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Low-Temperature Behavior of the Planar Heisenberg Ferromagnet*†

J. S. Semura‡

Department of Physics, University of Wisconsin, Madison, Wisconsin 53706
and Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

D. L. Huber

Department of Physics, University of Wisconsin, Madison, Wisconsin 53706

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We have examined the low-temperature properties of the cubic-planar Heisenberg ferromagnet with nearest-neighbor exchange which is defined by the Hamiltonian $\mathcal{H}=-\sum_{i,j} J_{ij}\vec{S}_i\cdot\vec{S}_j+\sum_{i,j}(J_{ij}-K_{ij})\,S_i^x S_j^x$, where $-J\le K\le J$ (J positive). We find that as the exchange-anisotropy parameter $\theta=(J-K)/J$ ranges over the planar ferromagnetic stability limits $0\le\theta\le 2$, the behavior of the system changes from that of the isotropic ferromagnet at $\theta=0$ into that of the isotropic antiferromagnet at $\theta=2$. The system's noninteracting-spin-wave frequency, ground-state energy, zero-point spin deviation, and lowest-order renormalized frequency scale between isotropic ferromagnetic and antiferromagnetic values as θ goes from zero to two. Over most of the system's stability range, the planar ferromagnet exhibits a mixture of properties combining characteristics of its intrinsic ferromagnetism with those of the antiferromagnet. This behavior is discussed in terms of an isomorphic mapping symmetry for nearest-neighbor exchange in loose-packed lattices which requires that in the limit $\theta=2$ the planar ferromagnet be unitarily equivalent to the isotropic antiferromagnet.

I. INTRODUCTION

The planar Heisenberg ferromagnet was first introduced and studied as a magnetic analog to the lattice-liquid model for the superfluid transition in ^{4}He.[1] More recently, however, there has been increased interest in its behavior as a purely magnetic system.[2,3] In this paper, we study the properties of the planar ferromagnet in the low-temperature spin-wave regime. We shall be concerned with the Hamiltonian

$$\mathcal{H}=-\sum_{i,j}\left(K_{ij}S_i^xS_j^x+J_{ij}S_i^yS_j^y+J_{ij}S_i^zS_j^z\right), \qquad (1.1)$$

where the sums run over the N lattice sites of a crystal with cubic symmetry and nearest-neighbor interactions. When the exchange constants satisfy the condition $-J< K<J$ (J positive), the system possesses an easy plane of magnetization perpendicular to the x axis.[4]

Although it is ferromagnetic at low temperatures, the planar ferromagnet exhibits a spin-wave dispersion curve linear in the wave vector at long wavelengths, a depression of the ground-state energy below that calculated in the molecular-field approximation, and a finite zero-point spin deviation. These typically *antiferromagnetic* properties are related to a well known isomorphism for loose-packed lattices between the planar ferromagnet and the planar antiferromagnet.[3,5] In particular, for $J=-K$, the system becomes unitarily equivalent to the isotropic antiferromagnet. The consequence of this symmetry is discussed in Sec. II.

In Sec. III we diagonalize the spin-wave equation of motion neglecting interactions. The ground-state energy and spin deviation as a function of the anisotropy parameter $\theta=(J-K)/J$ are evaluated in Sec. IV in the noninteracting-spin-wave approximation. In Sec. V, the renormalized spin-wave spectrum is obtained through a random-phase-approximation (RPA) linearization of the magnon interactions. In each case, we show that when $(J-K)/J=0$ and 2, the results for the planar system reduce to the corresponding results for the isotropic Heisenberg ferromagnets and antiferromagnets, respectively. Thus, in the planar ferromagnet, one has the opportunity to examine the mixed ferromagnetic and antiferromagnetic behavior of a magnetic system as it changes from the isotropic Heisenberg ferromagnet in one limit ($\theta=0$) into a system equivalent to the isotropic Heisenberg antiferromagnet in the other limit ($\theta=2$). In Sec. VI, we conclude with a brief discussion of our results.

II. MAPPING SYMMETRY

Some insight can be gained into the relationship between the planar ferromagnet and the isotropic antiferromagnet through an examination of the sublattice rotational properties of the Hamiltonian (1.1). If it is assumed that there are only nearest-neighbor interactions and a loose-packed lattice structure which can be separated into two equivalent interpenetrating sublattices such that all nearest neighbors of a site on sublattice A belong to sublattice B (and vice versa), then it is well known that a Hamiltonian of the form (1.1) can be transformed into one in which $J\to -J$, $K\to K$.[3,5] Thus, if U_R is the unitary operator which rotates all spins of sublattice A by an angle π about the x axis in (1.1), then a spin $\vec{S}_j=(S_j^x,\ S_j^y,\ S_j^z)$ at site j on either sublattice transforms according to $U_R\vec{S}_j U_R^\dagger=(S_j^x,\ \eta_j S_j^y,\ \eta_j S_j^z)$, where $\eta_j=-1$ if site j is on sublattice A and $\eta_j=+1$ if site j is on sublattice B. The Hamiltonian (1.1) then becomes

$$U_R\mathcal{H}U_R^\dagger\equiv\mathcal{H}'=-\sum_{i,j}[\eta_i\eta_j J_{ij}(S_i^yS_j^y+S_i^zS_j^z)+K_{ij}S_i^xS_j^x]\ . \tag{2.1}$$

The effect of the unitary transformation is to map the original Hamiltonian as a function of the exchange parameters $(J_{ij},\ K_{ij})$ into a Hamiltonian with the parameters $(\eta_i\eta_jJ_{ij},\ K_{ij})=(-J_{ij},\ K_{ij})$ when i and j are nearest-neighbor sites.

A convenient way to give the above discussion a simple pictorial representation is to plot the coupling parameters J and K (see Fig. 1). The line $J=-K$ divides the J, K plane into ferromagnetic and antiferromagnetic half-planes, with the right-upper half-plane representing ferromagnetic systems and the left-lower half-plane representing antiferromagnetic systems. The regions $|K|>|J|$ and $|K|<|J|$ correspond, respectively, to easy-axis or easy-plane systems. The line $J=K$ corresponds to the isotropic Heisenberg ferromagnet or antiferromagnet for positive or negative coupling constants, respectively, while the J and K axes similarly represent the Ising or y-z mag-

