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Symmetry Properties of the Transport Coefficients of Magnetic Crystals

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A discussion of the symmetry properties of tensors describing the transport properties of magnetic crystals, or of nonmagnetic crystals in applied magnetic fields, is given both from the macroscopic and microscopic points of view. A prescription for the simplification of the form of such a tensor is derived (prescription C) which is based on the use of Onsager's theorem and of Neumann's principle both for the unitary symmetry operations and also, in a modified form, for the antiunitary symmetry operations. This is different from the methods used previously by Birss (prescription A) and by Kleiner (prescription B). While prescription A ignores the antiunitary symmetry operations, the use of these operations is included in both prescriptions B and C. However, prescriptions B and C often lead to different results and it is suggested that experiments based on the Hall effect could be used to determine which of these two prescriptions is correct.

I. INTRODUCTION

The reader is assumed to be familiar with the general outlines of the classic book by Nye[1] on the use of symmetry to simplify the forms of tensors describing various physical properties of nonmagnetic crystals.

The first writings in English on the extension of these ideas to the case of crystals exhibiting magnetic ordering are those of Birss,[2,3] based on the quite considerable amount of scattered work of several Russian workers. As far as tensors that describe macroscopic static, or equilibrium, properties are concerned, the treatment given by Birss has found complete acceptance. However, when it comes to the case of transport properties, which describe only quasiequilibrium states or dynamic equilibrium states of a crystal, the tensors involved (the transport coefficients) depend on time, implicitly at least, and there has been some criticism[4–7] of the treatment given by Birss.

In the present paper we shall try to identify the exact nature of the disagreement which has arisen, to establish to what extent it is a matter of semantics or a matter of physics, and to offer some further suggestions on the matter of the symmetry properties of transport coefficients and of any microscopic description of physical phenomena in magnetic crystals.

II. SYMMETRY PROPERTIES OF TENSORS FOR MAGNETIC CRYSTALS–MACROSCOPIC APPROACH

For a nonmagnetic crystal, in the absence of any external magnetic field, there are two separate arguments that can be used to simplify the forms of the tensors representing the transport properties of the crystal (see, for example, Ref. 1). The first is based on the use of Onsager's theorem and the second is based on the use of Neumann's principle. From the use of Onsager's theorem it is possible to show that σ_{ij} and κ_{ij}, the electrical- and thermal-conductivity tensors, are symmetric; that is,

$$\sigma_{ij}=\sigma_{ji}\ ,\qquad \kappa_{ij}=\kappa_{ji}\ , \tag{1}$$

and also to obtain a relation between the Peltier tensor π_{ij} and the Seebeck tensor α_{ij}:

$$(1/T)\pi_{ij}=\alpha_{ji}\ . \tag{2}$$

The conditions expressed in Eqs. (1) and (2) apply to any crystal regardless of any crystallographic symmetry that it may possess. The use of Neumann's principle enables one to make use of any crystallographic symmetry to impose further restrictions on the transport coefficients in addition to those given in Eqs. (1) and (2). For any second-rank tensor d_{ij}, Neumann's principle leads to

$$d_{ij}=\sum_p \sum_q R_{ip}R_{jq}d_{pq}\ . \tag{3}$$

In this equation, R_{ip} is the orthogonal matrix that represents the action of a point-group symmetry operation R of the crystal on a vector $\vec{x}_p[=(x_1, x_2, x_3)]$, that is,

$$x_i'=\sum_p R_{ip}x_p\ . \tag{4}$$

In the conventional treatment, the symmetry operations R belong to the classical point group $\vec{G}$ of the crystal, rather than the "gray" point group, and therefore do not include operations involving θ, the operation of time inversion.

If one now considers either a crystal that exhibits spontaneous magnetic ordering or a nonmagnetic crystal that is subjected to an external magnetic field, the details of the application of Onsager's theorem and Neumann's principle have to be reexamined. If a crystal is subjected to a magnetic field $\vec{H}$, one that may be an external field or may arise as an internal field in the crystal, it is then possible to show[1,5,8,9] that Eqs. (1) and (2) have to be replaced by

$$\sigma_{ij}(\vec{H}) = \sigma_{ji}(-\vec{H}) ,\quad \kappa_{ij}(\vec{H}) = \kappa_{ji}(-\vec{H}) , \tag{5}$$

and

$$(1/T)\pi_{ij}(\vec{H}) = \alpha_{ji}(-\vec{H}) . \tag{6}$$

In general, the symmetry of a magnetically ordered crystal or of a nonmagnetic crystal in an applied magnetic field $\vec{H}$ will have to be described by some magnetic point group $\vec{M}$ which can be written in the form

$$\vec{M} = \vec{G} + A\vec{G} , \tag{7}$$

where $\vec{G}$ is the subgroup of the point-group operations that do not involve θ and $A\vec{G}$ is the coset of point-group operations that do involve θ (see, for example, Ref. 10). The set $A\vec{G}$ may be null, in which case $\vec{M}$ will be a classical point group; that is, a type-I Shubnikov point group. If $A\vec{G}$ is not null, it will contain the same number of elements as group $\vec{G}$ and none of the elements of $A\vec{G}$ will be θ on its own; that is, $\vec{M}$ will be a type-III Shubnikov point group. If $\vec{M}$ is a type-I Shubnikov point group, Eq. (3) can be used for all the elements of $\vec{M}$ just as before.

So far it is agreed by all that the procedure outlined in the previous paragraph is correct. The difficulty arises when $\vec{M}$ is a type-III Shubnikov group. Although $\vec{M}$ contains antiunitary operations that involve θ, it does not contain the element θ on its own. The disputation between Kleiner and Birss is concerned with the problem of how to treat the antiunitary elements, which comprise the set $A\vec{G}$, in an equation like Eq. (3). The procedure adopted by Birss and referred to as "prescription A" by Kleiner[6] consists of the following: (i) Use Eqs. (5) and (6) based on Onsager's theorem, (ii) use Eq. (3), based on Neumann's principle, only for the unitary elements of $\vec{M}$, and (iii) ignore the antiunitary elements of $\vec{M}$, i.e., the elements in $A\vec{G}$.

The justification of this procedure is discussed in considerable detail in the book by Birss.[3] In particular, it should be noted that (iii) means that one is only using part of the symmetry information that one has available for the crystal (see Ref. 3, pp. 111 and 149).

