(\omega_0+is)=\eta_e\,, \tag{4.4}$$

which have the solution

$$\chi^{+-}(\omega_0+is)=\chi_s^{+-}(\omega_0+is)+\chi_e^{+-}(\omega_0+is)$$
$$=\frac{(\epsilon_e-\omega_0)\eta_s+\zeta_s\eta_e+(\epsilon_s-\omega_0)\eta_e+\zeta_e\eta_s}{(\epsilon_s-\omega_0)(\epsilon_e-\omega_0)-\zeta_e\zeta_s}\,, \tag{4.5}$$

where

$$\epsilon_s=\omega_s+\lambda g_sM_e^z-\Sigma_{\mathrm{se}}^{(2)}(\omega_0+is)-\lambda(g_s/g_e)[i\,\mathrm{Im}\Sigma_{\mathrm{es}}(\omega_0+is)\chi_e^0+\Xi_{\mathrm{es}}(\omega_0)]\,,$$
$$\zeta_s=g_s\lambda M_s^z-(g_s/g_e)\Sigma_{\mathrm{es}}^{(2)}(\omega_0+is)-\lambda[i\,\mathrm{Im}\Sigma_{\mathrm{se}}(\omega_0+is)\chi_s^0+\Xi_{\mathrm{se}}(\omega_0)]\,,$$
$$\eta_s=g_sM_s^z-\Xi_{\mathrm{se}}(\omega_0)+(g_s/g_e)\Xi_{\mathrm{es}}(\omega_0)-i\,\mathrm{Im}\Sigma_{\mathrm{se}}(\omega_0+is)\chi_s^0+i(g_s/g_e)\mathrm{Im}\Sigma_{\mathrm{es}}(\omega_0+is)\chi_e^0\,,$$
$$\epsilon_e=\omega_e+\lambda g_eM_s^z-\Sigma_{\mathrm{es}}^{(2)}(\omega_0+is)-\Sigma_{\mathrm{el}}(\omega_0+is)-i[v_F^2/3\Sigma_i(\omega_0+is)]\,q^2-\lambda(g_e/g_s)[i\,\mathrm{Im}\Sigma_{\mathrm{se}}(\omega_0+is)\chi_s^0+\Xi_{\mathrm{se}}(\omega_0)]\,, \tag{4.6}$$
$$\zeta_e=g_e\lambda M_e^z-(g_e/g_s)\Sigma_{\mathrm{se}}^{(2)}(\omega_0+is)-\lambda(\{i\,\mathrm{Im}[\Sigma_{\mathrm{es}}(\omega_0+is)+\Sigma_{\mathrm{el}}(\omega_0+is)]+i[v_F^2/3\Sigma_i(\omega_0+is)]q^2\}\chi_e^0+\Xi_{\mathrm{es}}(\omega_0))\,,$$
$$\eta_e=g_eM_e^z-\Xi_{\mathrm{es}}(\omega_0)+(g_e/g_s)\Xi_{\mathrm{se}}(\omega_0)-\{i\,\mathrm{Im}[\Sigma_{\mathrm{es}}(\omega_0+is)+\Sigma_{\mathrm{el}}(\omega_0+is)]$$
$$+i[v_F^2/3\Sigma_i(\omega_0+is)]q^2\}\chi_e^0+i(g_e/g_s)\,\mathrm{Im}\Sigma_{\mathrm{se}}(\omega_0+is)\chi_s^0\,,$$

where the various two-particle self-energy contributions have been combined according to $\Sigma_{\mathrm{es}\uparrow}(p_F,\ \omega_0+is)+\Sigma_{\mathrm{es}\downarrow}(p_F,\ \omega_0+is)=\Sigma_{\mathrm{es}}(\omega_0+is)$. The combination of the spin self-energies has been permitted owing to the high-temperature ($kT\gg\omega_0$) approximation. In the opposite temperature limit, a similar set of equations can be written with $\chi^0_{s\uparrow} \neq \chi^0_{s\downarrow}$. The form of the diffusion term, such as in Eq. (4.6), is dependent upon a strong potential scattering and small $|\vec{q}|$.

B. Bloch Equations

In order to compare Eqs. (4.3) and (4.4) with the phenomenological Bloch equations, it is necessary not only to consider the high temperature limit $kT > \omega_0$, ω_s, but also the second-order logarithmic terms must be unimportant. Then Eqs. (4.3) and (4.4) are equivalent to a pair of the Bloch equations:

$$\begin{aligned}\frac{d}{dt}\vec{M}_s = {} & g_s[\vec{M}_s \times (\vec{H}_{\text{ext}} + \lambda\vec{M}_e)] \\ & - (1/T_{se})[\vec{M}_s - \chi^0_s(\vec{H}_{\text{ext}} + \lambda\vec{M}_e)] \\ & + (g_s/g_e T_{es})[\vec{M}_e - \chi^0_e(\vec{H}_{\text{ext}} + \lambda\vec{M}_s)] ,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\frac{d}{dt}\vec{M}_e = {} & g_e[\vec{M}_e \times (\vec{H}_{\text{ext}} + \lambda\vec{M}_s)] \\ & - (1/T_{es} + 1/T_{e1} - D\nabla^2)[\vec{M}_e - \chi^0_e(\vec{H}_{\text{ext}} + \lambda\vec{M}_s)] \\ & + (g_e/g_s T_{se})[\vec{M}_s - \chi^0_s(\vec{H}_{\text{ext}} + \lambda\vec{M}_e)] ,\end{aligned} \tag{4.8}$$

where

$$\begin{aligned} 1/T_{se} &= 4i\pi(\rho J)^2 kT , \\ 1/T_{es} &= 2i\pi c\rho J^2[S(S+1) + \langle S^2_z\rangle] , \\ 1/T_{e1} &= 2i\pi c'\rho H^2[2(\langle h^2_z\rangle - c'\langle h_z\rangle^2) + (\langle h^2_x\rangle + \langle h^2_y\rangle)] , \\ D &= \tfrac{1}{3}T_i v^2_F , \quad 1/T_i = 2i\pi c'\rho A^2 . \end{aligned} \tag{4.9}$$

These equations are identical to those proposed by Cottet *et al.*,[2] except the scattering-in terms are multiplied by g-factor ratios, that is, $1/T_{es}$ in the spin equation by g_s/g_e, $1/T_{se}$ in the electronic equation by g_e/g_s. The diffusion term is included. It is trivial to show that the static limits of both $\chi^{+-}_e(0)$ and $\chi^{+-}_s(0)$ are individually real. The pair of the Bloch equations (4.7) and (4.8) has solutions Eqs. (4.3)–(4.6) with the real second-order terms omitted. Consistent with the molecular-field approximation, the unenhanced susceptibilities may be rewritten

$$\chi^0_e = g_e M^z_e/(\omega_e + \lambda g_e M^z_s)$$

and

$$\chi^0_s = g_s M^z_s/(\omega_s + \lambda g_s M^z_e) .$$

Then the static ($\omega_0 = 0$) limit yields

$$\chi^{+-}_e(0) = g_e M^z_e/\omega_e = \chi^0_e(1 + \lambda\chi^0_s)/(1 - \lambda^2\chi^0_e\chi^0_s)^{-1}$$

and

$$\chi^{+-}_s(0) = g_s M^z_s/\omega_s = \chi^0_s(1 + \lambda\chi^0_e)/(1 - \lambda^2\chi^0_e\chi^0_s) ,$$

which are the molecular-field results.

C. Local Internal Field

The relaxation terms in Eqs. (4.1) and (4.2) have the form of the relaxation towards the local instantaneous internal field. Since there is some controversy over the inclusion of the molecular field in the destination vectors, it is of some importance to show from where it has arisen. Below a brief demonstration is given for the high-temperature limit, and ignoring logarithmic Kondo terms.

If instead of the full vertex equation shown in Fig. 7 only the first diagram of the right-hand side (the simple outgoing-spin external-field vertex) were included, then the result would contain only the first and second terms on the right-hand side of Eq. (3.4); that is

$$\begin{aligned}(\omega_s - \omega_0 - is)\chi^{+-}_s(\omega_0 + is) = {} & g_s M^z_s + i\,\text{Im}[\Sigma_{se}(\omega_0 + is) \\ & + \Sigma_{se\downarrow}(\omega_0 + is)][\chi^{+-}_s(\omega_0 + is) - \chi^0_s] \\ & + \text{Re}[\Sigma_{se\uparrow}(\omega_0) + \Sigma_{se\downarrow}(\omega_0)]\chi^{+-}_s(\omega_0 + is) - \Xi_{se}(\omega_0) .\end{aligned} \tag{4.10}$$

Ignoring the second-order real parts, which is equivalent to a molecular-field approximation, the result for $\chi^{+-}(\omega_0 + is)$ would read

$$\chi^{+-}_s(\omega_0) = \frac{\chi^0_s[(1 + \lambda\chi_e)\omega_s - i/T_{se}]}{(1 + \lambda\chi_e)\omega_s - i/T_{se} - \omega_0 - is} , \tag{4.11}$$

where $(1/T_{se}) = \text{Im}[\Sigma_{se\uparrow}(\omega_0 + is) + \Sigma_{se\downarrow}(\omega_0 + is)]$, and where the molecular-field result $g_s M^z_s = \chi^0_s(\omega_s + \lambda g_s M^z_e) = \omega_s\chi^0_s(1 + \lambda\chi_e)$ has been used. The sum of the first-order self-energies, that is, the "Knight shift" would equal in this case $\lambda\chi_e\omega_s$. Equations (4.10) and (4.11) are of the form of the relaxation towards the instantaneous *external* field, i.e., the destination field does not include the internal field. Equation (4.11) trivially produces the expected static result that $\chi^{+-}_s(\omega_0 = 0) = \chi^0_s$.

The third term in Fig. 7 represents the direct spin-flip off-diagonal coupling. Its value is equal to the first term in Fig. 7 times $\lambda\chi^{+-}_e(\omega_0 + is)$, where λ is the molecular-field constant $2J/g_e g_s$. Including this term, instead of Eq. (4.11), the following is obtained:

$$\begin{aligned}[\omega_s(1 + \lambda\chi_e) - \omega_0 - is]\chi^{+-}_s(\omega_0 + is) & \\ = \chi^0_s(1 + \lambda\chi_e)[1 + \lambda\chi^{+-}_e(\omega_0 + is)] & \\ + (i/T_{se})\{\chi^{+-}_s(\omega_0 + is) - \chi^0_s[1 + \lambda\chi^{+-}_e(\omega_0 + is)]\} & .\end{aligned} \tag{4.12}$$

This equation is of the form of relaxation towards the local instantaneous field, including the internal exchange field, and it has arisen from the inclusion of the direct spin-flip coupling of the spin and electronic bubbles. To obtain the resonance condition, one has to solve a *pair* of coupled equations such as (4.3) and (4.4).

D. Static Limit

The static limits of each diagram in the vertex equation in Fig. 7 must be real. If the so-called strip integration method (integrations along discontinuity cuts of the analytically continued integrands into the complex ν plane) for the evaluation of the diagrams were used, a value of the integrand for each diagram would reduce at zero external frequency to a sum of complex conjugate terms, that is, a real quantity.

Thus, the static limit of Eqs. (3.4)–(3.6) are exactly real. This implies that the static limit of the bracketed term $\{\chi_e^{+-}(\omega_0+is)-\chi_e^0[1+\lambda\chi_s^{+-}(\omega_0+is)]\}$, and of all similar terms in Eqs. (3.14)–(3.16) (they describe the relaxation destination) should identically reduce to zero. However, in the replacements (3.10)–(3.13), the internal field is evaluated up to the first order in J only, that is, the second-order off-diagonal terms of the vertex equations have not been included in the evaluation of the relaxation destination vectors. Hence, the bracketed terms will reduce to zero only up to the first order in J. For these terms to vanish exactly, and hence for the static limits of each diagram to be manifestly real, it is necessary to evaluate the relaxation destination terms to higher orders in J.