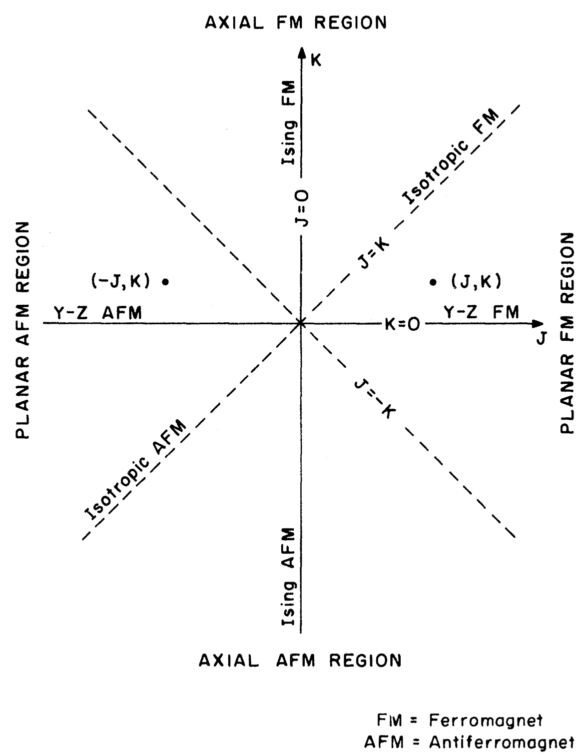

FIG. 1. Relationship between the easy-axis, easy-plane, and isotropic systems defined by Hamiltonian (1.1). The system represented by the point (J,K) maps isomorphically into the system represented by $(-J,K)$ for nearest-neighbor interactions in loose-packed lattices.

nets.[6] As previously pointed out, the net effect of the transformation (2.1) is to establish an isomorphic mapping which holds for all temperatures between every system represented by the point (J, K) in the right half of the J, K plane in Fig. 1 and its image system [represented by the point $(-J, K)$ in the left half-plane]. Of special interest to this work is the isomorphism between the isotropic antiferromagnet and the $J=-K$ limit of the planar ferromagnet. As the value of the exchange parameter J approaches $-K$, the planar ferromagnet, constrained by the isomorphism, exhibits a mixture of properties combining characteristics of its intrinsic ferromagnetism with those of the antiferromagnet. In particular, when $J=-K$, all properties of the planar ferromagnet determined by the equilibrium thermodynamics are exactly identical with the corresponding staggered properties of the isotropic antiferromagnet since thermodynamic averages remain unchanged under a unitary transformation.

III. NONINTERACTING SPIN WAVES

Fourier transforming Hamiltonian (1.1) gives

$$\mathcal{H}=-\frac{1}{N}\sum_{\vec{k}} J(\vec{k})\,\vec{S}_{-\vec{k}}\cdot\vec{S}_{\vec{k}}+\frac{1}{N}\sum_{\vec{k}}\alpha(\vec{k})\,S^x_{\vec{k}}S^x_{-\vec{k}}\ , \qquad (3.1)$$

where

$$S^\alpha_{\vec{k}}=\sum_j e^{i\vec{k}\cdot\vec{r}_j}S^\alpha_j \qquad (\alpha=x,y,z)\ ,$$

$$\alpha(\vec{k})=J(\vec{k})-K(\vec{k})\ ,\quad J(\vec{k})=JZ\gamma_{\vec{k}}\ , \qquad (3.2)$$

$$K(\vec{k})=KZ\gamma_{\vec{k}}\ ,\qquad \gamma_{\vec{k}}=\frac{1}{Z}\sum_j{}' e^{i\vec{k}\cdot\vec{r}_j}\ .$$

The prime in (3.2) indicates a summation over the Z nearest neighbors. We take the z axis to lie along the direction of maximum spin alignment. The equation of motion for the spin raising and lowering operators is then given by

$$\hbar\dot{S}^\pm_{\vec{q}}=\pm\frac{2i}{N}\sum_{\vec{k}}[J(\vec{k})-J(\vec{q}-\vec{k})]\,S^\pm_{\vec{k}}S^z_{-\vec{k}+\vec{q}}$$
$$\mp\frac{2i}{N}\sum_{\vec{k}}\frac{1}{2}\,\alpha(\vec{k})\,(S^+_{\vec{k}}+S^-_{\vec{k}})\,S^z_{-\vec{k}+\vec{q}}\ , \qquad (3.3)$$

where $\hbar\dot{A}=i[\mathcal{H},A]$.

We now make use of the spin-wave approximation

$$S^\pm_{\vec{k}}\simeq(2SN)^{1/2}\,b^\mp_{\pm\vec{k}}\ ,\quad S^z_{\vec{k}}=NS\delta_{\vec{k},0}-\sum_{\vec{p}}b^+_{\vec{p}}b^-_{\vec{p}+\vec{k}}\ , \qquad (3.4)$$

where the operators $b^\pm_{\vec{k}}$ obey the boson commutation relations

$$[b^-_{\vec{k}},b^+_{\vec{q}}]=\delta_{\vec{k},\vec{q}}\ . \qquad (3.5)$$

Using (3.4), the equation of motion (3.3) can be written in the form

$$i\hbar\begin{pmatrix}\dot{b}^-_{\vec{q}}\\ \dot{b}^+_{-\vec{q}}\end{pmatrix}=\begin{pmatrix}r_{\vec{q}}, & -s_{\vec{q}}\\ s_{\vec{q}}, & -r_{\vec{q}}\end{pmatrix}\begin{pmatrix}b^-_{\vec{q}}\\ b^+_{-\vec{q}}\end{pmatrix}+\begin{pmatrix}\mathrm{s}^-_{\vec{q}}\\ \mathrm{s}^+_{\vec{q}}\end{pmatrix}\ , \qquad (3.6)$$

where

$$r_{\vec{q}}=2S[J(0,\vec{q})+\tfrac{1}{2}\alpha(\vec{q})]\ , \qquad (3.7)$$

$$s_{\vec{q}}=-S\alpha(\vec{q})\ , \qquad (3.8)$$

$$\mathrm{s}^\mp_{\vec{q}}=\pm\frac{2}{N}\sum_{\vec{k},\vec{p}}J(\vec{k},\vec{q}-\vec{k})\,b^\mp_{\pm\vec{k}}b^+_{\vec{p}}b^-_{\vec{p}-\vec{k}+\vec{q}}$$
$$\mp\frac{2}{N}\sum_{\vec{k},\vec{p}}\frac{1}{2}\,\alpha(\vec{k})\,(b^-_{\vec{k}}+b^+_{-\vec{k}})\,b^+_{\vec{p}}b^-_{\vec{p}-\vec{k}+\vec{q}}\ , \qquad (3.9)$$

$$J(\vec{q},\vec{k})=J(\vec{q})-J(\vec{k})\ .$$

It should be noted that in obtaining Eq. (3.6), the spin-wave approximation is inserted into the exact equation of motion for $S^\pm_{\vec{q}}$. This particular approach is chosen for the sake of consistency with the low-temperature calculations of the spin-wave damping where it has been shown[7] that this method, coupled with the RPA, reproduces Dyson's dynamical interaction in the first Born approximation for the isotropic ferromagnet.