Kleiner[5,6] objected to the procedure described in prescription A on the grounds that by ignoring the antiunitary elements, it does not exploit the full symmetry of the situation; we would agree with this criticism, although there is more to Birss's discussion of transport properties than just considering the symmetry properties of the conductivity tensor (see Sec. 5.4 of Ref. 3). However, it was also alleged by Kleiner (Ref. 5, p. 326; Ref. 6) that prescription A "is inconsistent with the existence of the extraordinary Hall effect in ferromagnets"; this is not the case, as will be seen from the example of ferromagnetic Co which will be considered in Sec. V [see, in particular, Eq. (34)]. Kleiner proposed an alternative procedure, which he called "prescription B," which states that while Eq. (3) is used for the unitary elements, a similar, but not identical, equation is used for the antiunitary elements [see Eq. (2.28) of Ref. 5]. By choosing a new group which is not just $\vec{M}$ but also includes operations that send $\vec{H}$ into $-\vec{H}$, Kleiner avoids having to use Onsager's theorem explicitly and is able to regard it as a consequence of the elements of this larger group. Since Kleiner's arguments involve some considerable discussion of microscopic theories, we shall postpone any further discussion of prescription B until Sec. III and continue to concentrate for the moment on the macroscopic approach.

To prevent our discussion from becoming too abstract, let us consider the particular example of the electrical-conductivity tensor σ_{ij} which is defined by

$$j_i = \sum_j \sigma_{ij} E_j , \tag{8}$$

where $\vec{E}$ is the electric field and $\vec{j}$ is the electric current density. θ will reverse the sign of $\vec{j}$, but will leave $\vec{E}$ unaltered; that is, j_i is a c tensor of rank one and E_j is an i tensor of rank one. Therefore, from Eq. (8) we must have

$$\theta\sigma_{ij} = -\sigma_{ij} . \tag{9}$$

It is important to understand precisely what this equation means; it is concerned with the application of the operation θ to the complete configuration of our material specimen, together with the external influence $\vec{E}$ and the response $\vec{j}$. Since θ is obviously not a symmetry operation of the complete configuration of specimen + $\vec{j}$ + $\vec{E}$, there is no question of any clash between Eq. (9) and Neumann's principle. Equation (9) simply tells us that if one wishes to produce the situation in Fig. 1(b) as a result of applying θ to the situation in Fig. 1(a),

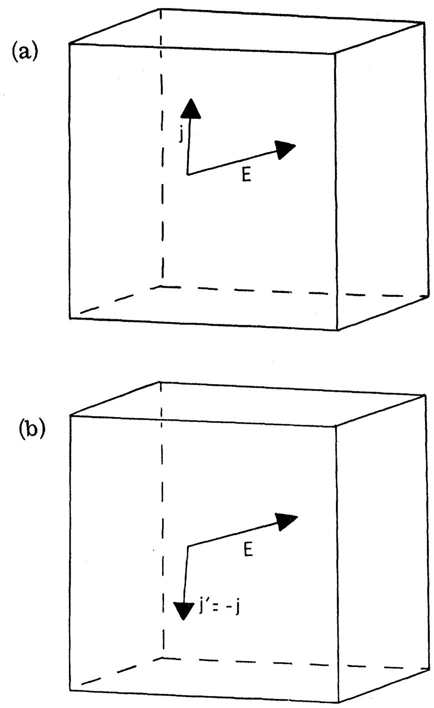

FIG. 1. The relation between $\vec{j}$, the electric current density, and $\vec{E}$, the electric field, in a crystal (a) before and (b) after the application of θ, the operation of time inversion.

one must have chosen a specimen for which Eq. (9) happens to be valid; that is, a specimen for which σ_{ij} happens to be a *c* tensor. Whether this can ever happen in practice remains to be seen.

Let us consider things in a slightly different manner. We can regard our specimen as a "black box," which contains some material entities and a magnetic field $\vec{H}$; we apply an electric field $\vec{E}$ to this black box and observe the response $\vec{j}$. If the black box has no crystallographic symmetry at all, then we have, from considerations of irreversible thermodynamics, the Onsager reciprocal relation

$$\sigma_{ij}(\vec{H}) = \sigma_{ji}(-\vec{H}) , \tag{10}$$

which we have already mentioned. $\vec{H}$ is the magnetic field in our "black-box" system; it may be an external field applied to a nonmagnetic crystal, an internal magnetic field in a magnetically ordered crystal, or a combination of internal and external fields.

If we have a crystal for which we know that θ is a symmetry operation, this means $\vec{H}=0$, so that

$$\sigma_{ij}(0) = \sigma_{ji}(0) , \tag{11}$$

and because θ is a symmetry operation of the black box, then

$$\theta\sigma_{ij} = \sigma_{ij} . \tag{12}$$

Equation (12) is not in conflict with Eq. (9). Equation (12) applies if we are given that θ is a symmetry operation of the black box, i.e., the symmetry of the specimen is described by one of the gray groups, and if we assume that Neumann's principle still holds for θ. Equation (9) refers to the application of θ to a larger system of which θ is not a symmetry operation, and tells us the condition that must be satisfied by σ_{ij} if Fig. 1(b) is to be produced as a result of the application of θ to Fig. 1(a). If Eq. (12) holds, then Eq. (9) cannot hold, because σ_{ij} would then be null.

If $\vec{H} \neq 0$, i.e., when our black box consists of a magnetically ordered crystal or consists of a nonmagnetic crystal situated in an applied magnetic field, we have a situation in which θ, by itself, is not a symmetry operation of our black box. However, there are certain antiunitary operations of the form θS, where S is a point-group operation, which are symmetry operations of the black box. Let us suppose that Neumann's principle still holds for these operations θS. We should then have

$$(\theta S)\sigma_{ij}(\vec{H}) = \sigma_{ij}(\vec{H}) . \tag{13}$$

To make use of the transformation properties of the tensor under these antiunitary operations, we must determine the result of the application of θ to $\sigma_{ij}(\vec{H})$. The effect of θ on the motions of all the particles in the black box will be to reverse the directions of their velocities and, therefore, the Lorentz forces $\mu\mu_0 e(\vec{v}\times\vec{H})$; therefore, the Hamiltonian will only remain invariant if $\vec{H}$ is also reversed. Consequently, the current j_i flowing in a given direction as a result of the application of an electric field E_j will be unaltered if the direction of $\vec{H}$ is also reversed; that is

$$\theta\sigma_{ij}(\vec{H}) = \sigma_{ij}(-\vec{H}) . \tag{14}$$

Equation (14) is simply a statement of the fact that the compound operation of time reversal + changing the sign of $\vec{H}$ is a symmetry operation of the system in the "black box." It is related to Onsager's theorem in the sense that Onsager's theorem is a consequence of the use of this compound symmetry operation in thermodynamic arguments based on the principle of microscopic reversibility. There is no thermodynamics involved in Eq. (14).