As is shown in Appendix B, this can be done by applying the present methods to the above mentioned second-order off-diagonal terms. This would yield an additional second-order (in J) frequency-dependent contribution to the relaxation destination vector, which is necessary for a correct analytical behavior of Eqs. (3.14)–(3.16). For the sake of simplicity, however, these missing terms (they are of the "overlapping" type of fourth order discussed in Appendix B) have been omitted here. This is consistent with the declared degree of accuracy, since the aim of the present paper is a comparison with the phenomenological Bloch equations. When this comparison is made, even the second-order real terms must be dropped since they have no simple interpretation in terms of these equations. In this case, the correct analytical form is again recovered.

However, this work does show that, in general, in order to obtain the correct analytic properties for the susceptibility, *the destination fields are as equally important as the self-energies* themselves and should be evaluated to the same degree of accuracy as the real parts of the self-energies, the Σ's, and the inhomogeneous terms, the Ξ's. If this is done, one obtains the physically correct result, that is, the relaxation bracketed terms in Eq. (3.14)–(3.16) vanish identically in the zero-frequency limit. It is then simple to determine the static values of any of the susceptibilities $\chi_e^{+-}(0)$ and $\chi_s^{+-}(0)$, for example, from the pair of Eqs. (3.14)–(3.16). Upon adding any such pair of equations in the zero-frequency limit, one obtains a simple result for the total static susceptibility, that is, $\chi^{+-}(0)=\chi_s^{+-}(0)+\chi_e^{+-}(0)=g_sM_s^z/\omega_s+g_eM_e^z/\omega_e$, where M_s^z are the magnetizations "dressed" with their self-energies.

V. DETAILED-BALANCE RELATION

In connection with the discussion of the phenomenological Bloch equations several variations of the so-called "detailed balance" condition have been proposed.[17]

Dividing the high-temperature ($kT \gg S\omega_s$) expressions for the exchange widths in Eq. (4.9) one obtains directly the ratio

$$T_{\text{se}}/T_{\text{es}}=c\,[S(S+1)+\langle S_z^2\rangle]/2\rho kT=g_e^2\chi_s^0/g_s^2\chi_e^0\,, \tag{5.1}$$

where $\chi_e^0=\frac{1}{2}g_e^2\rho$ and $\chi_s^0=c\,g_s^2\,S(S+1)/3kT$ are the high-temperature values for the unenhanced static electronic and local-moment susceptibilities. The relationship given by Eq. (5.1) contains the ratio of the g factor squared. This is because spin is conserved rather than magnetization, and consequently [see discussion following the Bloch equations (4.7)–(4.8)], the scattering-in terms are also modified by the g-factor ratio.

However, there do exist more general relationships between the exchange widths, and also the real parts of the second-order self-energies. Figure 12 shows the four simplest diagrams involved in the off-diagonal coupling. By rearrangement, each of them can be shown to be equal to that one obtained from it by a rotation about 180°, including all matrix elements at diagram vertices. For evaluation, use is made of the identity (3.1), or one can compare the above diagrams with Figs. 7(a) and 7(b) and Eqs. (3.4) and (3.5).

For the sum of these four diagrams one obtains

$$\begin{aligned}(\omega_s-i\nu_0)^{-1}\sum_{p,\nu}\,&[\Sigma^{(2)}_{\text{es}\uparrow}(i(\nu+\nu_0))+\Sigma^{(2l)}_{\text{es}\uparrow}(i(\nu+\nu_0))\\&-\Sigma_{\text{es}\downarrow}(i\nu)-\Sigma^{(2l)}_{\text{es}\downarrow}(i\nu)]G^0_{p\uparrow}(i(\nu+\nu_0))\,G^0_{p\downarrow}(i\nu)\,,\end{aligned} \tag{5.2}$$

and for the sum of the rotated ones:

$$\begin{aligned}(\omega_e-i\nu_0)^{-1}(e^{-\beta\lambda}/Z_s)\sum_{\nu}\,&[\Sigma^{(2)}_{\text{se}\uparrow}(i(\nu+\nu_0))\\&+\Sigma^{(2l)}_{\text{se}\uparrow}(i(\nu+\nu_0))-\Sigma^{(2)}_{\text{se}\downarrow}(i\nu)\end{aligned}$$

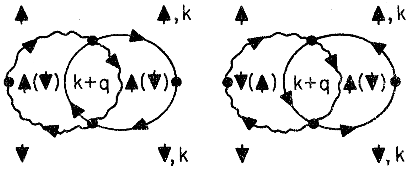

FIG. 12. Off-diagonal transfer of magnetization.

$$-\Sigma^{(2)}_{\mathrm{se}\downarrow}(i\nu)]C^0_\uparrow(i(\nu+\nu_0))\,C^0_\downarrow(i\nu)\ . \qquad (5.3)$$

Performing the sum over the interval frequency in the same way as in Sec. III, and then equating Eqs. (5.2) and (5.3), the following *exact* relationship is obtained:

$$[1/g_e^2(\omega_s-\omega_0-is)]\sum_p\{i\,\mathrm{Im}[\Sigma^{(2)}_{\mathrm{es}\uparrow}(p,\ \omega_0+is)]\,[\chi_p(\omega_0+is)-\chi^0_{p\uparrow}]+i\,\mathrm{Im}[\Sigma^{(2)}_{\mathrm{es}\downarrow}(p,\ \omega_0+is)]\,[\chi_p(\omega_0+is)-\chi^0_{p\downarrow}]$$

$$+\,\mathrm{Re}[\Sigma^{(2)}_{\mathrm{es}\uparrow}(p,\ \omega_0)+\Sigma^{(2)}_{\mathrm{es}\downarrow}(p,\ \omega_0)]\,\chi_p(\omega_0+is)-\Xi_{\mathrm{es}}(\omega_0)\}$$

$$=[1/g_s^2(\omega_e-\omega_0-is)]\{i\,\mathrm{Im}\Sigma^{(2)}_{\mathrm{se}\uparrow}(\omega_0+is)\,[\chi_s(\omega_0+is)-\chi^0_{s\uparrow}]+\mathrm{Im}\Sigma^{(2)}_{\mathrm{se}\downarrow}(\omega_0+is)\,[\chi_s(\omega_0+is)-\chi^0_{s\downarrow}]$$

$$+\,\mathrm{Re}[\Sigma^{(2)}_{\mathrm{se}\uparrow}(\omega_0)+\Sigma^{(2)}_{\mathrm{se}\downarrow}(\omega_0)]\,\chi_s(\omega_0+is)-\Xi_{\mathrm{se}}(\omega_0)\}\ , \qquad (5.4)$$

where the definitions of Sec. III have been used, and where

$$\chi_p(\omega_0+is)=(\tfrac{1}{2}g_e^2)\,(f^+_{p\downarrow}-f^+_{p\uparrow})/(\omega_e-\omega_0)\ , \qquad (5.5a)$$

$$\chi_s(\omega_0+is)=(g_s^2/2Z_s)\,(e^{-\beta\omega_s/2}-e^{\beta\omega_s/2})/(\omega_s-\omega_0)\ . \qquad (5.5b)$$

Equation (5.4) represents a general relationship between the second-order self-energies. Again, for sufficiently high temperatures ($\epsilon_F \gg kT \gg \omega_s, \omega_0$) the following two equations for the real and imaginary parts are obtained:

$$[1/g_e^2(\omega_s-\omega_0-is)]\,\mathrm{Im}\Sigma^{(2)}_{\mathrm{es}}(\omega_0+is)\,[\chi_e(\omega_0+is)-\chi^0_e]$$

$$=[1/g_s^2(\omega_e-\omega_0-is)]$$

$$\times\mathrm{Im}\Sigma^{(2)}_{\mathrm{se}}(\omega_0+is)\,[\chi_s(\omega_0+is)-\chi^0_s], \qquad (5.6)$$

$$[1/g_e^2(\omega_s-\omega_0-is)]\,[\mathrm{Re}\Sigma^{(2)}_{\mathrm{es}}(\omega_0)\,\chi_e(\omega_0+is)-\Xi_{\mathrm{es}}(\omega_0)]$$

$$=[1/g_s^2(\omega_e-\omega_0-is)]$$

$$\times[\mathrm{Re}\Sigma^{(2)}_{\mathrm{se}}(\omega_0)\,\chi_s(\omega_0+is)-\Xi_{\mathrm{se}}(\omega_0)]\ , \qquad (5.7)$$

where

$$\chi_e(\omega_0+is)=\chi^0_e\ \omega_e(\omega_e-\omega_0-is)^{-1},$$

$$\chi_s(\omega_0+is)=\chi^0_s\,\omega_s(\omega_s-\omega_0-is)^{-1}$$

and where all other quantities have been defined in Secs. III and IV. Now, the difference $\chi_e(\omega_0+is)-\chi^0_e$ gives the transient part of the dynamic transverse susceptibility,

$$\chi_e(\omega_0+is)-\chi^0_e=\omega_0\chi^0_e(\omega_e-\omega_0-is)^{-1},$$

and for sufficiently high temperatures the same holds for the local moments

$$\chi_s(\omega_0+is)-\chi^0_s=\omega_0\chi^0_s(\omega-\omega_0-is)^{-1}.$$

Equation (5.6) reduces then to Eq. (5.1). However, the general relationship (5.4) is valid even at low temperatures.

It is seen that the so-called detailed-balance relationship for the relaxation rates is a dynamical relationship [at $\omega_0=0$ both sides of (5.6) reduce to zero]. The diagrams in Fig. 12, from which the relationship has been calculated, have associated with them one longitudinal and one transversal vertex in each diagram. Longitudinal or transversal vertices alone *are not* sufficient for these off-diagonal second-order-in-J couplings in the vertex equations. This detailed-balance relation is thus a relation involving the full relaxation rates, both longitudinal (S^z, σ^z) and transverse (S^x, σ^x, S^y, σ^y).

VI. DISCUSSION AND COMPARISON

It is easy to identify standard contributions to the relaxation rates according to the spin matrix elements involved in appropriate vertices. The frequency modulation part $1/T_2'$ of the localized-spin-resonance linewidth as calculated by the Redfield formalism based on the evaluation of the rate of change of the density matrix[18] yields, at high temperatures,

$$1/T_2'\simeq|\langle S|\mathcal{H}_{\mathrm{es}}|S\rangle-\langle S-1|\mathcal{H}_{\mathrm{es}}|S-1\rangle|^2$$

$$\simeq 2i\pi(\rho J)^2kT\,|\langle S|\vec{S}_z|S\rangle-\langle S-1|\vec{S}_z|S-1\rangle|^2\ . \qquad (6.1a)$$

The vertex corrections to the spin susceptibility correspond to the crossterms in the expression (6.1a), while the one-fermion self-energies give rise to those contributions to Eq. (6.1a) which contain the matrix elements squared. The spin-flip part of the exchange Hamiltonian $\mathcal{H}_{\mathrm{es}}$ contributes to the self-energies only, and give the transverse contribution $1/T_1'$ to the linewidth,

$$1/T_1'\simeq|\langle S|\mathcal{H}_{\mathrm{es}}|S-1\rangle|^2+|\langle S-1|\mathcal{H}_{\mathrm{es}}|S\rangle|^2$$

$$\simeq i\pi(\rho J)^2kT(|\langle S|\vec{S}^+|S-1\rangle|^2+|\langle S-1|\vec{S}^-|S\rangle|^2)\ . \qquad (6.1b)$$

In Eqs. (6.1), the levels $|S\rangle$ and $|S-1\rangle$ are those between which the resonance experiment is performed.