As a first approximation, the interaction terms $\mathrm{s}^\pm_{\vec{q}}$ in Eq. (3.6) are neglected and the resultant equation of motion is easily diagonalized, giving

$$\begin{pmatrix}\dot{A}^-_{\vec{q}}\\ \dot{A}^+_{-\vec{q}}\end{pmatrix}=\begin{pmatrix}-i\omega^{(0)}_{\vec{q}} & \\ & i\omega^{(0)}_{-\vec{q}}\end{pmatrix}\begin{pmatrix}A^-_{\vec{q}}\\ A^+_{-\vec{q}}\end{pmatrix}\ , \qquad (3.10)$$

where

$$\begin{pmatrix}A^-_{\vec{q}}\\ A^+_{-\vec{q}}\end{pmatrix}=\hat{U}(\vec{q})\begin{pmatrix}b^-_{\vec{q}}\\ b^+_{-\vec{q}}\end{pmatrix}\ , \qquad (3.11)$$

$$\hat{U}(\vec{q})=\begin{pmatrix}u_{\vec{q}}, & -v_{\vec{q}}\\ -v_{\vec{q}}, & u_{\vec{q}}\end{pmatrix}\ , \qquad (3.12)$$

$$\omega^{(0)}_{\vec{q}}=(r^2_{\vec{q}}-s^2_{\vec{q}})^{1/2}\ . \qquad (3.13)$$

The coefficients of the transformation are given by the relations

$$u^2_{\vec{k}}=\frac{r_{\vec{k}}}{2\hbar\omega^{(0)}_{\vec{k}}}+\frac{1}{2}\ , \qquad (3.14)$$

$$v^2_{\vec{k}}=\frac{r_{\vec{k}}}{2\hbar\omega^{(0)}_{\vec{k}}}-\frac{1}{2}\ , \qquad (3.15)$$

$$u_{\vec{k}}v_{\vec{k}}=\frac{s_{\vec{k}}}{2\hbar\omega^{(0)}_{\vec{k}}}\ , \qquad (3.16)$$

where $u_{\vec{k}}$ is chosen to be positive and $v_{\vec{k}}$ is negative.

Using Eqs. (3.7), (3.8), and (3.13), the noninteracting-spin-wave energy becomes

$$\hbar\omega^{(0)}_{\vec{q}}=2SJ(0)[1+(\theta-2)\gamma_{\vec{q}}+(1-\theta)\gamma^2_{\vec{q}}]^{1/2}\ , \qquad (3.17)$$

where $\theta=(J-K)/J$. In the limits $\theta=0$ and $\theta=2$, the frequency $\omega^{(0)}_{\vec{q}}$ reduces to the noninteracting-spin-wave frequencies of the isotropic ferromagnets and antiferromagnets, respectively. The dispersion

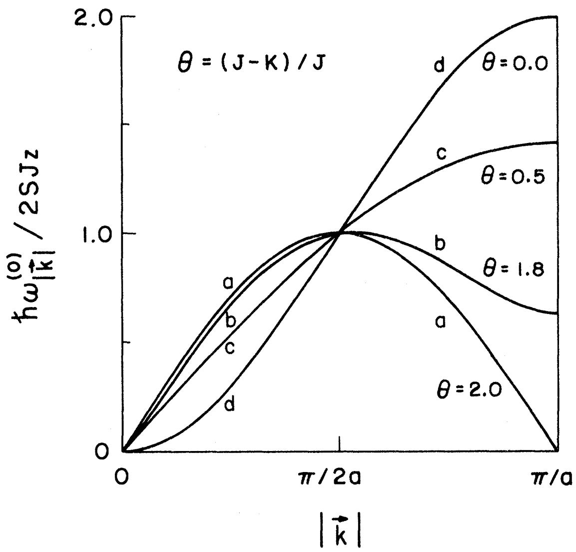

FIG. 2. $\hbar\omega^{(0)}_{|\vec{k}|}/2SJZ$ vs $|\vec{k}|$ for spin waves traveling in the [111] direction in a simple cubic lattice for several values of the anisotropy θ.

curve for several values of the anisotropy θ is plotted in Fig. 2 for spin waves traveling in the [111] direction in a simple cubic lattice. For $\frac{4}{3} < \theta < 2$, the dispersion curve has a maximum when $\gamma_{\vec{q}} = (1-\frac{1}{2}\theta)/(1-\theta)$ and a local minimum at the Brillouin-zone boundary which results from the compromise that the system is forced to make between ferromagnetic and antiferromagnetic behavior. For $\theta > 2$, the square root in (3.17) becomes imaginary at some point in the Brillouin zone, corresponding to an instability in the planar ordering originally postulated. The range of exchange anisotropy over which the planar ordering exhibits spin-wave stability is therefore given by $0 \leq \theta \leq 2$.

In the finite anisotropy, long-wavelength limit the frequency becomes

$$\hbar\omega^{(0)}_{\vec{q}} \simeq D|\vec{q}a| , \quad |\vec{q}a| \to 0 \tag{3.18}$$

for cubic lattices, where a is the lattice constant and $D = 2SJ(0)(\theta/Z)^{1/2}$. Due to the existence of an easy-ordering plane, the planar system does not exhibit the energy gap in the uniform mode which is characteristic of an easy-axis system.

For finite anisotropy θ and temperatures low enough so that the bulk of the excited magnons are linear in the wave vector, the system's internal energy will vary as T^4 while the heat capacity and magnetization varies as T^3 and T^2, respectively, just as in the isotropic antiferromagnet.

IV. GROUND-STATE FLUCTUATIONS

The planar-magnetic-ground-state problem for finite anisotropies is closely analogous to that of the well-known antiferromagnetic case.[8,9] The finite-exchange anisotropy that gives rise to the planar ordering simultaneously destroys the rotational symmetry of the system about the z axis, so that the total magnetization is not an exact constant of the motion. Although this fact does not invalidate the use of the spin-wave formalism as a good approximation for low temperatures, it does mean that for finite θ the system will be characterized by zero-point fluctuations. The zero-temperature isotropic antiferromagnetic and planar cases are intrinsically different in this respect from that of the isotropic ferromagnetic where rotational symmetry rigorously assures that $[\mathcal{H}, \sum_j S^z_j] = 0$. The fact that the magnetization is an exact constant of the motion in the latter case allows for a well-defined labeling of the isotropic-ferromagnetic ground state (i.e., all spins ordered parallel in the z direction).

Given these considerations, and the behavior of $\omega^{(0)}_{\vec{q}}$ as a function of the anisotropy, one might reasonably expect the effects of the planar zero-point motion, calculated from the noninteracting-spin-wave frequency, will scale smoothly between those of the isotropic ferromagnet and the isotropic antiferromagnet as θ goes from zero to two. We now show this to be the case for the zero-point energy and spin deviation.