Equations (12) and (14) only apply when we are concerned with direct current. If we are concerned with alternating current, the electric field $\vec{E}$ and the current density $\vec{j}$ take the forms $\vec{E}_0 e^{i\omega t}$ and $\vec{j}_0 e^{i\omega t}$, respectively, so that $\theta\vec{E} = \vec{E}_0 e^{-i\omega t} = \vec{E}^*$ and $\theta\vec{j} = \vec{j}_0 e^{-i\omega t} = \vec{j}^*$. By applying θ to both sides of Eq. (8), we see that for alternating current we have to replace Eqs. (12) and (14) by

$$\theta\sigma_{ij} = \sigma_{ij}^* \tag{15}$$

and

$$\theta\sigma_{ij}(\vec{H}) = \sigma^*_{ij}(-\vec{H}) \ . \tag{16}$$

From now on we shall use Eqs. (15) and (16) rather than Eqs. (12) and (14) so as to keep the treatment general.

We can use Eq. (16) in Eq. (13) to give

$$\begin{aligned}\sigma_{ij}(\vec{H}) &= \theta S\,\sigma_{ij}(\vec{H}) = S\theta\,\sigma_{ij}(\vec{H})\\ &= S\sigma^*_{ij}(-\vec{H})\\ &= \sum_p \sum_q S_{ip}S_{jq}\sigma^*_{pq}[S^{-1}(-\vec{H})]\\ &= \sum_p \sum_q S_{ip}S_{jq}\sigma^*_{pq}(\vec{H}) \ ,\end{aligned}$$

i.e.,

$$\sigma_{ij}(\vec{H}) = \sum_p \sum_q S_{ip}S_{jq}\sigma^*_{pq}(\vec{H}) \ , \tag{17}$$

where θS is any antiunitary symmetry operation of our black box, and it does not matter whether $\vec{H}$ is an internal field in a magnetically ordered crystal or an external field applied to a nonmagnetic crystal or some combination of internal and external fields. Thus for an antiunitary element θS in a type-III Shubnikov group we use Eq. (17) to simplify the form of the tensor $\sigma_{ij}(\vec{H})$ in a similar manner to the use of Eq. (3) for the unitary elements. If we return for a moment to the case of a gray group when θ by itself is a symmetry operation of the crystal, and therefore $\vec{H} = 0$, we can use θE as θS in Eq. (17) when we obtain $\sigma_{ij}(0) = \sigma^*_{ij}(0)$ which implies that, in the absence of a magnetic field, the electrical conductivity is real. We therefore arrive at what we shall call "prescription C" which we may write as follows: (i) Use Eqs. (5) and (6) based on Onsager's theorem, (ii) use Eq. (3), based on Neumann's principle, for the unitary elements R of $\vec{M}$, and (iii) use Eq. (17), based on a modified form of Neumann's principle, for the antiunitary elements, θS of $\vec{M}$. Parts (i) and (ii) are identical to prescription A. Consequently, prescription A and prescription C will always lead to the same results, except that, as a result of (iii), prescription C may lead to some further simplification of the form of σ_{ij} beyond that achieved by prescription A. The relation between prescription C and prescription B from Kleiner is rather more obscure and we postpone that discussion until Sec. III. Although prescription C was developed for the electrical conductivity σ_{ij}, its extension to a tensor representing any other transport property is trivial.

III. SYMMETRY PROPERTIES OF TENSORS FOR MAGNETIC CRYSTALS–MICROSCOPIC APPROACH

Kleiner has objected to the procedure adopted by Birss (see prescription A described in Sec. II) on the grounds that no use was being made of the antiunitary symmetry operations of a magnetic crystal. Towards the end of Sec. II we discussed the transformation of a tensor under these symmetry operations from a macroscopic point of view. The approach adopted by Kleiner[5–7] was based on studying the effect of the antiunitary symmetry operations from a microscopic point of view. This led to a procedure described by Kleiner as "prescription B." We now examine the relationship between prescription B and prescription C.

In the ordinary application of group theory to the quantum-mechanical treatment of the microscopic description of a physical system that is not magnetically ordered, it is usual to construct the group $\vec{G}$ of the symmetry operations of the Hamiltonian $\mathcal{H}$ of the system. $\vec{G}$ is sometimes called the Schrödinger group of the system. For a crystal, one would usually expect $\vec{G}$ to be the same as the classical point group or space group of the crystal, although sometimes one may happen to choose, for reasons based on a knowledge of the "physics of the situation," a Hamiltonian with more (but not with less) symmetry than $\vec{G}$. That is, physical arguments may be used to justify neglecting certain terms in $\mathcal{H}$, and this may accidentally increase the symmetry of $\mathcal{H}$. Having chosen a Hamiltonian with symmetry of $\vec{G}$, it follows from the well-known theorem of Wigner[11] that the eigenfunctions of $\mathcal{H}$ belong to, i.e., transform according to, the irreducible representations of $\vec{G}$. The degeneracies of the corresponding eigenvalues of $\mathcal{H}$ will be determined by the degeneracies of irreducible representations of $\vec{G}$. Abundant evidence of the validity of Wigner's theorem has been provided over several decades from large amounts of work on atomic, molecular, and crystal physics.

Now consider a system with magnetic ordering and let us suppose that the form of the magnetic ordering is such that the group $\vec{M}$ of the symmetry operations of the crystal is one of the type-III Shubnikov point groups. To obtain a quantum-mechanical description of this system, we construct a Hamiltonian $\mathcal{H}_{\vec{M}}$ that has the symmetry of the crystal. $\vec{M}$ contains antiunitary operations of the form θS, where θ is the operation of time inversion and S is a crystallographic point-group operation. It is shown in Wigner's book on group theory[12] that the eigenfunctions of $\mathcal{H}_{\vec{M}}$ belong to, i.e., transform according to, the irreducible corepresentations of $\vec{M}$. There is experimental evidence of this in the fact that the degeneracies of the spin-wave dispersion relations, at various points in the Brillouin zone, have now been predicted correctly by the use of corepresentation theory for a number of different magnetically ordered crystals. In passing we should note that for a given crystal we are still free to choose, on physical grounds, a Hamiltonian with more (but not with less) symmetry than $\vec{M}$, in which case extra degeneracies may occur in the spin-wave dispersion relations (see the papers by Brinkman and Elliott[13] on "spin space groups").