The replacement of the spin levels in the above expressions by the conduction electrons levels $|k_\uparrow\rangle$ and $|k_\downarrow\rangle$, respectively, will give a similar expression for the conduction-electron counterpart. Thus, for $S=\frac{1}{2}$ after summing over all electrons,

$$1/T_2'\simeq i\pi\rho J^2|\langle\uparrow|\vec{S}_z|\uparrow\rangle-\langle\downarrow|\vec{S}_z|\downarrow\rangle|^2 \qquad (6.2a)$$

and

$$1/T_1' \simeq (\tfrac{1}{2} i\pi\rho J^2)(|\langle\uparrow|\vec{S}^{+}|\downarrow\rangle|^2 + |\langle\downarrow|\vec{s}^{-}|\uparrow\rangle|^2). \tag{6.2b}$$

These identifications become more involved for the higher spin magnitudes. For the local spin $S=\frac{1}{2}$, vertex corrections are found to double the longitudinal self-energies, yielding high-temperature rotational invariance of the relaxation rates $1/T_2$ and $1/T_1$.

The resonance condition is given by the denominator of the coupled susceptibility. Different resonance roots are found according to whether one is in bottlenecked or unbottlenecked limit.[19] The unbottlenecked region is characterized by the inequality $(1/T_{es}+1/T_{se}) \ll (1/T_{el}$ or $|\omega_s-\omega_e|)$; i.e., the time associated with the coupling between the two spin systems is relatively long as compared to the electronic-lattice relaxation and to the rate of dephasing (owing to the differences in the Zeeman energies) between the localized and conduction electrons. In this limit, the conduction electrons relax rapidly to the equilibrium polarization and two resonance roots are found. The width of the commonly observed resonance root associated with the local-moment magnetization is $1/T_{se}$ (the Korringa width) which is proportional to kT at high temperatures. For $1/T_{el} \gg 1/T_{es}+1/T_{se}$ the resonance frequency is $\omega_s(1+\lambda\chi_e^0)$, plus the relevant Kondo terms. This resonance is observable at liquid-helium temperatures for suitable alloys.

In the strong bottlenecked limit where $1/T_{es}+1/T_{se} \gg 1/T_{el}$ or $|\omega_e-\omega_s|$, a single sharp resonance is found with a width which may be substantially less than Korringa width $1/T_{se}$. In the extreme bottleneck, i.e., for $\omega_e=\omega_s$ and $1/T_{el}=0$, the resonance becomes a δ function at ω_s regardless of the temperature region, as can be seen from Eqs. (3.14) and (3.15). The Knight and Kondo shift are suppressed by the bottleneck.

It is claimed that the present methods correspond to the two-particle character of the response function, and represent the local moments correctly. In support of this statement, the fluctuation-dissipation theorem can be rederived using the present methods and shown to be of exactly the same form as presented by Izuyama *et al.*[10] The denominators of the two-particle self-energies (consistent with the energy conservation requirement for the processes involved in the relaxation of the system) are identical with those obtained by Orbach and Spencer[3] for corresponding quantities, using the direct equation-of-motion approach for the two-particle propagator.

The cross section for the spin-flip neutron scattering as calculated by Izuyama *et al.*[10] is essentially the spin-flip contribution to the local-moment relaxation rate. Their result can be recovered and the longitudinal equivalent calculated by expressing the appropriate integrands in a different way. The "electronic bubbles" corresponding to the $\vec{q}$-dependent electronic transverse susceptibilities $\chi_e^{+-}(\vec{q},\omega_0)$ and $\chi_e^{-+}(\vec{q},\omega_0)$ are involved in Fig. 5(c). The localized self-energy for the spin-fermion with the spin label $S=\frac{1}{2}$ can then be written in the following form:

$$\Sigma_{\mathrm{se}\uparrow}^{(2t)}(i(\nu+\nu_0)) = (J/N)^2 \sum_{q,\nu'} \frac{\chi_e^{+-}(q,-i\nu')}{\lambda-\frac{1}{2}\omega_s - i(\nu+\nu_0+\nu')}\ , \tag{6.3}$$

and a similar expression involving the susceptibility $\chi_e^{-+}(q,i\nu')$ for the self-energy of the spin fermion $S=-\frac{1}{2}$. Evaluation of these self-energies can be easily done using the strip-integration (i.e., the integration along the discontinuity cuts of the analytically continued integrand into the complex frequency plane) method.[20] Integration around the singularities of the integrand involved in summation in Eq. (6.3) yields, then,

$$\Sigma_{\mathrm{se}\uparrow}^{(2t)}(i(\nu+\nu_0)) = \left(\frac{J}{N}\right)^2 \frac{1}{2}\sum_q \left[-f^+\left(\lambda-\frac{\omega_s}{2}\right)\chi_e^{+-}\left(q,\ i(\nu+\nu_0)-\lambda+\frac{\omega_s}{2}\right) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\omega' b^+(\omega')\,\mathrm{Im}\chi_e^{+-}(q,-\omega'+is)}{\lambda-\frac{1}{2}\omega_s - i(\nu+\nu_0)-\omega'+is}\right]\ , \tag{6.4}$$

where b^+ is the usual Bose function, $b^+(x)=(e^{\beta x}-1)^{-1}$. The appropriate arguments for the *two-particle* self-energies are obtained as a result of the algebraic rearrangements described earlier in Sec. III and Appendix B. Owing to the final projection, the singularity of the spin propagator [first term in Eq. (6.4)] has vanishing residue, so that for the transverse contribution to the relaxation rate one obtains

$$\begin{aligned}&\mathrm{Im}[\Sigma_{\mathrm{se}\uparrow}^{(2t)}(\lambda-\tfrac{1}{2}\omega_s+\omega_0+is) - \Sigma_{\mathrm{se}\downarrow}^{(2t)}(\lambda+\tfrac{1}{2}\omega_s-\omega_0-is)]\\ &\quad = \tfrac{1}{2}(J/N)^2 \sum_q (e^{-\beta\omega_0}-1)^{-1}\,\mathrm{Im}[\chi_e^{+-}(q,\omega_0+is) + \chi_e^{-+}(q,\omega_0+is)]\ ,\end{aligned} \tag{6.5}$$

which is exactly the result obtained by Izuyama *et al.* [Ref. 10, Eq. (10), Appendix I]. This agreement confirms the correctness of the two-particle character of the relaxation rates as well as the projection method.

The corresponding expression for the longitudinal self-energy is

$$\begin{aligned}&2\,\mathrm{Im}[\Sigma_{\mathrm{se}\uparrow}^{(2l)}(\lambda-\tfrac{1}{2}\omega_s+\omega_0+is) - \Sigma_{\mathrm{se}\downarrow}^{(2l)}(\lambda+\tfrac{1}{2}\omega_s-\omega_0-is)]\\ &\quad = \tfrac{1}{2}(J/N)^2 \sum_q (e^{-\beta(\omega_0-\omega_s)}-1)^{-1}\end{aligned}$$

$$\times \mathrm{Im}[2\chi_e^z(q,\ \omega_0 - \omega_s + is)]\ , \quad (6.6)$$

where $\chi_e^z(q,\ \omega_0)$ is the longitudinal electronic susceptibility.

There is a numerical error[15] of factor 2 in the Orbach and Spencer[3] expression for the longitudinal electronic self-energy; also, the spatial averaging should become $c(\langle S_z^2\rangle - c\langle S_z\rangle^2)$ instead of their $c(\langle S_z^2\rangle - \langle S_z\rangle^2)$. The present authors believe that most of the decoupling in their paper can be justified on the grounds that they are exact factorizations to that order in the perturbation theory. The only exception is the decoupling

$$\langle\langle S_i^z S_j^-(t),\ S_j^+\rangle\rangle \simeq \langle S_i^z\rangle \langle\langle S_j^-(t),\ S_j^+\rangle\rangle \quad \text{for} \quad i = j\ .$$

If instead the $S = \frac{1}{2}$ result that

$$\langle\langle S_i^z S_j^-(t),\ S_j^+\rangle\rangle = -\tfrac{1}{2}\langle\langle S_j^-(t),\ S_j^+\rangle\rangle$$

is used, the structure of their equations remains the same and their self-energies appear to agree with those obtained here. Their electronic self-energies are also momentum dependent. They do not obtain terms equivalent to the Ξ's of the present paper, because they evaluated the expectation values of operators which form the inhomogeneous terms using the unperturbed Hamiltonian.

Spencer[13] has omitted self-energies corresponding to diagrams 6(f) of this paper and vertex corrections diagram 4c ii of his paper. Upon inclusion of these omitted diagrams, his self-energies would agree with those in this paper. The only discrepancies which remain are those appropriate to the localized-spin self-energies at very low temperature. This is believed to be due to his using a method where the singular self-energies of the form discussed in the Sec. II cause trouble.

Langreth and Wilkins have derived a set of Bloch equations, limited to the case of equal g factors and the semiclassical regime, where time and special variations are assumed to be slow. In this regime, and if the second-order real Kondo terms are ignored, their equations do agree with the present equations. However, the Kondo terms obtained here contain terms (which are *not* small) proportional to the rf frequency. Owing to the semiclassical nature of their method, the frequency dependence of these terms could not be determined by Langreth and Wilkins, although they chose "energy shells" in such a way that their shifts agree with ours at the resonance.

The low temperature problem associated with the momentum dependence of the electronic self-energies is suppressed in the work of Götze and Wölfe[5] owing to their *Ansatz* for the susceptibility, and their calculation was not carried to sufficient degree of accuracy to determine whether the destination vectors contain the molecular field.

In summary we wish to emphasize the following points.

(a) Without assuming slow time and spatial variations, and without *a priori* restrictions that the temperature be high, we derive a set of $S = \frac{1}{2}$ spin transport equations which are sufficiently accurate to be compared with Hasegawa's phenomenological theory. In particular, care is taken to include all of the terms which represent the "destination vectors" in the relaxation terms, *including* the internal-field term.

(b) The transport equations are derived by the well-known method involving the evaluation of a set of vertex equations. In connection with this method we have developed some new mathematical and manipulative techniques which we feel considerably simplify the construction of this type of theory, especially in the nonclassical regime.

(c) For equal g factors and sufficiently high temperatures ($kT > \omega_s$) and in the absence of the second-order real Kondo terms, the spin-transport equations reduce to Bloch equations designated by Hasegawa as "case B." When $g_s \neq g_e$, the present theory differs from that of Hasegawa and Cottet in that the scattering-in term in the local-moment equation, which is proportional to $1/T_{es}$, must be multiplied by the ratio g_s/g_e, while the scattering-in term in the conduction-electron equation must be multiplied by the ratio g_s/g_e. This reflects the fact that the exchange interaction conserves spin and *not* magnetization, and has the important effect of recovering the correct static limit for the individual susceptibilities $\chi_e^{+-}(\omega_0 = 0)$ and $\chi_s^{+-}(\omega_0 = 0)$. Because of their structure, the "Kondo g shifts" *cannot* be incorporated into the Bloch equation in any simple way.

(d) At low temperatures ($kT < \omega_s$), it is found that it is *not* possible to write a single linear equation for the total electronic susceptibility. This is because at low temperatures the strongly energy-dependent electronic spin-flip relaxation rate proportional to $\langle S_z\rangle$ is comparable to other, energy-independent, terms, and thus a separate equation must be written for each contribution to the electronic susceptibility that comes from electrons with a different momenta.