The ground-state energy E_0 is identified with the thermal-expectation value of the Hamiltonian at $T = 0$:

$$E_0 = \langle \mathcal{H} \rangle_{T=0} , \tag{4.1}$$

where $\langle A \rangle = \mathrm{Tr}e^{-\beta\mathcal{H}}A/\mathrm{Tr}e^{-\beta\mathcal{H}}$, $\beta = (k_BT)^{-1}$, and k_B is Boltzmann's constant. The thermal average of the Hamiltonian through terms bilinear in the normal modes is given by

$$\langle \mathcal{H} \rangle = -NJ(0)S(S+1) + \sum_{\vec{k}} (n_{\vec{k}} + \tfrac{1}{2})\omega^{(0)}_{\vec{k}} , \tag{4.2}$$

where $n_{\vec{k}} \equiv \langle A^{\dagger}_{\vec{k}} A_{\vec{k}} \rangle = [\exp(\beta\hbar\omega^{(0)}_{\vec{k}}) - 1]^{-1}$. Passing to the zero-temperature limit gives

$$E_0 = -NJ(0)S^2(1 + B/ZS) ,$$

where (4.3)

$$B = Z - (1/2JNS)\sum_{\vec{k}} \omega^{(0)}_{\vec{k}} .$$

The quantity B is a measure of the extent to which the noninteracting-spin-wave approximation reduces the energy below the estimate obtained in the molecular-field approximation from the ground-state wave function $|0\rangle = \prod_i |S^z_i = S\rangle$. The value of B as a function of θ was calculated numerically over the exact Brillouin zone for the three cubic lattices and the result tabulated in Table I. It is seen that the ground-state energy is significantly depressed by the anisotropy below that obtained from the molecular field approximation. For $\theta = 2$, our result 0.58 for the sc and bcc lattices

TABLE I. Effect of the anisotropy $(J-K)/J$ on the ground-state-energy parameter B [Eq. (4.3)] for the three cubic lattices.

$(J-K)/J$	B sc	B bcc	B fcc
0	0	0	0
0.2	0.01	0.01	0.01
0.4	0.03	0.03	0.04
0.6	0.06	0.06	0.08
0.8	0.10	0.10	0.12
1.0	0.15	0.15	0.18
1.2	0.21	0.21	0.24
1.4	0.28	0.28	0.30
1.6	0.36	0.36	0.38
1.8	0.46	0.46	0.47
2.0	0.58	0.58	0.56

agrees with that found for these lattices in the non-interacting-spin-wave calculations for the isotropic antiferromagnet by Anderson[8] and Kubo.[9]

The average ground-state spin deviation per site is given by

$$(\Delta S)_{T=0} = (1/N)\sum_{\vec{k}} v_{\vec{k}}^2 , \tag{4.4}$$

where $v_{\vec{k}}^2$ is given by Eq. (3.15). Although $v_{\vec{k}}^2$ diverges at long wavelengths as $|\vec{k}|^{-1}$, the divergence is integrable in three dimensions. Equation (4.4) was integrated numerically over the exact Brillouin zone for the sc and bcc lattices, with the results tabulated in Table II as a function of anisotropy. Once again the value of $(\Delta S)_{T=0}$ scales between that of the isotropic ferromagnet and the isotropic antiferromagnet[8–10] in the appropriate anisotropy limits.

V. RENORMALIZATION

In Secs. III and IV, the properties of the planar system were calculated by ignoring the effects of the magnon-magnon interactions that appear in the equation of motion (3.6). In this section we obtain the leading temperature-dependent correction to the spin-wave frequency through an RPA linearization of the interactions in the equation of motion.

From Eqs. (3.6), (3.11), and (3.12), the full equation of motion, including interactions, can be written

$$\begin{pmatrix} \left(\dfrac{d}{dt} + i\omega_{\vec{q}}^{(0)}\right) A_{\vec{q}}^- \\ \left(\dfrac{d}{dt} - i\omega_{-\vec{q}}^{(0)}\right) A_{-\vec{q}}^+ \end{pmatrix} = -\frac{i}{\hbar}\begin{pmatrix} u_{\vec{q}} S_{\vec{q}}^- - v_{\vec{q}} S_{\vec{q}}^+ \\ -v_{\vec{q}} S_{\vec{q}}^- + u_{\vec{q}} S_{\vec{q}}^+ \end{pmatrix} \tag{5.1}$$

We shall summarize the linearization of the right-hand side of Eq. (5.1). The calculations are given in greater detail in Appendix A.

In its original form, the interaction term $S_{\vec{q}}^{\pm}$ was given by (3.9) as sums containing terms of third order in the $b_{\vec{q}}^{\pm}$ operators. Using transformation (3.11) to rewrite $S_{\vec{q}}^{\pm}$ in terms of the spin-wave amplitudes $A_{\vec{q}}^{\pm}$, the right-hand side of (5.1) is expressed as sums containing terms of third order in $A_{\vec{q}}^{\pm}$. Following Mori and Kawasaki[7] and Tani[11] we reduce these third-order terms to first order in $A_{\vec{q}}^{\pm}$ by the use of the RPA. For example, operators of the form $A_{\vec{k}}^+ A_{\vec{q}}^+ A_{\vec{p}}^-$ are replaced by

$$A_{\vec{k}}^+ A_{\vec{q}}^+ A_{\vec{p}}^- = \langle A_{\vec{p}}^+ A_{\vec{p}}^- \rangle A_{\vec{q}}^+ \delta_{\vec{k},\vec{p}} + \langle A_{\vec{p}}^+ A_{\vec{p}}^- \rangle A_{\vec{k}}^+ \delta_{\vec{q},\vec{p}} ,$$

and the thermal averages are replaced by the Bose function $\langle A_{\vec{k}}^+ A_{\vec{k}}^- \rangle = n_{\vec{k}}$.

It is well known that the RPA is essentially equivalent to the time-dependent Hartree theory in that it introduces an effective harmonic potential and neglects the effects of rapid short-lived fluctuations in the driving terms in the equation of motion. In Sec. VI, we comment on the effects not included in the RPA.

Carrying out the linearization (see Appendix A), Eq. (5.1) can be written in the form

$$\begin{pmatrix} \dot{A}_{\vec{q}}^- \\ \dot{A}_{-\vec{q}}^+ \end{pmatrix} = -\frac{i}{\hbar}\begin{pmatrix} [\hbar\omega_{\vec{q}}^{(0)} + F_1(\vec{q},T)] & [F_2(\vec{q},T)] \\ [-F_2(\vec{q},T)] & [-\hbar\omega_{-\vec{q}}^{(0)} - F_1(\vec{q},T)] \end{pmatrix}\begin{pmatrix} A_{\vec{q}}^- \\ A_{-\vec{q}}^+ \end{pmatrix}, \tag{5.2}$$

where $F_1(\vec{q}, T)$ and $F_2(\vec{q}, T)$ are defined by Eqs. (A7) and (A8). The off-diagonal elements $\pm F_2(\vec{q}, T)$ are of relative order $(2S)^{-1}$ in comparison with $\pm F_1(\vec{q}, T)$. Diagonalizing Eq. (5.2), and calling the second-order diagonalized eigenmodes $B_{\vec{q}}^{\pm}$, we get

$$\begin{pmatrix} \dot{B}_{\vec{q}}^- \\ \dot{B}_{-\vec{q}}^+ \end{pmatrix} \begin{pmatrix} -i\omega_{\vec{q}}(T) & 0 \\ 0 & +i\omega_{-\vec{q}}(T) \end{pmatrix}\begin{pmatrix} B_{\vec{q}}^- \\ B_{-\vec{q}}^+ \end{pmatrix} . \tag{5.3}$$