Kleiner's approach is based on starting with a particular expression for a transport coefficient, due to Kubo,[14]

$$\tau_{B_\mu A_\nu}(\omega, \vec{\mathrm{H}}) = \int_0^\infty dt\, e^{-i\omega t} \int_0^\beta d\lambda\, \mathrm{Tr}\rho(\vec{\mathrm{H}}) A_\nu B_\mu(t+i\hbar\lambda;\ \vec{\mathrm{H}}) . \tag{18}$$

We leave aside for the moment any discussion of the question of the validity of the Kubo formalism (see, for example, Ref. 15). The effect of the unitary operations in $\vec{\mathrm{M}}$ on the coefficient $\tau_{B_\mu A_\nu}(\omega, \vec{\mathrm{H}})$ was determined by transforming the right-hand side of Eq. (18). Then Kleiner obtained

$$\tau_{B_\mu A_\nu}(\omega, \vec{\mathrm{H}}) = \sum_\kappa \sum_\lambda \tau_{B_\kappa A_\lambda}(\omega, \vec{\mathrm{H}}) D^{(B)}(u)_{\kappa\mu} D^{(A)}(u)_{\lambda\nu} \tag{19}$$

[see Eq. (2.17) of Ref. 5], where the notation is fully defined by Kleiner.[5] The matrices $D^{(B)}(u)_{\kappa\mu}$ and $D^{(A)}(u)_{\lambda\nu}$ correspond to R_{ip} and R_{jq} in Eq. (3). Equation (19) is the commonly accepted equation based on the definition of a tensor of rank two and the use of Neumann's principle, and is identical, apart from the notation, to Eq. (3), thereby confirming that the coefficients defined by Eq. (18) do form a tensor.

The effect of the antiunitary elements in $\vec{\mathrm{M}}$ on the coefficients $\tau_{B_\mu A_\nu}(\omega, \vec{\mathrm{H}})$ was also determined by Kleiner in a similar manner by transforming the right-hand side of Eq. (18). If θS is one of these antiunitary operations (= a in Kleiner's notation) the result of this transformation is given by Eq. (2.28) of Ref. 5:

$$\tau_{B_\mu A_\nu}(\omega, \vec{\mathrm{H}}) = \sum_\kappa \sum_\lambda \tau_{A_\lambda^\dagger B_\kappa^\dagger}(\omega, \vec{\mathrm{H}}_a) D^{(B)}(\theta S)^*_{\kappa\mu} D^{(A)}(\theta S)^*_{\lambda\nu} . \tag{20}$$

Equation (20) differs from Eq. (17) in that the transformation matrices on the right-hand side of Eq. (20) are the complex conjugates of those in Eq. (17). This does not matter, because these matrices are all real orthogonal matrices. The transposition of the suffixes on the right-hand side of Eq. (20) is an important feature that does not occur when one considers unitary symmetry [see Eq. (19)]. This transposition leads, for example, in Kleiner's treatment, to the relations between the Seebeck and Peltier coefficients.

In Kleiner's treatment, relations among transport coefficients determined by the use of Neumann's principle are obtained by using in Eqs. (19) and (20) [Eqs. (2.17) and (2.28) of Ref. 5] the unitary and antiunitary operations, respectively, that are contained in the group $\vec{\mathrm{M}}$ [or $\mathcal{G}$ or $\mathcal{G}(\vec{\mathrm{H}})$ in Kleiner's notation]; $\vec{\mathrm{M}}$ is the group of the operations that leave the Hamiltonian $\mathcal{H}(\vec{\mathrm{H}})$ invariant. In dealing with Onsager's principle, Kleiner has a different procedure from that given in part (i) of prescriptions A and C. In Kleiner's procedure, instead of using the group $\vec{\mathrm{M}}$ that includes only the operations which leave the Hamiltonian $\mathcal{H}(\vec{\mathrm{H}})$ of the system invariant, another group $\mathcal{K}(\vec{\mathrm{H}})$ is constructed. In addition to the operations of the group $\vec{\mathrm{M}}$, the group $\mathcal{K}(\vec{\mathrm{H}})$ also includes all the operations that send $\mathcal{H}(\vec{\mathrm{H}})$ into $\mathcal{H}(-\vec{\mathrm{H}})$. $\mathcal{K}(\vec{\mathrm{H}})$ will therefore be some supergroup of $\vec{\mathrm{M}}$, and the various possible structural relations between $\mathcal{K}(\vec{\mathrm{H}})$ and $\vec{\mathrm{M}}$ (= $\mathcal{G}$) are identified in Table I of Ref. 5. Having constructed the group $\mathcal{K}(\vec{\mathrm{H}})$, the procedure that is described by Kleiner,[6] "prescription B," consists of the following: (i) Use Eq. (19) for the unitary elements of $\mathcal{K}(\vec{\mathrm{H}})$, and (ii) use Eq. (22) for the antiunitary elements of $\mathcal{K}(\vec{\mathrm{H}})$.

By using the (possibly) larger group $\mathcal{K}(\vec{\mathrm{H}})$ instead of $\vec{\mathrm{M}}$, Kleiner's formalism enables the generalized Onsager reciprocal relations to be derived as a consequence of requiring the tensor to be invariant under the operations of the group $\mathcal{K}(\vec{\mathrm{H}})$. In this prescription the modified Onsager reciprocal relations are not used explicitly, because they are assumed to be covered by the use of $\mathcal{K}(\vec{\mathrm{H}})$.

IV. RELATION BETWEEN PRESCRIPTION B AND PRESCRIPTION C

In Secs. II and III we have considered the problem of the exploitation of antiunitary symmetry operations in simplifying the form of a tensor that represents the transport coefficients of a magnetic crystal or of a nonmagnetic crystal in the presence of a magnetic field $\vec{\mathrm{H}}$. We did not concern ourselves very much with the problem of actually identifying the nonunitary group that applies to a given physical situation. We now turn our attention to this problem and use the term "magnetic crystal" loosely to mean any system involving a crystal which is described by a Hamiltonian involving a magnetic field $\vec{\mathrm{H}}$ that may be external in origin or may originate within the crystal as a result of some magnetic ordering.