(e) In the low-temperature regime modifications to the semiclassical Bloch equations occur. For $kT < \omega_0$, the radio frequency, the local-moment susceptibility χ_s^0 which appears in the destination vectors, but not elsewhere, is modified. It is replaced by

$$\chi_{s\uparrow}^0 = \tfrac{1}{2} g_s^2 (e^{-\beta\omega_0} n_{s\downarrow}^0 - n_{s\uparrow}^0)(\omega_s - \omega_0)^{-1}$$

or

$$\chi_{s\downarrow}^0 = \tfrac{1}{2} g_s^2 (n_{s\downarrow}^0 - n_{s\uparrow}^0 e^{\beta\omega_0})(\omega_s - \omega_0)^{-1}\ ,$$

depending on which contribution to $1/T_{se}$ the relaxation term involves.

The well-known low-temperature frequency dependence of the widths and shifts, indicating retarded relaxation and non-Lorentzian absorption, are displayed. These widths are evaluated using undressed propagators, justifiable for sufficiently high temperatures such that kT is greater than the single-particle widths. In the limit of low temperatures, the self-energies must be evaluated self-consistently, and the reported dramatic non-Lorentzian shapes of resonance lines[21] at very low temperatures seem questionable to us.

(f) The algebraic equality of the scattering-out and the cross-magnetization relaxation rates in the pair of Bloch equations (4.10) and (4.11) is dependent upon rotational symmetry of the interaction Hamiltonian, as discussed in Sec. III. If the exchange were to reflect the axial symmetry, then it would be of the form

$$-(1/N)\sum_{q,j} e^{-i\vec{q}\cdot\vec{R}_j}\left[J_{\parallel}S^z\sigma_q^z+\tfrac{1}{2}J_{\perp}(S^+\sigma_q^-+S^-\sigma_q^+)\right].$$

Then the local-moment scattering-out rate is given by

$$(1/T_{se})_{out}=2\pi[(\rho J_{\perp})^2+(\rho J_{\parallel})^2]kT\ ,$$

the expected result. However, the corresponding scattering-in term in the electronic equation is given by

$$(1/T_{se})_{in}=4\pi\rho^2J_{\perp}J_{\parallel}kT\ .$$

Clearly, $(1/T_{se})_{out}$ and $(1/T_{se})_{in}$ are equal only if $J_{\parallel}=J_{\perp}$. In order to recover the simple form of the Bloch equations, the relaxation rate $(1/T_{se})_{out}$ must be replaced by $(1/T_{se})_{in}$ and a contribution

$$2\pi\rho^2(J_{\perp}-J_{\parallel})^2kT$$

is added to the spin-lattice relaxation rate $1/T_{se}$. Similar modifications must be made to the electronic relaxation rates. These modifications reflect the fact that the total magnetization commutes with the isotropic exchange interaction, and thus for the nonisotropic case, and even in the absence of other relaxation processes, the well-known bottleneck result that the exchange effects exactly cancel no longer obtains.

(g) For the first time a microscopic method for the calculation of the detailed balance relationships is given. In the high-temperature region the detailed balance reduces to the relationship

$$g_e^2\chi_s^0/T_{se}=g_s^2\chi_e^0/T_{es}\ .$$

Also, for the first time a relationship between the real parts of the second-order self-energies is given.

(h) In the text the conduction-electron–lattice relaxation is introduced by adding to the exchange Hamiltonian a term representing a random spacially inhomogeneous magnetic field. In Appendix D it is shown that the spin-orbit-scattering amplitude $iB[\vec{k}\times\vec{k}']\cdot\vec{\sigma}$ leads to an exactly similar term, but only if the total electronic-impurity-scattering cross section is much larger than that which arises from the spin-orbit scattering alone. This result is not new; however, the present calculation does show clearly the reason for this restriction, namely, that the large impurity scattering is required to rapidly redistribute electronic momentum vector over the Fermi sphere, thereby averaging out the directionally dependent components of the spin-orbit scattering.

(i) As mentioned in (b), some new diagrammatic methods have been developed in this paper. Using these methods (cf. the earlier methods used for example in Refs. 8 and 20), it is possible to show the following:

i. A set of coupled vertex equations may be exactly reduced to a set of coupled linear equations.

ii. It is possible to deduce exactly the arguments ("energy shells") appropriate to the self-energy expressions which appear in the susceptibility functions. The resulting "two-particle" self-energies reflect correctly the energy conservation requirements of the two-particle processes to which they correspond.

iii. These *two-particle* self-energies are in one-to-one correspondence with another set of terms (here called "inhomogeneous"). It is argued in the text and appendices that these terms which arise in a completely natural way are typically of the form corresponding to an inhomogeneous term in the exact equations of motion for the susceptibility function.

iv. It is possible to cast into the same form the problem of evaluating the terms involving self-energies and leading to scattering-out terms, and the terms arising from the last two diagrams in the vertex equation illustrated in Fig. 7 which couple the local-moment and conduction-electron vertex equations, and lead to scattering-in terms. The vertex-correction terms can also be treated in a similar fashion. This not only halves the work involved, since only scattering-out terms need be evaluated, but also demonstrates in a very clear fashion the well-known result that, although both spin-dependent and spin-independent scattering broadens each level, the spin-dependent vertex correction doubles the longitudinal-spin relaxation rate, while the spin-independent correction exactly cancels the spin-independent relaxation rate.

ACKNOWLEDGMENTS

The authors wish to thank Professor R. Orbach for his interest and assistance. The authors also wish to thank H. Shiba for suggesting the method used to perform the configuration averages and M. B. Walker, who pointed out to us the importance of large impurity scattering in simplifying the treatment of the spin-orbit interaction.

APPENDIX A

The most formidable expression resulting from the first step of the algebraic rearrangement seems to be that in Eq. (3.5a), which has arisen from the second-order-in-J off-diagonal coupling in the electron vertex equation in Fig. 7. These, the fourth and fifth terms in Fig. 7, are represented by algebraic expressions as follows:

$$\begin{aligned}
&\sum_{\nu,\nu',\nu'',\sigma,q} \{(g_e e^{\beta\lambda}/\sqrt{2} Z_s) G_{p\uparrow}(i(\nu+\nu_0)) G_{p\downarrow}(i\nu)(cJ^2/2N)[G_{p+q,\sigma}(i(\nu+\nu_0)) C_\sigma(i(\nu'+\nu''))\\
&\quad - G_{p+q,-\sigma}(i(\nu+\nu_0-\nu'')) C_\sigma(i(\nu'+\nu''))]\Lambda_s(i\nu', i(\nu'+\nu_0))\}\\
&= (\omega_e - i\nu_0) \sum_{\nu,\dots} (g_e e^{\beta\lambda}/2g_s Z_s)(J/N)^2 [G_{p\downarrow}(i\nu) - G_{p+q\uparrow}(i(\nu+\nu_0))][G_{p+q,\sigma}(i(\nu+\nu'')) C_\sigma(i(\nu'+\nu''))\\
&\quad - G_{p+q,-\sigma}(i(\nu+\nu_0-\nu'')) C_\sigma(i(\nu'+\nu''))](cNg_s/\sqrt{2})\Lambda_s(i\nu', i(\nu'+\nu_0))\\
&\qquad + (\omega_e - i\nu_0)^{-1} \sum_{\nu,\dots} [\Sigma_{es\uparrow}(i(\nu+\nu_0)) + \Sigma_{el\uparrow}(i(\nu+\nu_0)) - \Sigma_{es\downarrow}(i\nu) - \Sigma_{el\downarrow}(i\nu)]\{\cdots\}\,,
\end{aligned} \tag{A1}$$

where the identity Eq. (3.1) has been used, and the last parenthesis $\{\cdots\}$ represent the starting expression in the same parenthesis on the right of the summation symbols. Summation over the internal frequency variables $i\nu$, $i\nu''$ finally yields

$$\begin{aligned}
&-\frac{g_e}{g_s(\omega_e - i\nu_0)} \frac{e^{\beta\lambda}}{2Z_s}\left(\frac{J}{N}\right)^2 \sum_{\nu,\nu',\nu'',q,\sigma} [G_{p\uparrow}(i(\nu+\nu_0)) G_{p+q,\sigma}(i(\nu+\nu'')) C_\sigma(i(\nu'+\nu''))\\
&\quad + G_{p\downarrow}(i\nu) G_{p+q,-\sigma}(i(\nu+\nu_0-\nu'')) C_\sigma(i(\nu'+\nu'')) - G_{p\downarrow}(i\nu) G_{p+q,\sigma}(i(\nu+\nu'')) C_\sigma(i(\nu'+\nu''))\\
&\quad - G_{p\uparrow}(i(\nu+\nu_0)) G_{p+q,-\sigma}(i(\nu+\nu_0-\nu'')) C_\sigma(i(\nu'+\nu''))]\frac{cg_sN}{\sqrt{2}}\Lambda_s(i\nu', i(\nu'+\nu_0))\\
&\quad + \frac{1}{\omega_e - i\nu_0}\sum_{\nu,\dots}[\Sigma_{es\uparrow}(i(\nu+\nu_0)) + \Sigma_{e\uparrow}(i(\nu+\nu_0)) - \Sigma_{es\downarrow}(i\nu) - \Sigma_{el\downarrow}(i\nu)]\{\cdots\}\\
&= \frac{-g_e}{g_s(\omega_e - i\nu_0)} \frac{e^{\beta\lambda}}{2Z_s}\left(\frac{J}{N}\right)^2 \sum_{\nu',q,\sigma}\left(\frac{f^+(\lambda+\frac{1}{2}\sigma\omega_s) f^+_{p\uparrow} f^-_{p+q,\sigma} + f^-(\lambda+\frac{1}{2}\sigma\omega_s) f^-_{p\uparrow} f^+_{p+q,\sigma}}{\epsilon_{p\uparrow} - \epsilon_{p+q,\sigma} + \lambda + \frac{1}{2}\sigma\omega_s - i(\nu'+\nu_0)}\right.\\
&\quad + \frac{f^+(\lambda-\frac{1}{2}\sigma\omega_s) f^+_{p+q,\sigma} f^-_{p\downarrow} + f^-(\lambda+\frac{1}{2}\omega_s) f^-_{p+q,\sigma} f^+_{p\downarrow}}{\epsilon_{p+q,\sigma} - \epsilon_{p\downarrow} + \lambda - \frac{1}{2}\sigma\omega_s - i(\nu'+\nu_0)} - \frac{f^+(\lambda+\frac{1}{2}\sigma\omega_s) f^+_{p\downarrow} f^-_{p+q,\sigma} + f^-(\lambda+\frac{1}{2}\sigma\omega_s) f^-_{p\downarrow} f^+_{p+q,\sigma}}{\epsilon_{p\downarrow} - \epsilon_{p+q,\sigma} + \lambda + \frac{1}{2}\sigma\omega_s - i\nu'}\\
&\quad \left. - \frac{f^+(\lambda-\frac{1}{2}\sigma\omega_s) f^+_{p+q,\sigma} f^-_{p\uparrow} + f^-(\lambda-\frac{1}{2}\sigma\omega_s) f^-_{p+q,\sigma} f^+_{p\uparrow}}{\epsilon_{p+q,\sigma} - \epsilon_{p\uparrow} + \lambda - \frac{1}{2}\sigma\omega_s - i\nu'}\right)\frac{cg_sN}{\sqrt{2}}\Lambda_s(i\nu', i(\nu'+\nu_0)) + (\omega_e - i\nu_0)^{-1}\sum_{\nu,\dots}[\cdots]\{\cdots\}
\end{aligned} \tag{A2}$$

where, for summation purposes, the dressed propagators have been replaced by their "bare" values.

The first bracketed term in Eq. (A2) is the explicitly written out expression in Eq. (3.5a) in the text, and the second terms, a contribution to the full vertex $\Lambda_p(i\nu, i(\nu+\nu_0))$, is incorporated into the second term on the right-hand side of Eq. (3.5a).