The renormalized frequency $\omega_{\vec{q}}(T)$, corrected to first order in $(2S)^{-1}$, can then be written

$$\omega_{\vec{q}}(T) = \omega_{\vec{q}}^{(0)}(1 - e_{\vec{q}}/2S) , \tag{5.4}$$

$$e_{\vec{q}} = e_{\vec{q}}(0) + e_{\vec{q}}(T) , \tag{5.5}$$

where the temperature-independent and -dependent parts of the correction $e_{\vec{q}}$ are given by

$$e_{\vec{q}}(0) = -2SF_1(\vec{q}, 0)/\hbar\omega_{\vec{q}}^{(0)} , \tag{5.6}$$

$$e_{\vec{q}}(T) = -2S[F_1(\vec{q}, T) - F_1(\vec{q}, 0)]/\hbar\omega_{\vec{q}}^{(0)} . \tag{5.7}$$

The evaluation of (5.6) and (5.7) is given in Appendix B for the simple cubic lattice for which the $\vec{q}$ and $\vec{k}$ behaviors in $F_1(\vec{q}, T)$ can be factored exactly. The results are

TABLE II. Effect of the anisotropy $(J-K)/J$ on the zero-point spin deviation $(\Delta S)_{T=0}$ [Eq. (4.4)] for the simple cubic and body-centered-cubic lattices.

$(J-K)/J$	$(\Delta S)_{T=0}$ sc	$(\Delta S)_{T=0}$ bcc
0	0	0
0.2	0.003	0.002
0.4	0.007	0.005
0.6	0.012	0.009
0.8	0.016	0.013
1.0	0.022	0.017
1.2	0.028	0.021
1.4	0.035	0.026
1.6	0.043	0.032
1.8	0.054	0.041
2.0	0.078	0.059

$$e_{\vec{q}}(0) = -1 + \frac{1}{1+(\theta-1)\gamma_{\vec{q}}} \frac{1}{N} \sum_{\vec{k}} \left((1+\tfrac{1}{2}\theta\gamma_{\vec{q}})\epsilon_{\vec{k}} + (\theta-2)\frac{\gamma_{\vec{q}}}{2\epsilon_{\vec{k}}}(1-\gamma_{\vec{k}})^2 \right), \quad (5.8)$$

$$e_{\vec{q}}(T) = \frac{1}{N}\sum_{\vec{k}} n_{\vec{k}} 2\epsilon_{\vec{k}} + \frac{\gamma_{\vec{q}}(2\theta-\theta^2)}{1+(\theta-1)\gamma_{\vec{q}}} \times \frac{1}{N}\sum_{\vec{k}} n_{\vec{k}}\gamma_{\vec{k}} \left(\frac{1-\gamma_{\vec{k}}}{1+(\theta-1)\gamma_{\vec{k}}}\right)^{1/2}, \quad (5.9)$$

where $\epsilon_{\vec{k}} = \hbar\omega_{\vec{k}}^{(0)}/2SJ(0)$.

Equations (5.8) and (5.9) are exact results for the simple cubic lattice within the spin-wave and random-phase approximations. We examine these results in two anisotropy regions.

A. $\theta=0$ and 2

In the limits $\theta=0$ and 2, Eqs. (5.8) and (5.9) reduce exactly to the corresponding results for the isotropic ferromagnet and isotropic antiferromagnet, respectively.[12] We get

$$e_{\vec{q}}(0) = \begin{cases} 0 & \text{when } \theta = 0 \\ -1 + \dfrac{1}{N}\sum_{\vec{k}} \epsilon_{\vec{k}} & \text{when } \theta = 2, \end{cases} \quad (5.10)$$

$$e_{\vec{q}}(T) = \frac{2}{N}\sum_{\vec{k}} n_{\vec{k}}\epsilon_{\vec{k}}, \qquad \theta = 0,\ 2, \quad (5.11)$$

where

$$\epsilon_{\vec{k}} = \begin{cases} 1-\gamma_{\vec{k}} & \text{when } \theta=0 \\ (1-\gamma_{\vec{k}}^2)^{1/2} & \text{when } \theta = 2. \end{cases}$$

As is the case in the RPA, the renormalization corrections are independent of the magnon wavelength $\vec{q}$.

B. $0<\theta<2$

In this anisotropy range it is convenient to pass to the small $\vec{q}$ limit in evaluating $e_{\vec{q}}(0)$, while still treating the $\vec{k}$ dependence exactly. This gives

$$e_0(0) = -1 + \frac{1}{\theta N}\sum_{\vec{k}} \left((1+\tfrac{1}{2}\theta)\epsilon_{\vec{k}} + (\theta-2)\frac{(1-\gamma_{\vec{k}})^2}{2\epsilon_{\vec{k}}} \right), \quad (5.12)$$

where

$$\epsilon_{\vec{k}} = [1+(\theta-2)\gamma_{\vec{k}} + (1-\theta)\gamma_{\vec{k}}^2]^{1/2}. \quad (5.13)$$

Equation (5.12) has been estimated numerically for $\theta=1$ with the result $e_0(0) \simeq -0.07$.

In evaluating $e_0(T)$, we note that for finite anisotropy, (5.13) is linear in $\vec{k}$ for $\vec{k}$ values which satisfy the condition

$$\tfrac{1}{6}|\theta-1|k^2a^2 \ll \theta. \quad (5.14)$$

We are interested in the lowest-order temperature-correction term to the frequency spectrum where it is predominantly the low-energy long-wavelength magnons that comprise the bulk of the excitations in the system. In this very-low-temperature region, owing to the nature of the Boltzmann factor, the summands in (5.9) will be sharply peaked near $|\vec{k}|=0$. Thus condition (5.14) will hold, and we evaluate (5.9) for both small $\vec{q}$ and $\vec{k}$, keeping terms quadratic in the wave vector in $\gamma_{\vec{k}}$. The result is

$$e_0(T) = \left(\frac{\theta+2}{\theta}\right)\frac{1}{N}\sum_{\vec{k}} n_{\vec{k}}\epsilon_{\vec{k}}, \quad (5.15)$$

where the frequency $\epsilon_{\vec{k}}$ is linear in $\vec{k}$ and is given by the long-wavelength limit of (5.13). Equation (5.15) is proportional to the internal energy $\sum_{\vec{k}} n_{\vec{k}}\hbar\omega_{\vec{k}}^{(0)}$ which varies as T^4 for very low temperatures and finite anisotropy. When the anisotropy is not too close to zero but small compared to two, Eq. (5.15) reduces to

$$e_0(T) \simeq \frac{1}{S[J(0)-K(0)]N}\sum_{\vec{k}} n_{\vec{k}}\hbar\omega_{\vec{k}}^{(0)}, \quad 0<\theta\ll 2, \quad (5.16)$$

a result published previously.[13]

When θ is very close to zero, the system acquires many of the characteristics of an isotropic ferromagnet, and condition (5.14) begins to break down. This is exhibited in the apparent θ^{-1} divergence of (5.16) as the system attempts to change the T^4 variation of $e(T)$ in order to agree with the $T^{5/2}$ behavior in the isotropic ferromagnetic limit. The behavior of $e_0(T)$ in this small anisotropy region becomes quite complicated; for its evaluation, one must return to expression (5.9) and retain terms of at least fourth order in the wave vector in $\gamma_{\vec{k}}$.