For the general Hamiltonian $\mathcal{H}_{\vec{\mathrm{M}}}$ of a magnetic crystal, the group which is of physical significance is the group of $\mathcal{H}_{\vec{\mathrm{M}}}$; that is, *the group of operations that leave the Hamiltonian* $\mathcal{H}_{\vec{\mathrm{M}}}$ *invariant*: this group may be either a type-I or a type-III Shubnikov group. If it is a type-I group, there are no antiunitary operations to be considered and, therefore, no difficulties. We are therefore concerned with the case when the group $\vec{\mathrm{M}}$ of the Hamiltonian is a type-III Shubnikov point group. In general it seems reasonable to suppose that $\mathcal{H}_{\vec{\mathrm{M}}}$ may contain both odd and even terms in $\vec{\mathrm{H}}$, and therefore *the operations of this group must leave* $\vec{\mathrm{H}}$ *invariant*. That is, in Kleiner's notation it is $\mathcal{G}$ and not $\mathcal{K}(\vec{\mathrm{H}})$ that is the group of the Hamiltonian of the magnetic crystal. However, by analogy with the case of spin space groups mentioned previously, if *there are some additional physical reasons for assuming that only even terms in* $\vec{\mathrm{H}}$ *appear in* $\mathcal{H}_{\vec{\mathrm{M}}}$, then one would be justified in using Kleiner's group $\mathcal{K}(\vec{H})$. The additional physical information which is available comes from the modified Onsager relations

$$\sigma_{ij}(\vec{H}) = \sigma_{ji}(-\vec{H}) ,$$
$$\kappa_{ij}(\vec{H}) = \kappa_{ji}(-\vec{H}) , \tag{5}$$

and

$$(1/T)\pi_{ij}(\vec{H}) = \alpha_{ji}(-\vec{H}) . \tag{6}$$

Equation (5) means that the diagonal components of the electrical- and thermal-conductivity tensors are even functions of $\vec{H}$. This is demonstrated in many metals when the transverse magnetoresistance is proportional to $\vec{H}^2$, provided the Fermi surface of the metal is not multiply connected. But Kleiner's treatment using $\mathcal{K}(\vec{H})$ rather than $\vec{M}$ (or $\mathcal{G}$) is equivalent to assuming that the Hamiltonian is an even function of $\vec{H}$. This is certainly a different condition from the modified Onsager relations which, although they impose other conditions, only require the diagonal components of the electrical and thermal conductivity to be even functions of $\vec{H}$. Kleiner's assumption would also seem to be doubtful because the Zeeman term is odd in $\vec{H}$.

In Kleiner's treatment, because the requirement of invariance of the tensor under $\mathcal{K}(\vec{H})$ leads to the right answer for the generalized Onsager reciprocal relations, it is then *assumed* that prescription B will lead to the correct simplification of any tensor describing a transport property of a crystal in a situation with symmetry described by one of the magnetic point groups. It is not obvious, to us at least, that this assumption is justified, and if it were so, it would be useful to see some general proof that prescription B is equivalent to prescription C. We submit, however, that no general proof exists and that prescription B is an artifice. This in itself does not matter *provided* it leads to the correct answers. However, by imposing a symmetry different from that actually possessed by the system, there would seem to be a danger that Kleiner's treatment may lead to an incorrect simplification of certain tensors. We can illustrate this by considering a simple example and, to facilitate comparison with the results of Kleiner, we assume that all the components of the electrical-conductivity tensor are real.

Consider two crystals with the point-group symmetry $[\mathcal{K}^L(0)]$ of 2′2′2 and 4′22′ in the absence of any external magnetic field. Suppose that each of these crystals is now placed in a magnetic field $\vec{H}$ that is directed parallel to the z axis in each case. We can identify the various groups involved when $\vec{H}$ is present by inspection of Table I of Ref. 7:

$\mathcal{K}^L(0)$	$\vec{H}$	$\mathcal{K}^L(\vec{H})$	$\mathcal{G}^L(\vec{H})$	Case and category
2′2′2	∥ 2	2′2′2	2′2′2	ii c
4′22′	∥ 4′	4′22′	2′2′2	v a.

For $\sigma_{ij}(\vec{H})$, the electrical conductivity in a nonzero magnetic field, we find from Table VI of Ref. 5 and Table IV of Ref. 7, respectively, for these two groups $\mathcal{K}^L(\vec{H})$

<u>2′2′2</u>

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{xx}(\vec{H}) & \sigma_{xy}(\vec{H}) & 0 \\ -\sigma_{xy}(\vec{H}) & \sigma_{yy}(\vec{H}) & 0 \\ 0 & 0 & \sigma_{zz}(\vec{H}) \end{pmatrix} , \tag{21}$$

<u>4′22′</u>

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{xx}(\vec{H}) & \sigma_{xy}(\vec{H}) & 0 \\ \sigma_{yx}(\vec{H}) & \sigma_{xx}(\vec{H}) & 0 \\ 0 & 0 & \sigma_{zz}(\vec{H}) \end{pmatrix} , \tag{22}$$

where we have used $\sigma(\vec{H}) = \sigma^e(\vec{H}) + \sigma^0(\vec{H})$.

To find the form of $\sigma_{ij}(\vec{H})$ using prescription C we need to use the group $\vec{M}$ [or $\mathcal{G}^L(\vec{H})$], which is 2′2′2 and has the unitary subgroup 2 (C_2). For 2 (C_2), $\sigma_{ij}(\vec{H})$ takes the form

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{xx}(\vec{H}) & \sigma_{xy}(\vec{H}) & 0 \\ \sigma_{yx}(\vec{H}) & \sigma_{yy}(\vec{H}) & 0 \\ 0 & 0 & \sigma_{zz}(\vec{H}) \end{pmatrix} , \tag{23}$$

where, of course, we have not required $\sigma_{ij}(\vec{H})$ to be a symmetric tensor. The antiunitary operations give, using, for example, C_{2x}, which has the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,$$

as S in Eq. (17),

$$\sigma_{xx}(\vec{H}) = \sigma_{xx}(\vec{H})^* ,$$
$$\sigma_{yy}(\vec{H}) = \sigma_{yy}(\vec{H})^* , \tag{24}$$
$$\sigma_{zz}(\vec{H}) = \sigma_{zz}(\vec{H})^* ,$$

and

$$\sigma_{xy}(\vec{H}) = -\sigma_{xy}(\vec{H})^* . \tag{25}$$

Equations (24) show that the diagonal elements of $\sigma_{ij}(\vec{H})$ must be real, while Eq. (25) shows that $\sigma_{xy}(\vec{H})$ must be imaginary. If we use Onsager's relation for the diagonal elements we find

$$\sigma_{xx}(\vec{H}) = \sigma_{xx}(-\vec{H}) ,$$
$$\sigma_{yy}(\vec{H}) = \sigma_{yy}(-\vec{H}) , \tag{26}$$
$$\sigma_{zz}(\vec{H}) = \sigma_{zz}(-\vec{H}) ,$$

which means that each of the diagonal components of the conductivity must be an even function of $\vec{H}$, i.e., a function of $\vec{H}^2$. This is consistent with the observed $\vec{H}^2$ behavior of the magnetoresistance of many metals. If we use Onsager's relation for the off-diagonal element $\sigma_{xy}(\vec{H})$ we obtain

$$\sigma_{xy}(\vec{H}) = \sigma_{yx}(-\vec{H}) \tag{27}$$

or

$$\sigma_{yx}(\vec{H}) = \sigma_{xy}(-\vec{H}) \ . \tag{28}$$

The final expression for $\sigma_{ij}(\vec{H})$ is therefore

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{xx}(\vec{H}^2) & \sigma_{xy}(\vec{H}) & 0 \\ \sigma_{xy}(-\vec{H}) & \sigma_{yy}(\vec{H}^2) & 0 \\ 0 & 0 & \sigma_{zz}(\vec{H}^2) \end{pmatrix} , \tag{29}$$

where the diagonal elements are real and the off-diagonal elements are imaginary. The thermal conductivity takes the same form. Equation (29) shows that the result of using prescription C is different from prescription B for $2'2'2$ [see Eq. (21)] and is more restrictive than, although not incompatible with, prescription B for $4'22'$ [see Eq. (22)].