After an additional summation over the momenta label and the spin σ, the first term of Eq. (A2) reduces simply to

$$\begin{aligned}
&[-g_e/g_s(\omega_e - i\nu_0)]\sum_{\nu}(e^{\beta\lambda}/Z_s)\\
&\quad\times[\Sigma^{(2)}_{se\uparrow}(i(\nu+\nu_0)) + \Sigma^{(2l)}_{se\uparrow}(i(\nu+\nu_0))\\
&\qquad - \Sigma^{(2)}_{se\downarrow}(i\nu) - \Sigma^{(2l)}_{se\downarrow}(i\nu)]\\
&\qquad\quad\times(cNg_s/\sqrt{2})\Lambda_s(i\nu, i(\nu+\nu_0))\,,
\end{aligned} \tag{A3}$$

which is the expression in Eq. (3.5b) in the text. Thus, proceeding either algebraically or, fully equivalently, diagrammatically as pictured in Fig. 9, all terms in the vertex equation Fig. 7 are expressible exactly in terms of the self-energy-like quantities solely. Had the summation been performed over the dressed propagators, the self-energies would have been dressed, i.e., evaluated in a self-consistent manner.

APPENDIX B

By the second step of the algebraic rearrangement, resulting in replacement equations (3.6)–(3.13), the self-energies of a two-particle character are factored out of the summation over internal frequencies appropriate to the dynamic susceptibility. They are expressed in terms of the one-particle self-energies with certain arguments, which can be interpreted in terms of the energy conservation requirements for the two-particle processes. They correspond to the believable arguments of the two-particle self-energies as obtained by direct calculations by Orbach and Spencer.[3] The inhomogeneous terms which arise are appropriate to the propagators associated with the intermediate states involved in the corresponding self-energies.

The summations to be carried out involve terms of the type

$$(cg_s e^{\beta\lambda}/\sqrt{2}Z_s)\sum_{\nu}\{\Sigma^{(2l)}_{\text{se}\uparrow}(i(\nu+\nu_0))\,\Lambda_s(i\nu,\, i(\nu+\nu_0))\}\,. \tag{B1}$$

In general, the second-order one-particle self-energies [Eqs. (3.1) and (3.2)] display frequency-dependent denominators, such as

$$\Sigma^{(2l)}_{\text{se}\uparrow}(i(\nu+\nu_0)) = \sum_{p,q,\sigma}\frac{N(\Sigma^{(2l)}_{\text{se}\uparrow})}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}\,, \tag{B2}$$

where $N(\Sigma^{(2l)}_{\text{se}\uparrow})$ stands for the appropriate combination of the Fermi functions in the numerator of self-energy $\Sigma^{(2l)}_{\text{se}\uparrow}$. A similar expression to Eq. (B2) is written for all other self-energy contributions.

The vertex function in Eq. (B1) can be written

$$\Lambda_s(i\nu,\, i(\nu+\nu_0)) = C_\uparrow(i(\nu+\nu_0))C_\downarrow(i\nu)\,\Gamma_s(i\nu,\, i(\nu+\nu_0))\,, \tag{B3}$$

where

$$\Gamma_s(i\nu,\, i(\nu+\nu_0)) = 1+(J/N)\sum_{p,\nu'}\Lambda_p(i\nu',\, i(\nu'+\nu_0)) + (J/N)^2(\cdots) \tag{B4}$$

and where the last term on the right-hand side is of the order J^2 and represents the vertex correction and the second-order off-diagonal diagrams.

The two-particle character of the self-energies is obtained by applying the identity (3.3) to the product of the denominator of the self-energy in Eq. (B1) and the propagator in Eq. (B3) which has spin opposite the self-energy itself. The particular exemplified term above then becomes

$$\begin{aligned}
&\frac{cg_s e^{\beta\lambda}}{\sqrt{2Z_s}}\sum_{\nu,p,q,\sigma}\frac{N(\Sigma^{(2l)}_{\text{se}\uparrow})}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0}\left(C_\downarrow(i\nu)-\frac{1+\Sigma_{\text{se}\downarrow}(i\nu)C_\downarrow(i\nu)}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}\right)C_\uparrow(i(\nu+\nu_0))\,\Gamma_s(i\nu,\, i(\nu+\nu_0))\\
&=\frac{cg_s e^{\beta\lambda}}{\sqrt{2}Z_s}\left\{\Sigma^{(2l)}_{\text{se}\uparrow}(i\nu_0+\lambda-\tfrac{1}{2}\omega_s)\sum_{\nu}\Lambda_s(i\nu,\, i(\nu+\nu_0))-\sum_{\nu,p,q,\sigma}\frac{N(\Sigma^{(2l)}_{\text{se}\uparrow})\,[1+\Sigma_{\text{se}\downarrow}(i\nu)C_\downarrow(i\nu)]}{(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0)\,[\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)]}\right.\\
&\qquad\left.\times C_\uparrow(i(\nu+\nu_0))\frac{g_s}{\sqrt{2}}\left[1+\frac{2J}{g_e g_s N}\sum_{p'}\frac{g_e}{\sqrt{2}}\Lambda_{p'}(i\nu',\, i(\nu'+\nu_0))+\left(\frac{J}{N}\right)^2(\ldots)\right]\right\}.
\end{aligned} \tag{B5}$$

For a single ν, p, q, and σ one now transfers the common denominator from the right- to the left-hand side, and after using again the identity (3.3) for the product of the denominators of the self-energies $\Sigma^{(2l)}_{\text{se}\uparrow}(i(\nu+\nu_0))$ and $\Sigma^{(i\nu)}_{\text{se}\downarrow}$, one then obtains

$$\begin{aligned}
&(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0)\,\frac{cg_s e^{\beta\lambda}}{2Z_s}\left(\frac{N(\Sigma^{2l}_{\text{se}\uparrow})}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}\,\Lambda_s(i\nu,\, i(\nu+\nu_0))\right)\\
&=\frac{cg_s e^{\beta\lambda}}{\sqrt{2}Z_s}\,N(\Sigma^{2l}_{\text{se}\uparrow})\left\{\Lambda_s(i\nu,\, i(\nu+\nu_0))-\frac{C_\uparrow(i(\nu+\nu_0))\,[1+(2J/g_e g_s N)\sum_{p'}(g_e/\sqrt{2})\Lambda_{p'}(i\nu',\, i(\nu'+\nu_0))+(J/N)^2(\cdots)]}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}\right.\\
&\quad-\left[\frac{\Sigma^{(1)}_{\text{se}\downarrow}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}+\sum_{p',q',\sigma'}\frac{N(\Sigma^{(2l)}_{\text{se}\downarrow})}{\epsilon_{p'+q',\sigma'}-\epsilon_{p',\sigma'}-\omega_s-\epsilon_{p+q,\sigma}+\epsilon_{p,\sigma}+i\nu_0}\right.\\
&\quad\times\left(\frac{1}{\lambda+\frac{1}{2}\omega_s+\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}-i(\nu+\nu_0)}-\frac{1}{\epsilon_{p'+q',\sigma'}-\epsilon_{p',\sigma'}+\lambda-\frac{1}{2}\omega_s-i\nu}\right)\\
&\quad\left.+\sum_{p',\sigma'}\frac{N(\Sigma^{(2t)}_{\text{se}\downarrow})}{\epsilon_{p'+q'\downarrow}-\epsilon_{p'\uparrow}-\epsilon_{p+q,\sigma}+\epsilon_{p,\sigma}+i\nu_0}\left(\frac{1}{\lambda+\frac{1}{2}\omega_s+\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}-i(\nu+\nu_0)}-\frac{1}{\epsilon_{p'+q'\downarrow}-\epsilon_{p'\uparrow}+\lambda+\frac{1}{2}\omega_s-i\nu}\right)\right]\\
&\qquad\left.\times\Lambda_s(i\nu,\, i(\nu+\nu_0))\right\}\,.
\end{aligned} \tag{B6}$$

On the right-hand side of Eq. (B6), a contribution involving the self-energy $\Sigma_{\text{se}\uparrow}$ with a fixed argument, that is,

$$\Sigma_{\text{se}\downarrow}(-i\nu_0+\epsilon_{p+q}-\epsilon_p+\lambda+\tfrac{1}{2}\omega_s)$$

can be identified. First, these terms are transferred to the left-hand side of Eq. (B6), which yields a bracketed term

$$[\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0+\Sigma_{\text{se}\downarrow}(-i\nu_0+\epsilon_{p+q}-\epsilon_p+\lambda+\tfrac{1}{2}\omega_s)]\,,$$

and secondly, the whole bracketed term is transferred back to the right-hand side. Then, upon performing final summations over ν, p, q, and σ, and rewriting the self-energy numerators explicitly, one obtains

$$
\begin{aligned}
&(cg_s e^{\beta\lambda}/\sqrt{2}Z_s)\sum_{\nu}\left[\Sigma^{(2l)}_{\mathrm{se}\uparrow}(i(\nu+\nu_0))\,\Lambda_s(i\nu,\,i(\nu+\nu_0))\right]\\
&\to \frac{e^{\beta\lambda}}{Z_s}\,\Sigma^{(2l)}_{\mathrm{se}\uparrow}(i\nu_0-\tfrac{1}{2}\omega_s+\lambda-\Sigma_{\mathrm{se}\downarrow}(-i\nu_0+\epsilon_{p+q}-\epsilon_p+\lambda+\tfrac{1}{2}\omega_s))\sum_{\nu}\frac{cg_s}{\sqrt{2}}\,\Lambda_s(i\nu,\,i(\nu+\nu_0))\\
&\quad-\frac{cg_s e^{\beta\lambda}}{\sqrt{2}Z_s}\sum_{p,q,\sigma}\left(\frac{J}{2N}\right)^2\frac{f^{+}(\lambda-\frac{1}{2}\omega_s)f^{+}_{p+q,\sigma}f^{-}_{p,\sigma}+f^{-}(\lambda+\frac{1}{2}\omega_s)f^{-}_{p+q,\sigma}f^{+}_{p,\sigma}}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\omega_s-i\nu_0+\Sigma_{\mathrm{se}\downarrow}(-i\nu_0+\epsilon_{p+q}-\epsilon_p+\lambda+\frac{1}{2}\omega_s)}\\
&\quad\times\sum_{\nu}\Bigg[(g_s/\sqrt{2})\,\frac{C_{\uparrow}(i(\nu+\nu_0))\left[1+(2J/g_eg_sN)\sum_{p'}(g_e/\sqrt{2})\,\Lambda_{p'}(i\nu',\,i(\nu'+\nu_0))+(J/N)^2(\dots)\right]}{\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma}+\lambda+\frac{1}{2}\omega_s-i(\nu+\nu_0)}\\
&\qquad-\sum_{p',q',\sigma'}\Bigg(\frac{(J/2N)^2\left[f^{+}(\lambda-\frac{1}{2}\omega_s)f^{+}_{p'+q',\sigma'}f^{-}_{p',\sigma'}+f^{-}(\lambda-\frac{1}{2}\omega_s)f^{-}_{p'+q',\sigma'}f^{+}_{p',\sigma'}\right.}{(\lambda-\frac{1}{2}\omega_s+\epsilon_{p'+q',\sigma'}-\epsilon_{p',\sigma'}-i\nu)\,(\epsilon_{p'+q',\sigma'}-\epsilon_{p',\sigma'}-\omega_s-\epsilon_{p+q,\sigma}+\epsilon_{p,\sigma}+i\nu_0)}\\
&\qquad\quad-\frac{(J/N)^2\left[f^{+}(\lambda+\frac{1}{2}\omega_s)f^{+}_{p+q\downarrow}f^{-}_{p\uparrow}+f^{-}(\lambda+\frac{1}{2}\omega_s)f^{-}_{p+q\downarrow}f^{+}_{p\uparrow}\right]}{(\lambda+\frac{1}{2}\omega_s+\epsilon_{p'+q'\downarrow}-\epsilon_{p'\uparrow}-i\nu)\,(\epsilon_{p'+q'\downarrow}-\epsilon_{p'\uparrow}-\epsilon_{p+q,\sigma}+\epsilon_{p,\sigma}+i\nu_0)}\Bigg)\Lambda_s(i\nu,\,i(\nu+\nu_0))\Bigg]\,,
\end{aligned}
\tag{B7}
$$

which is Eq. (3.6) in the text.