As a function of temperature, Eq. (5.15) can be expected to hold for low enough temperatures such that $k_BT < D|\vec{k}^*|a$, where $\vec{k}^*$ is the wave vector where the linear approximation to the magnon spectrum begins to break down and D is given in (3.18). For temperatures higher than $D|\vec{k}^*|a/k_B$, the renormalization for small anisotropies resembles that of the isotropic ferromagnet.

In the Keffer and Loudon[14] theory of renormalization, for the isotropic ferromagnets and antiferromagnets the correction factor $e_{\vec{q}}(T)$ is equal to twice the internal energy per spin of the two respective systems. This is the exact result for the planar system [Eqs. (5.10)] in the $\theta = 0$ and 2 limits. For anisotropies intermediate between these two extremes, $e_0(T)$ scales as the internal energy [Eq. (5.15)] but with a proportionality factor that depends on the anisotropy.

The term $e_{\vec{q}}(0)$ gives rise to a temperature-independent frequency shift and represents a correction to the ground-state energy calculated in Sec. IV. Due to the rotational symmetry considerations discussed in Sec. IV, the correction factor $e_{\vec{q}}(0)$ vanishes for the isotropic ferromagnet but remains finite for the isotropic antiferromagnet. This fact is reflected in Eqs. (5.10).

VI. DISCUSSION

Perhaps the most interesting feature of our results is the close relationship exhibited between the planar ferromagnet and the antiferromagnet. This relationship arises from the constraint imposed on the planar system by the isomorphic mapping symmetry discussed in Sec. II. Indeed, from the viewpoint of quantum mechanics, for the loose-packed lattices and nearest-neighbor exchange, the planar ferromagnet in the limit $J = -K$ is *completely equivalent* to the isotropic antiferromagnet. Thus, in a certain sense, the planar ferromagnet might be viewed as an intermediate system which transforms smoothly as a function of anisotropy from the isotropic ferromagnet into the isotropic antiferromagnet, and which at intermediate points possesses characteristics of both systems.

Finally, two points deserve comment in regard to our normal modes. First, the steps leading from (5.2) to (5.3) mean that the off-diagonal element $F_2(\vec{q}, T)$ in the equation of motion (5.2) can be neglected to order $(2S)^{-1}$ in the renormalization factor $(1 - e_{\vec{q}}/2S)$, i.e., setting $F_2(\vec{q}, T) = 0$ reduces (5.2) to (5.3). In other words, in the first-order renormalization correction, the modes $B^{\pm}_{\vec{q}}$ are identical with $A^{\pm}_{\vec{q}}$, although we have retained the difference in notation in order to avoid confusing the $B^{\pm}_{\vec{q}}$ modes, whose frequencies are temperature dependent, with the operators $A^{\pm}_{\vec{q}}$ used in the noninteracting-spin-wave calculations.

The second point is more important. The modes $B^{\pm}_{\vec{q}}$ were obtained as the result of a two-step diagonalization procedure beginning with the original equation of motion (3.6). The most important limiting step in this process is the random-phase approximation, which, as previously stated, replaces the true potential by an average which leads to a purely harmonic equation of motion, Eq. (5.3). This harmonicity is destroyed by including the effects of the fluctuating interactions left out of the RPA, and the modes $B^{\pm}_{\vec{q}}$ will then be characterized by finite lifetimes. We would, however, expect these corrections to be small. We have estimated the damping of the renormalized modes and find that, in agreement with this assumption of well-defined spin waves (in the temperature-frequency regimes examined), the magnitude of the imaginary part of the magnon frequency is very much smaller than that of the real part.[15]

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APPENDIX A: RPA LINEARIZATION

In this Appendix we derive Eq. (5.2). Using (3.9) for $S^{\pm}_{\vec{q}}$, the right-hand side of (5.1) can be rewritten, giving

$$\begin{pmatrix} \left(\frac{d}{dt} + i\omega^{(0)}_{\vec{q}}\right) A^{-}_{\vec{q}} \\ \left(\frac{d}{dt} + i\omega^{(0)}_{-\vec{q}}\right) A^{+}_{-\vec{q}} \end{pmatrix} = -\frac{i}{\hbar} \begin{pmatrix} \frac{2}{N} \sum_{\vec{k},\vec{p}} \{[u_{\vec{q}} J(\vec{k}, \vec{q}-\vec{k}) - (u_{\vec{q}} + v_{\vec{q}}) \tfrac{1}{2}\alpha(\vec{k})]\, b^{-}_{\vec{k}} b^{+}_{\vec{p}} b^{-}_{\vec{p}-\vec{k}+\vec{q}} \\ \quad + [v_{\vec{q}} J(\vec{k}, \vec{q}-\vec{k}) - (u_{\vec{q}} + v_{\vec{q}}) \tfrac{1}{2}\alpha(\vec{k})]\, b^{+}_{-\vec{k}} b^{\pm}_{\vec{p}} b^{-}_{\vec{p}-\vec{k}+\vec{q}}\} , \\ \frac{2}{N} \sum_{\vec{k};\vec{p}} \{[-v_{\vec{q}} J(\vec{k}, \vec{q}-\vec{k}) + (u_{\vec{q}} + v_{\vec{q}}) \tfrac{1}{2}\alpha(\vec{k})]\, b^{-}_{\vec{k}} b^{+}_{\vec{p}} b^{-}_{\vec{p}-\vec{k}+\vec{q}} \\ \quad + [-u_{\vec{q}} J(\vec{k}, \vec{q}-\vec{k}) + (u_{\vec{q}} + v_{\vec{q}}) \tfrac{1}{2}\alpha(\vec{k})]\, b^{+}_{-\vec{k}} b^{+}_{\vec{p}} b^{-}_{\vec{p}-\vec{k}+\vec{q}}\} \end{pmatrix} \tag{A1}$$