For completeness we should also mention that using prescription A for the cases in our example would involve using just the unitary subgroup of $\vec{M}$ [or $\mathcal{G}(\vec{H})$], which in this case is the point group 2 (C_2), followed by the use of Eq. (5). This leads to the same form for $\sigma_{ij}(\vec{H})$ as that given in Eq. (29), but without the restrictions of the diagonal and off-diagonal elements as to being real and imaginary, respectively.

There are several points to note about the results of using prescriptions B and C. The first is the fact that $\vec{M}$ [or $\mathcal{G}(\vec{H})$] is the same for both cases, so that, whereas prescription B leads to two different forms for $\sigma_{ij}(\vec{H})$ for these two examples, prescription C (or, of course, prescription A) leads to a common form for $\sigma_{ij}(\vec{H})$ in both cases. The second point is that neither prescription B nor prescription C (nor, of course, prescription A) predicts the vanishing of the two off-diagonal components $\sigma_{xy}(\vec{H})$ and $\sigma_{yx}(\vec{H})$. Prescription B appears not to predict the behavior of the diagonal components as even functions of $\vec{H}$. Finally, prescriptions B and C predict different answers for both examples.

Having shown, by considering a simple example, that prescriptions B and C may give different answers, it is important to try to decide which one is actually correct. This is not easy because the differences between Eqs. (21), (22), and (29) are relatively difficult to distinguish experimentally. Such an experiment might involve measurements of the Hall effect for a crystal with the symmetry of $2'2'2$ or $4'22'$ with the magnetic field applied in the z direction to distinguish between Eqs. (22) and (29). There are various difficulties in the way of such an experiment because the symmetry predictions will only apply if $\vec{H}$ is *exactly* along the z axis. Neglecting operations that involve the space inversion I, the point groups $2'2'2$ and $4'22'$ describe the symmetry of antiferromagnetic NiF_2 (in a nonstandard crystallographic orientation) and antiferromagnetic MnF_2, respectively. Thus, on symmetry grounds, suitable crystals for such Hall-effect measurements would be antiferromagnetic NiF_2 or MnF_2. However, on practical grounds, the very low conductivity of such crystals would make these experiments very difficult to perform.

In view of the fact that prescriptions B and C may lead to different forms for the tensor representing a given property of a magnetic crystal—and also in view of the fact that results for tensors for the electrical conductivity, thermal conductivity, and thermoelectric effects have been tabulated previously by Kleiner[5,7]—it would seem to be useful to tabulate the results for the same properties using prescription C instead of prescription B. These results are presented in Table I.

There are some theoretical objections to prescription B. These include the objection that there is no direct physical justification for the use of Eq. (20) for those antiunitary elements of $\mathcal{K}(\vec{H})$ that are not also in $\vec{M}$ [or $\mathcal{G}(\vec{H})$]. Another objection to prescription B is the statement on p. 321 of Ref. 5 that if $\mathcal{K}(\vec{H})$ is in category b "there is consequently no Onsager reciprocity relation." This is a rather disturbing conclusion. Suppose that we consider a crystal with only the symmetry of the trivial point group 1 (C_1). If we had never thought of considering crystallographic (or magnetic) symmetry at all and had just gone through the usual considerations of irreversible thermodynamics, we would presumably have concluded, as usual, that Onsager's relation would apply. However, if one reduces the symmetry from that of any one of the many possible category-b groups (see Table I of Ref. 7) to no point-group symmetry at all, the Onsager reciprocal relation which, according to Kleiner, would not apply in the higher-symmetry situation, would suddenly reappear when the symmetry is removed. This would make it possible for a *reduction* in the crystallographic symmetry to lead to a reduction, rather than an increase, in the number of independent tensor components that have to be specified. This is rather curious and is the reverse of what happens in every other example of the application of group theory in physics; a reduction in the number of independent parameters needed to describe a physical situation always occurs as a result of an *increase*, not a decrease, in the symmetry of the system. We therefore regard Kleiner's conclusion that there is no Onsager reciprocal relation for category-b magnetic groups as highly suspect.

V. EXTRAORDINARY HALL EFFECT IN FERROMAGNETIC Co

The question of the existence of the extraordinary Hall effect in a ferromagnetic metal, that is, the contribution to the Hall effect arising from the internal field in such a metal, has played a significant part in the previous discussions of the symmetry properties of transport coefficients for magnetic crystals.[3–7] The discussions were particularly concerned with the example of ferromagnetic Co, which has the hcp structure. The symmetry predictions of the form of $\sigma_{ij}(\vec{H})$ will depend on the group $\vec{M}$ of the symmetry operations of the magnetically ordered phase, which in any given magnetic domain of a single-crystal specimen will depend on the relative orientation of the magnetization of that domain and the crystallographic axes of the specimen. The group $\vec{M}$ will be the intersection of the crystallographic point group $6/mmm$, or $6/mmm1'$, and the noncrystallographic group ∞/mm' that describes the symmetry of the axial vector representing the magnetization, or the internal field, in the domain under consideration. If the orientation of the magnetization is arbitrary relative to the crystallographic axes, the intersection $\vec{M}$ will be the almost trivial point group $\bar{1}$ (C_i) which contains only the identity E and the space inversion I. For this group $\vec{M}$ the conductivity tensor $\sigma_{ij}(\vec{H})$ is therefore

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{11}(\vec{H}) & \sigma_{12}(\vec{H}) & \sigma_{13}(\vec{H}) \\ \sigma_{21}(\vec{H}) & \sigma_{22}(\vec{H}) & \sigma_{23}(\vec{H}) \\ \sigma_{31}(\vec{H}) & \sigma_{32}(\vec{H}) & \sigma_{33}(\vec{H}) \end{pmatrix} \tag{30}$$

or, if we make use of Onsager's theorem,

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{11}(\vec{H}^2) & \sigma_{12}(\vec{H}) & \sigma_{13}(\vec{H}) \\ \sigma_{12}(-\vec{H}) & \sigma_{22}(\vec{H}^2) & \sigma_{23}(\vec{H}) \\ \sigma_{13}(-\vec{H}) & \sigma_{23}(-\vec{H}) & \sigma_{33}(\vec{H}^2) \end{pmatrix} . \tag{31}$$