The interpretation of the second-order terms, as well as the products of the third order in J, is given in the text.

From their explicit form, as displayed for some of them in Eq. (B7), it is possible to infer the meaning of the fourth-order terms. The numerator and denominator of the very last fourth-order term in Eq. (B7) can be identified with the energy conservation requirements of the overlapping (in time) scattering processes, occurring to each of the two particles involved in the two-particle propagator, as illustrated in Fig. 13(a). Such a diagram can be interpreted as an "indirect" fourth-order two-particle self-energy, similar to the "direct" two-particle self-energies which arise from the single-particle self-energy illustrated in Fig. 13(b). Since the latter self-energies were not included in the analysis from the beginning, it is consistent with the declared degree of accuracy to omit these fourth-order terms.

The fourth-order terms which arise from the overlap of the self-energies and second-order *off-diagonal* vertices [therefore contributing in Eq. (B7) a term proportional to $(J/N)^4\Lambda_p(i\nu,\,i(\nu+\nu_0))$, which is not explicitly displayed, but hinted by the parenthesis $(J/N)^2(\cdots)$], if evaluated, would make second-order contributions to the internal field. Overlap of the scattering processes, represented by the vertex corrections and the self-energies, gives effects in exact parallel to those arising from the overlap of two self-energies, displayed in Eq. (B7) and discussed above.

Following the same procedure for all other terms analogical to Eq. (B1), the corresponding equations, similar to Eq. (B7), are derived. Upon taking limit $\lambda\to\infty$ and using the approximations as described in the text, the replacement equations (3.10)–(3.13) are obtained, where the values of the inhomogeneous terms involved are as follows:

$$\Xi^{(2l)}_{\mathrm{se},\sigma}(\lambda-\sigma(\tfrac{1}{2}\omega_s-i\nu_0))=\frac{cg_s^2}{2}\,\mathrm{sgn}\sigma\sum_{p,q,\sigma'}\frac{(J/2N)^2n^0_{s,\sigma}\,(f^{+}_{p+q,\sigma'}-f^{+}_{p,\sigma'})/(\epsilon_{p+q,\sigma'}-\epsilon_{p,\sigma'})}{\epsilon_{p+q,\sigma'}-\epsilon_{p,\sigma'}+\omega_s-i\nu_0}\,,$$

$$\Xi^{(2t)}_{\mathrm{se},\sigma}(\lambda-\sigma(\tfrac{1}{2}\omega_s-i\nu_0))=\frac{cg_s^2}{2}\,\mathrm{sgn}\sigma\sum_{p,q}\frac{(J/N)^2\,(f^{-}_{p+q\uparrow}f^{+}_{p\downarrow}n^0_{s\uparrow}-f^{+}_{p+q\uparrow}f^{-}_{p\downarrow}n^0_{s\downarrow})/(\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-\omega_s)}{\epsilon_{p+q\uparrow}-\epsilon_{p\downarrow}-i\nu_0}\,,$$

$$\Xi^{(2l)}_{\mathrm{es},p,\sigma}(\epsilon_{p,-\sigma}+i\sigma\nu_0)=\frac{cg_e^2J^2}{2N}\,\mathrm{sgn}\sigma\langle S_z^2\rangle\sum_{q}\frac{(f^{+}_{p,\sigma}-f^{+}_{p+q,\sigma})/(\epsilon_{p+q}-\epsilon_p)}{\epsilon_{p+q,\sigma}-\epsilon_{p,-\sigma}-i\sigma\nu_0}\,,$$

$$\Xi^{(2t)}_{\mathrm{es},p,\sigma}(\epsilon_{p,-\sigma}+i\sigma\nu_0)=\frac{cg_e^2J^2}{2N}\sum_{q}\frac{\langle S_x^2+S_y^2\rangle\,(f^{+}_{p,\sigma}-f^{+}_{p+q,-\sigma})-\mathrm{sgn}\sigma\langle S_z\rangle\,(f^{+}_{p,\sigma}f^{-}_{p+q,-\sigma}+f^{-}_{p,\sigma}f^{+}_{p+q,-\sigma})}{(\epsilon_{p+q,-\sigma}-\epsilon_{p,\sigma}+\sigma\omega_s)\left[\epsilon_{p+q}-\epsilon_p+\sigma(\omega_s-i\nu_0)\right]}\,,$$

$$\Xi^{(2l)}_{\mathrm{el},p,\sigma}(\epsilon_{p,-\sigma}+i\sigma\nu_0)=\frac{c'g_e^2H^2}{2N}\operatorname{sgn}\sigma\sum_q\frac{(\langle h_z^2\rangle-c'\langle h_z\rangle^2)(f^+_{p,\sigma}-f^+_{p+q,\sigma})/(\epsilon_{p+q,\sigma}-\epsilon_{p,\sigma})}{\epsilon_{p+q,\sigma}-\epsilon_{p,-\sigma}-i\sigma\nu_0}\,,$$

$$\Xi^{(2t)}_{\mathrm{el},p,\sigma}(\epsilon_{p,-\sigma}+i\sigma\nu_0)=\frac{c'g_e^2H^2}{2N}\sum_q\frac{\langle h_x^2+h_y^2\rangle(f^+_{p,\sigma}-f^+_{p+q,-\sigma})/(\epsilon_{p+q,-\sigma}-\epsilon_{p,\sigma})}{\epsilon_{p+q}-\epsilon_p-i\sigma\nu_0}\,. \tag{B8}$$

All other quantities in the replacement equations (3.10)–(3.13) are defined in the text.

Interpretation of the terms Ξ's and Σ's goes along the same lines as for the terms $\Xi_\uparrow^{(2l)}$ and $\Sigma_\uparrow^{(2l)}$ in the text. The longitudinal parts of the local-spin self-energies have associated with them as an intermediate state the propagator $\langle a^\dagger_{p+q,\sigma}a_{p\sigma}\times S^+(\tau)S^-\rangle$, the transversal parts the propagator $\langle a^\dagger_{p+q\uparrow}a_{p\downarrow}S^z(\tau)S^-\rangle$. Similarly, for the electronic-local-spin self-energies, the intermediate states associated with the longitudinal and transversal contributions are characterized by the propagators $\langle S^z\sigma_q^+(\tau)\sigma^-\rangle$ and $\langle S^+\sigma_q^z(\tau)\sigma^-\rangle$, respectively. Finally, for the electronic-lattice terms the propagators $\langle h_z\sigma_q^+(\tau)\sigma^-\rangle$ and $\langle h^+\sigma_q^z(\tau)\sigma^-\rangle$ are relevant.

APPENDIX C

Low-temperature evaluation (ω_0, $\omega_e \ll kT$) of the quantities appearing in Eqs. (3.20)–(3.22) yields

$$\Sigma^{(2l)}_{\mathrm{se}\uparrow}(\omega_0+is)=(\rho J)^2\left[(\omega_0-\omega_s)\ln\left|D/(\omega_0-\omega_s)\right|+(2D-\omega_0+\omega_s)\ln2+i\pi(\omega_0-\omega_s)/(1-e^{-\beta(\omega_0-\omega_s)})\right],$$

$$\Sigma^{(2t)}_{\mathrm{se}\uparrow}(\omega_0+is)=(\rho J)^2\left[\omega_0\ln\left|D/\omega_0+(2D-\omega_0+\omega_e)\ln2+i\pi\omega_0/(1-e^{-\beta\omega_0})\right],$$

$$\Sigma^{(2l)}_{\mathrm{se}\downarrow}(\omega_0+is)=(\rho J)^2\left[(\omega_0-\omega_s)\ln\left|D/\omega_0-\omega_s\right|-(2D-\omega_s+\omega_0)\ln2+i\pi(\omega_s-\omega_0)/(1-e^{-\beta(\omega_s-\omega_0)})\right],$$

$$\Sigma^{(2t)}_{\mathrm{se}\downarrow}(\omega_0+is)=(\rho J)^2\left[\omega_0\ln\left|D/\omega_0\right|-(2D-\omega_e+\omega_0)\ln2-i\pi\omega_0/(1-e^{\beta\omega_0})\right],$$

$$\Sigma^{(2l)}_{\mathrm{es}\uparrow}(p,\ \omega_0+is)=\Sigma^{2l}_{\mathrm{es}\downarrow}(p,\ \omega_0+is)=2i\pi c\rho J^2\langle S_z^2\rangle\,,$$

$$\Sigma^{(2t)}_{\mathrm{es},\sigma}(p,\ \omega_0+is)=c\rho J^2\left[-2\langle S_z\rangle\ln\left|D/(\omega_0-\omega_s+\sigma\epsilon_{p,-\sigma})\right|+i\pi\langle S_x^2+s_y^2\rangle-i\pi\langle S_z\rangle\tanh\left(\tfrac{1}{2}(\omega_0-\omega_s+\epsilon_{p,-\sigma})\right)\right],$$

$$\Sigma_{\mathrm{el},\sigma}(p,\ \omega_0+is)=i\pi c'H^2\left[2(\langle h_z^2\rangle-c'\langle h_z\rangle^2)+\langle h_x^2+h_y^2\rangle\right],$$

$$\Xi_{\mathrm{se}}=cg_s^2(\rho J)^2\,2\langle S_z\rangle/(\omega_s-\omega_0)\left[\omega_s\ln\left|D/\omega_s\right|-\omega_0\ln\left|D/\omega_0\right|-(\omega_s-\omega_0)\ln2\right],$$

$$\Xi_{\mathrm{es}}=c\,g_e^2(\rho J)^2\,2\langle S_z\rangle/(\omega_e-\omega_0)\left[\omega_s\ln\left|D/\omega_s\right|+(\omega_0-\omega_s-\omega_e)\ln\left|D/(\omega_0-\omega_s-\omega_e)\right|+(\omega_e-\omega_0)\ln2\right],$$

$$\Xi_{\mathrm{el}}=0\,.$$

APPENDIX D

The aim of this appendix is twofold. First, it shows the equivalence of the spin-orbit scattering Hamiltonian and the model Hamiltonian $\mathcal{H}_{\mathrm{el}}$ introduced in Sec. II in the way it invokes relaxation effects in the present problem in the presence of large-impurity-potential scattering. Second, the diffusion term which was introduced in Sec. IV is derived. The two calculations are combined not because they are interrelated but because of the mathematical similarity of the two problems.

The amplitude of the impurity-potential scattering and the spin-flip scattering due to the spin-orbital interaction is given by[11]

$$W(\vec{k},\ \vec{k}')=A(\hat{k},\ \hat{k}')\,\hat{1}+iH(\hat{k},\ \hat{k}')\,[\hat{k}\times\hat{k}']\cdot\vec{\sigma}\,, \tag{D1}$$

where $\hat{1}$ is the unit matrix, the σ's are the Pauli matrices for the electronic spin, $\alpha=x,\ y,\ z$, and $\hat{k}$ and $\hat{k}'$ are unit vectors of the electronic momenta. The scattering amplitudes $A(\hat{k},\ \hat{k}')$ and $H(\hat{k},\ \hat{k}')$ will be assumed to be independent of $\hat{k}$ and $\hat{k}'$, first, because in practice the interactions are expected to be short range in the configuration space, and second, for mathematical simplicity.