Using Eqs. (3.11) and (3.12), and applying the RPA as outlined in Sec. V, then for the sum containing the operators $b^{-}_{\vec{k}} b^{+}_{\vec{p}} b^{-}_{\vec{p}-\vec{k}+\vec{q}}$ in the top line of the right-hand side of Eq. (A1), we get

$$\frac{2}{N} \sum_{\vec{k},\vec{p}} [u_{\vec{q}} J(\vec{k}, \vec{q}-\vec{k}) - (u_{\vec{q}} + v_{\vec{q}}) \tfrac{1}{2}\alpha(\vec{k})]\, b^{-}_{\vec{k}} b^{+}_{\vec{p}} b^{-}_{\vec{p}-\vec{k}+\vec{q}} = F^{-}(u_{\vec{q},\vec{k}}, v_{\vec{q},\vec{k}}) A^{-}_{\vec{q}} + F^{+}(u_{\vec{q},\vec{k}}, v_{\vec{q},\vec{k}}) A^{+}_{-\vec{q}} , \tag{A2}$$

where

$$F^{-}(x_{\vec q,\vec k},\, y_{\vec q,\vec k}) = \frac{2}{N}\sum_{\vec k}\{x_{\vec q}[x_{\vec q}J(\vec q, 0) - (u_{\vec q}+v_{\vec q})\tfrac{1}{2}\alpha(\vec q)]\,[n_{\vec k}(u_{\vec k}^2+v_{\vec k}^2)+v_{\vec k}^2] + v_{\vec q}[x_{\vec q}J(\vec k, \vec q-\vec k) - (u_{\vec q}+v_{\vec q})\tfrac{1}{2}\alpha(\vec k)]$$
$$\times[n_{\vec k}(x_{\vec k}v_{\vec k}+y_{\vec k}u_{\vec k})+x_{\vec k}v_{\vec k}]\; u_{\vec q}[x_{\vec q}J(\vec k, \vec q-\vec k) - (u_{\vec q}+v_{\vec q})\tfrac{1}{2}\alpha(\vec k)]\,[n_{\vec k}(x_{\vec k}u_{\vec k}+y_{\vec k}v_{\vec k})+x_{\vec k}u_{\vec k}]\}\,, \tag{A3}$$

$$F^{+}(x_{\vec q,\vec k},\, y_{\vec q,\vec k}) = \frac{2}{N}\sum_{\vec k}\{y_{\vec q}[x_{\vec q}J(\vec q, 0) - (u_{\vec q}+v_{\vec q})\tfrac{1}{2}\alpha(\vec q)][n_{\vec k}(u_{\vec k}^2+v_{\vec k}^2)+v_{\vec k}^2] + u_{q}[x_{\vec q}J(\vec k, \vec q-\vec k) - (u_{\vec q}+v_{\vec q})\tfrac{1}{2}\alpha(\vec k)]$$
$$\times[n_{\vec k}(x_{\vec k}v_{\vec k}+y_{\vec k}u_{\vec k})+x_{\vec k}v_{\vec k}] + v_{\vec q}[x_{\vec q}J(\vec k, \vec q-\vec k) - (u_{\vec q}+v_{\vec q})\tfrac{1}{2}\alpha(\vec k)][n_{\vec k}(x_{\vec k}u_{\vec k}+y_{\vec k}v_{\vec k})+x_{\vec k}u_{\vec k}]\}\,. \tag{A4}$$

The notation $F^{\pm}(u_{\vec q,\vec k},\, v_{\vec q,\vec k})$ means $x_{\vec q}=u_{\vec q}$, $x_{\vec k}=u_{\vec k}$, $y_{\vec q}=v_{\vec q}$, $y_{\vec k}=v_{\vec k}$ in (A3) and (A4).

The corresponding expression for the sum containing the product $b^{+}_{-\vec k}b^{+}_{\vec p}b^{-}_{\vec p-\vec k+\vec q}$ in the equation for $\dot A^{-}_{\vec q}$ in (A1) is given by

$$\frac{2}{N}\sum_{\vec k,\vec p}[v_{\vec q}J(\vec k, \vec q-\vec k) - (u_{\vec q}+v_{\vec q})\tfrac{1}{2}\alpha(\vec k)]\, b^{+}_{-\vec k}b^{+}_{\vec p}b^{-}_{\vec p-\vec k+\vec q}$$
$$= F^{-}(v_{\vec q,\vec k},\, u_{\vec q,\vec k})A^{-}_{\vec q} + F^{+}(v_{\vec q,\vec k},\, u_{\vec q,\vec k})A^{+}_{-\vec q}\,. \tag{A5}$$

Therefore equation of motion (A1) becomes

$$\begin{pmatrix} \left(\dfrac{d}{dt}+i\omega^{(0)}_{\vec q}\right)A^{-}_{\vec q} \\ \left(\dfrac{d}{dt}-i\omega^{(0)}_{-\vec q}\right)A^{+}_{-\vec q} \end{pmatrix} = -\frac{i}{\hbar}\begin{pmatrix} F_1(\vec q, T), & F_2(\vec q, T) \\ F_3(\vec q, T), & F_4(\vec q, T) \end{pmatrix}\begin{pmatrix} A^{-}_{\vec q} \\ A^{+}_{-\vec q} \end{pmatrix}\,, \tag{A6}$$

where

$$F_1(\vec q, T) = F^{-}(u_{\vec q,\vec k},\, v_{\vec q,\vec k}) + F^{-}(v_{\vec q,\vec k},\, u_{\vec q,\vec k})\,, \tag{A7}$$

$$F_2(\vec q, T) = F^{+}(u_{\vec q,\vec k},\, v_{\vec q,\vec k}) + F^{+}(v_{\vec q,\vec k},\, u_{\vec q,\vec k})\,. \tag{A8}$$

Since Hermitian conjugation and the replacement $\vec q \to -\vec q$ transforms the equation of motion of $A^{-}_{\vec q}$ into that for $A^{+}_{-\vec q}$, and since both $F_1(\vec q, T)$ and $F_2(\vec q, T)$ are even functions of $\vec q$, in (A6) we must have

$$F_3(\vec q, T) = -F_2(\vec q, T)\,, \qquad F_4(\vec q, T) = -F_1(\vec q, T)\,. \tag{A9}$$

Hence, using (A6) and (A9), we get the result (5.2) used in Sec. V with the quantities $F_1(\vec q, T)$ and $F_2(\vec q, T)$ given by (A3), (A4), (A7), and (A8).

APPENDIX B: RENORMALIZATION CORRECTION TERMS

The renormalization correction terms, Eqs. (5.6) and (5.7), are evaluated exactly in this Appendix for the case of the simple cubic lattice.