Clearly, therefore, for an arbitrary orientation of the magnetization relative to the crystallographic axes, the extraordinary Hall effect is permitted by symmetry considerations and, indeed, appears to have been observed in practice.[17,18]

TABLE I. Forms of tensors for thermogalvanomagnetic coefficients for type-III Shubnikov point groups. (a) The elements in each of the type-III Shubnikov point groups can be identified, for example, from Table 7.1 of Ref. 16. (b) To facilitate comparison with the tables of Kleiner (Refs. 5–7) we have restricted the tensor components to being real. (c) Only the forms of $\sigma_{ij}(\vec{H})$ and $\alpha_{ij}(\vec{H})$ are given. The form of $\kappa_{ij}(\vec{H})$ is the same as that of $\sigma_{ij}(\vec{H})$, while the form of $\pi_{ij}(\vec{H})$ can be obtained from that of $\alpha_{ij}(\vec{H})$ via Eq. (6).

Point groups	$\sigma_{ij}(\vec{H})$	$\alpha_{ij}(\vec{H})$
Triclinic: $\bar{1}'$	$\begin{pmatrix} \sigma_{11}(\vec{H}^2) & \sigma_{12}(\vec{H}) & \sigma_{13}(\vec{H}) \\ \sigma_{12}(-\vec{H}) & \sigma_{22}(\vec{H}^2) & \sigma_{23}(\vec{H}) \\ \sigma_{13}(-\vec{H}) & \sigma_{23}(-\vec{H}) & \sigma_{33}(\vec{H}^2) \end{pmatrix}$	$\begin{pmatrix} \alpha_{11}(\vec{H}) & \alpha_{12}(\vec{H}) & \alpha_{13}(\vec{H}) \\ \alpha_{21}(\vec{H}) & \alpha_{22}(\vec{H}) & \alpha_{23}(\vec{H}) \\ \alpha_{31}(\vec{H}) & \alpha_{32}(\vec{H}) & \alpha_{33}(\vec{H}) \end{pmatrix}$
Monoclinic: $2'$; m'; $2/m'$; $2'/m$; $2'/m'$	$\begin{pmatrix} \sigma_{11}(\vec{H}^2) & \sigma_{12}(\vec{H}) & 0 \\ \sigma_{12}(-\vec{H}) & \sigma_{22}(\vec{H}^2) & 0 \\ 0 & 0 & \sigma_{33}(\vec{H}^2) \end{pmatrix}$	$\begin{pmatrix} \alpha_{11}(\vec{H}) & \alpha_{12}(\vec{H}) & 0 \\ \alpha_{21}(\vec{H}) & \alpha_{22}(\vec{H}) & 0 \\ 0 & 0 & \alpha_{33}(\vec{H}) \end{pmatrix}$
Orthorhombic: $2'2'2$; $m'm'2$; $m'm2'$; $m'm'm'$; mmm'; $m'm'm$	$\begin{pmatrix} \sigma_{11}(\vec{H}^2) & 0 & 0 \\ 0 & \sigma_{22}(\vec{H}^2) & 0 \\ 0 & 0 & \sigma_{33}(\vec{H}^2) \end{pmatrix}$	$\begin{pmatrix} \alpha_{11}(\vec{H}) & 0 & 0 \\ 0 & \alpha_{22}(\vec{H}) & 0 \\ 0 & 0 & \alpha_{33}(\vec{H}) \end{pmatrix}$
Trigonal, tetragonal, and hexagonal[a]	$\begin{pmatrix} \sigma_{11}(\vec{H}^2) & 0 & 0 \\ 0 & \sigma_{11}(\vec{H}^2) & 0 \\ 0 & 0 & \sigma_{33}(\vec{H}^2) \end{pmatrix}$	$\begin{pmatrix} \alpha_{11}(\vec{H}) & 0 & 0 \\ 0 & \alpha_{11}(\vec{H}) & 0 \\ 0 & 0 & \alpha_{33}(\vec{H}) \end{pmatrix}$
Cubic: $m'3$; $\bar{4}'3m'$; $4'32'$; $m'3m'$; $m'3m$; $m3m'$	$\begin{pmatrix} \sigma_{11}(\vec{H}^2) & 0 & 0 \\ 0 & \sigma_{11}(\vec{H}^2) & 0 \\ 0 & 0 & \sigma_{11}(\vec{H}^2) \end{pmatrix}$	$\begin{pmatrix} \alpha_{11}(\vec{H}) & 0 & 0 \\ 0 & \alpha_{11}(\vec{H}) & 0 \\ 0 & 0 & \alpha_{11}(\vec{H}) \end{pmatrix}$

[a] $4'$; $\bar{4}'$; $42'2'$; $4'22'$; $4/m'$; $4'/m'$; $4'/m$; $4m'm'$; $4'mm'$; $\bar{4}2'm'$; $\bar{4}'2m'$; $\bar{4}'m2'$; $4/m'm'm'$; $4/m'mm$; $4'/mmm'$; $4'/m'm'm$; $4/mm'm'$; $32'$; $3m'$; $\bar{6}'$; $\bar{6}m'2'$; $\bar{6}'m2'$; $\bar{6}'m'2$; $6'$; $\bar{3}'$; $\bar{3}m'$; $\bar{3}'m$; $\bar{3}'m'$; $62'2'$; $6'2'2$; $6/m'$; $6'/m'$; $6'/m$; $6m'm'$; $6'm'm$; $6'/mmm'$; $6'/m'm'm$; $6/m'm'm'$; $6/m'mm$; $6/mm'm'$.