The corresponding Hamiltonian is written

$$\mathcal{H}_{\mathrm{el}}=(A/N)\sum_{i,\sigma,\vec{k},\vec{q}}a^\dagger_{\vec{k}+\vec{q},\sigma}a_{\vec{k}\sigma}e^{-i\vec{q}\cdot\vec{R}_i}$$

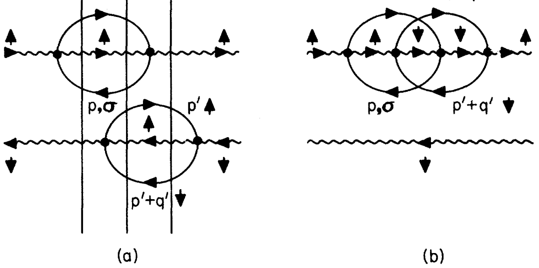

FIG. 13. (a) Overlapping self-energies; (b) fourth-order one-fermion self-energies.

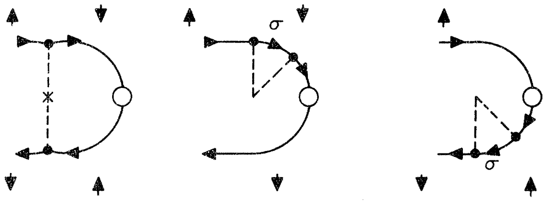

FIG. 14. Noncontributing vertices.

$$+ (H/N) \sum_{i,\vec{k},\vec{q}} \{[(\cos\theta \sin\theta' e^{-i\varphi'} - \sin\theta\cos\theta' e^{-i\varphi}) \tfrac{1}{2}\vec{\sigma}_q^{-} + \text{c.c.}] + i[\sin\theta\sin\theta' \sin(\varphi' - \varphi)]\vec{\sigma}^{z}_{\vec{q}}\} e^{-i\vec{q}\cdot\vec{R}_i} , \quad \text{(D2)}$$

where the vector $\vec{q}$ is given by $\vec{q} = \vec{k} - \vec{k}'$ and where the spherical coordinates for the vectors $\hat{\vec{k}}$ and $\hat{\vec{k}}'$ have been used $[\hat{\vec{k}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)]$. All other quantities have been defined in the text.

The Hamiltonian (D2) gives rise to self-energies and various vertex-correction-like diagrams. By induction, none of the iterated spin-*flip* type vertices shown in Fig. 14 contributes to the susceptibility. The self-energies and those vertex corrections which do not flip spin are similar to the self-energies and the vertex correction diagrams induced by the rotationally invariant model electron-lattice Hamiltonian, Eq. (3.4) in the text. However, owing to the matrix elements involved, the spin-orbit vertex scattering is found to be angularly dependent.

In order to investigate the diffusion term, the $\vec{q} \neq 0$ susceptibility $\chi^{-+}(\omega_0, \vec{q})$ must be calculated. It will be assumed that $|\vec{q}| \ll k_F$ and that $|\vec{q}\cdot\vec{k}_F| \ll$ (the total single-electron scattering rate), which includes all spin-dependent and spin-independent contributions.

The rederivation of the equations of Sec. III is then straightforward. The modification needed to evaluate the $\vec{q}$-dependent susceptibility using the present mathematical framework is the substitution $\epsilon_{\vec{p}\uparrow} \rightarrow \epsilon_{\vec{p}+\vec{q}\uparrow}$ everywhere it appears in the vertex equations. For small $|\vec{q}|$, this replacement is approximated by $\epsilon_{\vec{p}\uparrow} \rightarrow \epsilon_{\vec{p}\uparrow} + \vec{v}_p \cdot \vec{q}\cos\alpha$, where $v_p = |\vec{p}|/m$ and α is the angle between the vectors $\vec{p}$ and $\vec{q}$. It is convenient to define susceptibility $\chi_e^{+-}(q, i\nu_0;\ \theta,\ \phi;\ \alpha)$ as the total susceptibility of the electrons with a given direction, i.e., summed over all lengths $|\vec{p}|$ of the electronic momenta. It should be noted that α is not an independent variable; for a fixed $\vec{q}$ it is a function of θ and ϕ. For simplicity and to avoid lengthy equations in this appendix, use will be made of the reader's knowledge of Sec. III. It will be then assumed that those parts of the exchange self-energies which are functions of the $|\vec{p}|$ are unimportant and they will be evaluated, as well as some other terms appearing in the resulting equation on the energy shell ϵ_F. The real parts of the self-energies will be incorporated into the single-particle energies. The off-diagonal electronic-local-spin terms remain unchanged. With the understanding $\omega_0 \rightarrow \omega_0 + is$, the following equation is then obtained:

$$\begin{aligned}
(\omega_e - \omega_0)\chi_e^{+-}(\vec{q},\ \omega_0;\ \theta,\ \varphi;\ \alpha) &= (g_e^2/8\pi)(\langle n_{e\downarrow}\rangle - \langle n_{e\uparrow}\rangle) - v_F\vec{q}\cos\alpha\{\chi_e^{+-}(\vec{q},\ \omega_0;\ \theta,\ \varphi,\ \alpha) - (1/4\pi)\chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} \\
&+ i\,\mathrm{Im}[\Sigma_{i\uparrow}(\epsilon_F + \omega_0) + \Sigma_{\mathrm{es}\uparrow}(\epsilon_F + \omega_0) - \Sigma_{i\downarrow}(\epsilon_F - \omega_0) - \Sigma_{\mathrm{es}\downarrow}(\epsilon_F - \omega_0)] \\
&\times\{\chi_e^{+-}(\vec{q},\ \omega_0;\ \theta,\ \varphi;\ \alpha) - (1/4\pi)\chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} - i\,\mathrm{Im}[\Sigma_{i\uparrow}(\epsilon_F + \omega_0) - \Sigma^{2l}_{\mathrm{es}\uparrow}(\epsilon_F + \omega_0) - \Sigma_{i\downarrow}(\epsilon_F - \omega_0) + \Sigma^{(2l)}_{\mathrm{es}\downarrow}(\epsilon_F - \omega_0)] \\
&\times(1/4\pi)\sum_{\theta'\varphi'}\{\chi_e^{+-}(\vec{q},\ \omega_0;\ \theta',\ \varphi';\ \alpha) - (\chi_e^0/4\pi)[1 + \lambda\chi_s^{+-}(\omega_0)]\} \\
&+ i\,\mathrm{Im}\{[\tfrac{1}{3}(1 + \cos^2\theta)][\Sigma^{(2t)}_{\mathrm{el}\uparrow}(\epsilon_F + \omega_0) - \Sigma^{(2t)}_{\mathrm{el}\downarrow}(\epsilon_F - \omega_0)] + [\tfrac{1}{3}(1 - \cos^2\theta)][\Sigma^{(2l)}_{\mathrm{el}\uparrow}(\epsilon_F + \omega_0) - \Sigma^{(2l)}_{\mathrm{el}\downarrow}(\epsilon_F - \omega_0)]\} \\
&\times\{\chi_e(\vec{q},\ \omega_0;\ \theta,\ \varphi;\ \alpha) - (1/4\pi)\chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} \\
&+ i\,\mathrm{Im}\sin^2\theta[\Sigma^{(2l)}_{\mathrm{el}\uparrow}(\epsilon_F + \omega_0) - \Sigma^{(2l)}_{\mathrm{el}\downarrow}(\epsilon_F - \omega_0)](1/4\pi)\sum_{\theta'\varphi'}\sin^2\theta'\sin^2(\varphi - \varphi')\{\chi_e^{+-}(\vec{q},\ \omega_0;\ \theta',\ \varphi';\ \alpha') \\
&- (1/4\pi)\chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} + \cdots , \qquad \text{(D3)}
\end{aligned}$$

where

$$\Sigma_{i,\sigma}(\omega_0) = (A^2/N)\sum_q (\epsilon_{p+q,\sigma} - \omega_0)^{-1} , \qquad \text{(D4)}$$

$$\Sigma^{(2l)}_{\mathrm{el},\sigma}(\omega_0) = \Sigma^{2t}_{\mathrm{el},-\sigma}(\omega_0) = (H^2/N)\sum_q (\epsilon_{p+q,\sigma} - \omega_0)^{-1} , \qquad \text{(D5)}$$

and where the remaining quantities are defined in the text. The approximations $\mathrm{Re}\Sigma_{\mathrm{el}} \simeq 0$, $\Xi_{\mathrm{el}} \simeq 0$, and $\Xi \simeq 0$ have been made, all of which are valid for ω_e, ω_0, $kT \ll \epsilon_F$. The first term in Eq. (D6) is the inhomogeneous term, the second term is the diffusion term. Its form is again that appropriate to the local instantaneous field, and it has arisen by making the approximation

$$\sum_{|p|}(f^{+}(\epsilon_{p\downarrow}) - f^{+}(\epsilon_{p+q\uparrow})) \simeq (1/4\pi)[\langle n_{e\downarrow} - n_{e\uparrow}\rangle + \rho\vec{v}\cdot\vec{q}] \qquad \text{(D6)}$$

in both the simple outgoing electronic vertex and the one coupled by the direct spin-flip to the local-moment vertex. Since the principal contributions to the summed equation come from around the Fermi sphere, the usual approximation has been made that $\vec{v}_p \to \vec{v}_F$. The third and the fourth terms in Eq. (D6) are the usual self-energy and vertex correction contributions due to the s-d interaction and the impurity scattering. In the following term, the factor $\frac{1}{3}$ has arisen from the angular averages over the vector $\hat{\vec{k}}'$ involved in the one-fermion self-energies. The angular dependence of the last vertex correction term indicates directional dependence of the spin-orbit scattering. The dots stand for the remaining off-diagonal electronic-local-spin terms [see, e.g., Eq. (3.5b)].

Consider first the case when $\vec{q}=0$. Some simplifications can be made. For a constant density of states near the Fermi surface

$$\mathrm{Im}\Sigma_{\mathrm{el}\uparrow}(\epsilon_F+\omega_0) = -\,\mathrm{Im}\Sigma_{\mathrm{el}\downarrow}(\epsilon_F-\omega_0)$$

for both longitudinal and spin-flip contributions. This equality permits the combination of the angular factors in the electronic-orbital self-energy term in Eq. (D6), the result of which is no net angular dependence of this term. By iteration, there is no dependence on the angle ϕ. Hence, the total angular dependence remaining is determined by the orbital vertex correction term solely and is given by the proportionality factor $\sin^2\theta$. The electronic susceptibility must be of the form

$$4\pi\chi_e^{+-}(\omega_0,\,\theta) \equiv \chi_e^{+-}(\omega_0) + \sin^2\theta\,\chi_\theta^{+-}(\omega_0)\,. \tag{D7}$$