Combining Eqs. (A3), (A4), (A7), and (A8), Eqs. (5.6) and (5.7) become

$$e_{\vec q}(0) = 2S(\hbar\omega^{(0)}_{\vec q})^{-2}[r_{\vec q}\psi_{\vec q}(0) + s_{\vec q}\phi_{\vec q}(0)]\,, \tag{B1}$$

$$e_{\vec q}(T) = 2S(\hbar\omega^{(0)}_{\vec q})^{-2}[r_{\vec q}\psi_{\vec q}(T) + s_{\vec q}\phi_{\vec q}(T)]\,, \tag{B2}$$

where

$$\psi_{\vec q}(0) = \frac{1}{N}\sum_{\vec k} J(0)\left\{\left[\frac{r_{\vec q}}{SJ(0)} + 2\gamma_{\vec q-\vec k} - \left(1+\frac{K}{J}\right)\gamma_{\vec k}\right]v_{\vec k}^2 + \theta u_{\vec k}v_{\vec k}\gamma_{\vec k}\right\}\,, \tag{B3}$$

$$\phi_{\vec q}(0) = \frac{1}{N}\sum_{\vec k} J(0)\left\{\left[2\gamma_{\vec q-\vec k} - \left(1+\frac{K}{J}\right)\gamma_{\vec k}\right]u_{\vec k}v_{\vec k} + \theta(\gamma_{\vec q}+\gamma_{\vec k})v_{\vec k}^2\right\}\,, \tag{B4}$$

$$\psi_{\vec q}(T) = \frac{1}{N}\sum_{\vec k}\frac{n_{\vec k}}{\omega^{(0)}_{\vec k}} J(0)\left\{\left[\frac{r_{\vec q}}{SJ(0)} + 2\gamma_{\vec q-\vec k} - \left(1+\frac{K}{J}\right)\gamma_{\vec k}\right]r_{\vec k} + \theta s_{\vec k}\gamma_{\vec k}\right\}\,, \tag{B5}$$

$$\phi_{\vec q}(T) = \frac{1}{N}\sum_{\vec k}\frac{n_{\vec k}}{\omega^{(0)}_{\vec k}} J(0)\left\{\left[2\gamma_{\vec q-\vec k} - \left(1+\frac{K}{J}\right)\gamma_{\vec k}\right]s_{\vec k} + \theta(\gamma_{\vec q}+\gamma_{\vec k})r_{\vec k}\right\}. \tag{B6}$$

For the simple cubic lattice, the $\vec q$ and $\vec k$ behavior contained in $\gamma_{\vec q-\vec k}$ can be factored exactly *within a sum over the Brillouin zone*.[16] It is useful to write the resulting identity in the form

$$\sum_{\vec k}\gamma_{\vec q-\vec k}f(\vec k) = \sum_{\vec k}[-1+\gamma_{\vec q}+\gamma_{\vec k}+(1-\gamma_{\vec q})(1-\gamma_{\vec k})]f(\vec k) = \gamma_{\vec q}\sum_{\vec k}\gamma_{\vec k}f(\vec k)\,, \tag{B7}$$

where $f(\vec k)$ is an even function of $\vec k$.

Temperature-Independent Correction

Using (B7) and the definitions of $r_{\vec q}$, $v_{\vec k}^2$, and $u_{\vec k}v_{\vec k}$ given in Sec. III, $\psi_{\vec q}(0)$ and $\phi_{\vec q}(0)$ can be simplified to give

$$\psi_{\vec q}(0) = -J(0)[1+(\tfrac{1}{2}\theta-1)\gamma_{\vec q}] + \psi^{A}_{\vec q}(0)\,, \tag{B8}$$

$$\phi_{\vec q}(0) = -\tfrac{1}{2}J(0)\theta\gamma_{\vec q} + \phi^{A}_{\vec q}(0)\,, \tag{B9}$$

where

$$\psi^{A}_{\vec q}(0) = \frac{1}{N}\sum_{\vec k} J(0)\left(\epsilon_{\vec k} + \frac{\gamma_{\vec q}}{2\epsilon_{\vec k}}[1+(\tfrac{1}{2}\theta-1)\gamma_{\vec k}] \times[(\theta-2)+2\gamma_{\vec k}]\right)\,, \tag{B10}$$

$$\phi^{A}_{\vec{q}}(0)=\frac{1}{N}\sum_{\vec{k}} J(0)\,\frac{\theta}{2\epsilon_{\vec{k}}}\,[\gamma_{\vec{k}}+\gamma_{\vec{q}}(1+(\tfrac{1}{2}\theta-1)\gamma_{\vec{k}}-\gamma^2_{\vec{k}})]\;. \qquad \text{(B11)}$$

The quantity $\epsilon_{\vec{k}}$ is a dimensionless energy defined by $\epsilon_{\vec{k}}=\hbar\omega^{(0)}_{\vec{k}}/2SJ(0)=(1-\gamma_{\vec{k}})^{1/2}[1+(\theta-1)\gamma_{\vec{k}}]^{1/2}$. Finally, inserting Eqs. (B8)–(B11) into (B1), collecting terms, and simplifying, we get

$$e_{\vec{q}}(0)=-1+\frac{1}{1+(\theta-1)\gamma_{\vec{q}}}\;\frac{1}{N}\sum_{\vec{k}}\left((1+\tfrac{1}{2}\theta\gamma_{\vec{q}})\epsilon_{\vec{k}}+(\theta-2)\,\frac{\gamma_{\vec{q}}}{2\epsilon_{\vec{k}}}\,(1-\gamma_{\vec{k}})^2\right). \qquad \text{(B12)}$$

This is the result (5.8) for the temperature-independent correction.

Temperature-Dependent Correction

The derivation of the temperature-dependent correction term is similar to that leading to (B12). Again $\gamma_{\vec{q}-\vec{k}}$ can be factored within the $\vec{k}$ summation in expressions (B5) and (B6). Simplifying the resulting expressions leads to

$$\psi_{\vec{q}}(T)=\frac{1}{N}\sum_{\vec{k}}\frac{n_{\vec{k}}}{\omega_{\vec{k}}(0)}\,J^2(0)S\{[2+(\theta-2)\gamma_{\vec{k}}][\theta\gamma_{\vec{q}}+\theta\gamma_{\vec{k}}+2(1-\gamma_{\vec{q}})(1-\gamma_{\vec{k}})]-\theta^2\gamma^2_{\vec{k}}\}\;, \qquad \text{(B13)}$$

$$\phi_{\vec{q}}(T)=\frac{1}{N}\sum_{\vec{k}}\frac{n_{\vec{k}}}{\omega_{\vec{k}}(0)}\,J^2(0)S\{[2+(\theta-2)\gamma_{\vec{k}}][\theta\gamma_{\vec{q}}+\theta\gamma_{\vec{k}}]+2\theta\gamma^2_{\vec{k}}(1-\gamma_{\vec{q}})-\theta^2\gamma^2_{\vec{k}}\}\;. \qquad \text{(B14)}$$

Inserting these two expressions into (B2), rearranging terms, and simplifying, we get

$$e_{\vec{q}}(T)=\frac{1}{N}\sum_{\vec{k}} n_{\vec{k}}2\epsilon_{\vec{k}}+\frac{(2\theta-\theta^2)\gamma_{\vec{q}}}{1+(\theta-1)\gamma_{\vec{q}}}\;\frac{1}{N}\sum_{\vec{k}} n_{\vec{k}}\gamma_{\vec{k}}\left(\frac{1-\gamma_{\vec{k}}}{1+(\theta-1)\gamma_{\vec{k}}}\right)^{1/2}. \qquad \text{(B15)}$$

This is the result given in (5.9).

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‡Present address: Department of Physics, University of Pittsburgh, Pittsburgh, Pa. 15213

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