If $\mathfrak{M}$, the magnetization of a domain, is in some special orientation relative to the crystallographic axes the number of elements in the group $\vec{M}$ may be increased. In particular, if $\mathfrak{M}$ is parallel to the [0001], [10$\bar{1}$0], or [11$\bar{2}$0] direction, $\vec{M}$ will be $6/mm'm'$, $mm'm'$, or $m'mm'$, respectively, or, ignoring the space inversion, $62'2'$, $22'2'$, or $2'22'$, respectively.[19] The subgroup of unitary elements for $62'2'$ is the point group 6 (C_6) which simplifies $\sigma_{ij}(\vec{H})$ to

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{xx}(\vec{H}) & \sigma_{xy}(\vec{H}) & 0 \\ -\sigma_{xy}(\vec{H}) & \sigma_{xx}(\vec{H}) & 0 \\ 0 & 0 & \sigma_{zz}(\vec{H}) \end{pmatrix} , \qquad (32)$$

and the use of the Onsager theorem shows that the diagonal elements are functions of $\vec{H}^2$ and that

$$\sigma_{xy}(\vec{H}) = \sigma_{yx}(-\vec{H}) \; . \qquad (33)$$

Therefore, according to prescription A, $\sigma_{ij}(\vec{H})$ reduces to

$$\sigma_{ij}(\vec{H}) = \begin{pmatrix} \sigma_{xx}(\vec{H}^2) & \sigma_{xy}(\vec{H}) & 0 \\ -\sigma_{xy}(\vec{H}) & \sigma_{xx}(\vec{H}^2) & 0 \\ 0 & 0 & \sigma_{zz}(\vec{H}^2) \end{pmatrix} , \qquad (34)$$

where $\sigma_{xy}(\vec{H})$ is an odd function of $\vec{H}$ [from Eqs. (32) and (33)]. The off-diagonal elements in Eq. (34) are nonzero and therefore, in contradiction to the statement made by Kleiner,[5,6] it appears that when prescription A is correctly interpreted and applied to $62'2'$ it does lead to nonvanishing off-diagonal components. The application of prescription B to this example also leads to $\sigma_{ij}(\vec{H})$ of the form in Eq. (32) with no further restrictions (see Ref. 6). Thus, both prescriptions A and B allow the existence of the extraordinary Hall effect in ferromagnetic Co magnetized parallel to [0001]. The example of $2'2'2$, which is equivalent to $22'2'$ or $2'22'$, has already been discussed [see Eqs. (21) and (29)]; in this case when prescriptions A and B are used, the off-diagonal component $\sigma_{xy}(\vec{H})$ is also nonvanishing and real so that these prescriptions both allow the existence of the extraordinary Hall effect in a domain of ferromagnetic Co magnetized parallel to [10$\bar{1}$0] or [11$\bar{2}$0]. If one uses prescription C for $\mathfrak{M}$ parallel to [0001], one obtains Eq. (34) as before and then makes use of, for example, θC_{2x} as the additional antiunitary generating element. Equations (24) and (25) will then apply again so that, as in the case of $2'2'2$ considered previously, we find that the diagonal elements of $\sigma_{ij}(\vec{H})$ in Eq. (34) must be real, while the off-diagonal elements must be imaginary. Therefore, for direct currents and with use of prescription C, $\sigma_{xy}(\vec{H})$ will be zero for a domain of ferromagnetic Co that is magnetized parallel to [0001], [10$\bar{1}$0], or [11$\bar{2}$0].

To summarize, therefore, the existence of the extraordinary Hall effect in a single crystal of ferromagnetic Co is allowed by prescriptions A and B for all relative orientations of the magnetization and the crystallographic axes and is also allowed by prescription C for all orientations except when the magnetization is parallel to [0001], [10$\bar{1}$0], or [11$\bar{2}$0]. The predictions of prescriptions A and B are, therefore, fully compatible with the experimental observation of the extraordinary Hall effect in ferromagnetic Co. It is not quite so easy to reconcile the predictions of prescription C with the experimental observation of the extraordinary Hall effect in ferromagnetic Co. However, it has to be remembered that the symmetry considerations predicting a null extraordinary Hall effect in Co, using prescription C, will *only apply for a single-crystal specimen that also constitutes a single magnetic domain* (i.e., is saturated) *and is magnetized exactly parallel to* [0001], [10$\bar{1}$0], *or* [11$\bar{2}$0]. One or two other points should also be noted. First, the symmetry predictions strictly refer to the total magnetic field in the metal which must therefore include any external field $\vec{H}_0$ that is present in addition to the internal field in the metal. Moreover, it is also assumed that the shape of the specimen is such that demagnetizing effects do not cause local deviations in the direction of $\vec{H}$, or of the magnetization, from the intended special orientation. A perusal of the published accounts of the experimental work involved[17,18,20] shows that it is far from obvious that all of these conditions have been satisfied in practice. It therefore seems that it would be an interesting and important experiment to test the predictions of prescription C by performing Hall-effect measurements on a single-domain (i.e., saturated) single-crystal specimen of ferromagnetic Co to see if the extraordinary component vanishes for the special magnetization directions mentioned above.

VI. CONCLUSION

We accept the general validity of the criticism made by Kleiner of the neglect by Birss of the antiunitary symmetry elements when simplifying the form of the tensor representing a transport coefficient of a magnetic crystal or of a nonmagnetic crystal in an applied magnetic field $\vec{H}$. However, we are not in agreement with some of the aspects of Kleiner's treatment. Since Kleiner's treatment (prescription B) and our own treatment (prescription C) lead to different predictions in certain cases, it should be possible to distinguish experimentally between these two prescriptions, for example, by studies of the extraordinary Hall effect, although such experiments may be difficult to perform.

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Low-Temperature Behavior of the Planar Heisenberg Ferromagnet*†

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We have examined the low-temperature properties of the cubic-planar Heisenberg ferromagnet with nearest-neighbor exchange which is defined by the Hamiltonian $\mathcal{H} = -\sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j + \sum_{i,j} (J_{ij} - K_{ij}) S_i^x S_j^x$, where $-J \le K \le J$ (J positive). We find that as the exchange-anisotropy parameter $\theta = (J-K)/J$ ranges over the planar ferromagnetic stability limits $0 \le \theta \le 2$, the behavior of the system changes from that of the isotropic ferromagnet at $\theta = 0$ into that of the isotropic antiferromagnet at $\theta = 2$. The system's noninteracting-spin-wave frequency, ground-state energy, zero-point spin deviation, and lowest-order renormalized frequency scale between isotropic ferromagnetic and antiferromagnetic values as θ goes from zero to two. Over most of the system's stability range, the planar ferromagnet exhibits a mixture of properties combining characteristics of its intrinsic ferromagnetism with those of the antiferromagnet. This behavior is discussed in terms of an isomorphic mapping symmetry for nearest-neighbor exchange in loose-packed lattices which requires that in the limit $\theta = 2$ the planar ferromagnet be unitarily equivalent to the isotropic antiferromagnet.

I. INTRODUCTION

The planar Heisenberg ferromagnet was first introduced and studied as a magnetic analog to the lattice-liquid model for the superfluid transition in ^{4}He.[1] More recently, however, there has been increased interest in its behavior as a purely magnetic system.[2,3] In this paper, we study the properties of the planar ferromagnet in the low-temperature spin-wave regime. We shall be concerned with the Hamiltonian

$$\mathcal{H} = -\sum_{i,j} (K_{ij} S_i^x S_j^x + J_{ij} S_i^y S_j^y + J_{ij} S_i^z S_j^z) , \qquad (1.1)$$