Substituting this form of the susceptibility into Eq. (D6), the comparison of the right- and left-hand sides of the resulting equation yields the following pair of the coupled linear equations:

$$\begin{aligned}(\omega_e-\omega_0)\,\chi_e^{+-}(\omega_0) = {} & (\tfrac{1}{2}g_e^2)\,(\langle n_{e\downarrow}\rangle - \langle n_{e\uparrow}\rangle)\\ & + \mathrm{Im}[\Sigma_{\mathrm{es}\uparrow} + \Sigma^{2l}_{\mathrm{es}\uparrow} + \Sigma_{\mathrm{es}\downarrow} + \Sigma^{2l}_{\mathrm{es}\downarrow}\\ & + \tfrac{4}{9}(\Sigma^{2t}_{\mathrm{el}\uparrow} + \Sigma^{2t}_{\mathrm{el}\downarrow}) + \tfrac{2}{9}(\Sigma^{2l}_{\mathrm{el}\uparrow} + \Sigma^{2l}_{\mathrm{el}\downarrow})]\\ & \times\{\chi_e^{+-}(\omega_0) - \chi_e^0[1+\lambda\chi_s^{+-}(\omega_0)]\}\\ & - (2i/3)\,\mathrm{Im}[\Sigma_{i\uparrow} + \Sigma_{i\downarrow} + \Sigma^{2l}_{\mathrm{es}\uparrow} + \Sigma^{2l}_{\mathrm{es}\downarrow}]\,\chi_\theta^{+-}(\omega_0) + \cdots\,,\end{aligned} \tag{D8b}$$

$$\begin{aligned}(\omega_e-\omega_0)\,\chi_\theta^{+-}(\omega_0) = {} & i\,\mathrm{Im}[\Sigma_{i\uparrow} + \Sigma_{i\downarrow} + \Sigma_{\mathrm{es}\uparrow} + \Sigma_{\mathrm{es}\downarrow}\\ & + \tfrac{4}{9}(\Sigma^{(2t)}_{\mathrm{el}\uparrow} + \Sigma^{(2t)}_{\mathrm{el}\downarrow}) + (\tfrac{2}{9} + \alpha^2)(\Sigma^{(2l)}_{\mathrm{el}\uparrow} + \Sigma^{(2l)}_{\mathrm{el}\downarrow})]\,\chi_e^{+}(\omega_0)\\ & + (i/3)\,\mathrm{Im}[\Sigma^{(2l)}_{\mathrm{el}\uparrow} + \Sigma^{(2l)}_{\mathrm{el}\downarrow}]\\ & \times\{\chi_e^{+-}(\omega_0) - \chi_e^0[1+\lambda\chi_s^{+-}(\omega_0)]\}\,,\end{aligned}$$

where

$$\alpha^2 = (1/4\pi)\sum_{\theta'\sigma'} \sin^4\theta'\,\sin^2(\phi-\phi')$$

and where the self-energies $\mathrm{Im}\Sigma_\uparrow$ and $\mathrm{Im}\Sigma_\downarrow$ stand for $[+\,\mathrm{Im}\Sigma_\uparrow(\epsilon_F+\omega_0)]$ and $[-\,\mathrm{Im}\Sigma_\downarrow(\epsilon_F-\omega_0)]$, respectively.

The set of Eqs. (D8) can be easily solved. If the key approximations,

$$\mathrm{Im}\Sigma_i \gg \mathrm{Im}\Sigma_{\mathrm{el}},\ \mathrm{Im}\Sigma_{\mathrm{es}},\ |\omega_e-\omega_0|\,,$$

i.e., that the single-particle cross section associated with the impurity scattering is much bigger than all other cross sections in the problem and also the difference between the external and resonance frequencies, all of which are well satisfied in practice, the following results are obtained:

$$\begin{aligned}(\omega_e-\omega_0)\,\chi_e^{+-}(\omega_0) \simeq {} & (\tfrac{1}{2}g_e^2)\,\langle n_{e\downarrow} - n_{e\uparrow}\rangle + i\,\mathrm{Im}(\Sigma_{\mathrm{es}} + \Sigma_{\mathrm{el}})\\ & \times\{\chi_e^{+-}(\omega_0) - \chi_e^0[1+\lambda\chi_s^{+-}(\omega_0)]\} + \cdots\,,\end{aligned} \tag{D9a}$$

$$\begin{aligned}\chi_\theta^{+-}(\omega_0) = {} & (-\tfrac{1}{3}\,\mathrm{Im}\Sigma_i)\,\mathrm{Im}(\Sigma^{(2l)}_{\mathrm{el}\uparrow} + \Sigma^{(2l)}_{\mathrm{el}\downarrow})\\ & \times\{\chi_e^{+-}(\omega_0) - \chi_e^0[1+\lambda\chi_s^{+-}(\omega_0)]\}\,,\end{aligned}$$

where

$$\begin{aligned}\mathrm{Im}\Sigma_{\mathrm{es}} &= \mathrm{Im}(\Sigma_{\mathrm{es}\uparrow} + \Sigma^{(2l)}_{\mathrm{es}\uparrow} + \Sigma_{\mathrm{es}\downarrow} + \Sigma^{(2l)}_{\mathrm{es}\downarrow})\,,\\ \mathrm{Im}\Sigma_{\mathrm{el}} &= \mathrm{Im}\tfrac{4}{9}(\Sigma^{(2t)}_{\mathrm{el}\uparrow} + \Sigma^{(2t)}_{\mathrm{el}\downarrow} + \Sigma^{(2l)}_{\mathrm{el}\uparrow} + \Sigma^{(2l)}_{\mathrm{el}\downarrow})\,,\\ \mathrm{Im}\Sigma_i &= \mathrm{Im}(\Sigma_{i\uparrow} + \Sigma_{i\downarrow})\,.\end{aligned} \tag{D9b}$$

From Eq. (D9b), it is seen that the angularly dependent component of the susceptibility is small as compared to the angularly independent susceptibility; the parameter of small size is the ratio $\Sigma_{\mathrm{el}}/\Sigma_i$. This result expresses the fact that a strong potential scattering redistributes the vectors of the electronic momentum quickly over the Fermi sphere, thus *suppressing* the directional dependence of the susceptibility which would have arisen due to the spin-orbit scattering. The result (D9a) would have been obtained had it been assumed *a priori* that the susceptibility (or vertex function) were angularly independent. Namely, it is found that the vertex correction term doubles the angularly averaged longitudinal self-energy contribution to the relaxation rate. The identical result is obtained from the electron-lattice model Hamiltonian introduced in Sec. II in the text if the averages involved are defined as

$$\begin{aligned}&\langle h^\alpha\rangle = 0\,,\\ &\langle (h^\alpha)^2\rangle = 2\left((1/4\pi)\sum_{\theta,\varphi}\cos^2\theta\right)^2 = \tfrac{2}{9}\end{aligned} \tag{D10}$$

for $\alpha = x,\ y,\ z$. This correspondence justifies the earlier statement made in Sec. II, that is, that in the present problem the model Hamiltonian $\mathcal{H}_{\mathrm{el}}$ represents correctly the relaxation effects due to the spin-orbit coupling of the conduction electrons.

The diffusion term $v_F q\cos\alpha$ is included using similar methods. For a given $\hat{\vec{q}}$, the vector $\hat{\vec{p}}$ can be thought of as determined by the angles θ and α. Hence, apart from the $\sin^2\theta$ dependence, the electronic susceptibility vertex can be expressed as a power series in $\cos\alpha$. Upon substituting such a series into Eq. (D6), an infinite set of equations is obtained. Before writing these equations down, it is observed that the θ summations remain unchanged and that the exchange and spin-orbit vertex correction terms do not contribute to the diffusion, being small when compared to the impurity scattering contribution by assumption. The electronic-lattice problem may be then solved independently, and hence for the diffusion problem the model Hamiltonian of Sec. II in the text can be used with the (D10) values for the averages involved. For the expansion

$$4\pi\chi_e^{+-}(\omega_0, \vec{q};\ \alpha) = \chi_e^{+-}(\omega_0, \vec{q}) + \cos\alpha\,\chi_\alpha^{+-}(\omega_0, \vec{q}) + \cos^2\alpha\,\chi_{2\alpha}^{+-}(\omega_0, \vec{q}), \tag{D11}$$

the following infinite set of equations is obtained:

$$\begin{aligned}(\omega_e - \omega_0)\chi_e^{+-}(\omega_0, \vec{q}) \simeq{}& (\tfrac{1}{2}g_e^2)\langle n_{e\downarrow} - n_{e\uparrow}\rangle \\ &+ i\,\mathrm{Im}(\Sigma_{es} + \Sigma_{el})\{\chi_e^{+-}(\omega_0, \vec{q}) - \chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} \\ &- (i/3\,\mathrm{Im}\Sigma_i)[\chi_{2\alpha}^{+-}(\omega_0, \vec{q}) + \tfrac{3}{5}\chi_{4\alpha}^{+-}(\omega_0, \vec{q})] + \cdots,\end{aligned} \tag{D12}$$

$$\begin{aligned}(\omega_e - \omega_0)\chi_\alpha^{+-}(\omega_0, \vec{q}) \simeq{}& -v_F q\{\chi_e^{+-}(\omega_0, \vec{q}) \\ &- \chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} + i\,\mathrm{Im}\Sigma_i\,\chi_\alpha^{+-}(\omega_0, \vec{q}),\end{aligned} \tag{D13}$$

$$\begin{aligned}(\omega_e - \omega_0)\chi_{2\alpha}^{+-}(\omega_0, \vec{q}) \simeq{}& -v_F q\,\chi_\alpha^{+-}(\omega_0, \vec{q}) \\ &+ i\,\mathrm{Im}\Sigma_i\,\chi_{2\alpha}^{+-}(\omega_0, \vec{q}),\end{aligned} \tag{D14}$$

where only the leading impurity vertex terms $(\mathrm{Im}\Sigma_i\,\chi_\alpha^{+-})$ have been kept, the products $\mathrm{Im}(\Sigma_{es} + \Sigma_{el})\chi_\alpha^{+-}$ being unimportant for diffusion. Higher-order angular components are determined by equations of the same form as Eq. (D14), their smallness is given by powers of the ratio $v_F q/\mathrm{Im}\Sigma_i$. For

$$\mathrm{Im}\Sigma_i \gg \mathrm{Im}\Sigma_{el}, \quad \mathrm{Im}\Sigma_{es}, \quad |\omega_e - \omega_0|, \ \vec{q}\cdot\vec{k}_F,$$

they are indefinitely small and the leading angularly independent susceptibility is given by

$$\begin{aligned}(\omega_e - \omega_0)\chi_e^{+-}(\omega_0, q) ={}& (\tfrac{1}{2}g_e^2)\langle n_{e\downarrow} - n_{e\uparrow}\rangle + iq^2v_F^2/(3\,\mathrm{Im}\Sigma_i) \\ &+ i\,\mathrm{Im}(\Sigma_{es} + \Sigma_{el})\{\chi_e^{+-}(\omega_0, q) \\ &- \chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} + \cdots.\end{aligned} \tag{D15}$$

Equation (D15) is of exactly the same form as Eq. (4.2) in the text, except the diffusion term

$$iq^2D\{\chi_e^{+-}(\omega_0) - \chi_e^0[1 + \lambda\chi_s^{+-}(\omega_0)]\} \tag{D16}$$

has been added to the right-hand side everywhere the self-energy Σ_{el} appears. Its form is again appropriate to the local field and the constant

$$D = v_F^2/(3\,\mathrm{Im}\Sigma_i) = \tfrac{1}{3}v_F^2\,T_i \tag{D17}$$

is the usual value for the diffusion constant.

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*Supported in part by the National Science Foundation under Grant No. 21 290 and the Office of Naval Research under Contract No. N00014-69-0200-4032.

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[14] One of the aims of this paper is to give precise expressions for the relaxation rates, which requires, with the present one-particle approach the determination of the arguments of the two-particle self-energy-like contributions unambiguously and the performance of the summation over the internal frequencies involved in the evaluation of the susceptibility. The standard methods (see Ref. 20) for solving vertex equations, which involve the partial fraction of the product of two Green's functions and the appearance of frequency dependent self-energies in the resulting denominator, the analytic continuation and complicated integrations, would have required too many restrictive approximations for the present study.

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[16] Compare also with calculations of Ref. 3.

[17] Hasegawa (see Ref. 1) in his original paper proposed the relationship $\bar{M}_e/T_{es} = \bar{M}_s/T_{se}$, where $\bar{M}_e$ and $\bar{M}_s$ are Hasegawa destination vectors. Cottet *et al.* (see Ref. 2) preferred, as a condition for a positive definite absorption for equal g factor, relation $\chi_e^0/T_{es} = \chi_s^0/T_{se}$, where χ_e^0 and χ_s^0 are the unenhanced static susceptibilities.

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