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Critical Behavior of a Classical Heisenberg Ferromagnet with Many Degrees of Freedom*

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The critical behavior of a classical Heisenberg ferromagnet is studied in the limit where the spin dimensionality N is large. Corrections of order $1/N$ to the spherical model are obtained as functions of a continuous dimension d, $2<d<4$. Particular attention is given to the behavior near the coexistence curve. The divergence of the magnetic susceptibility below T_c as the external field vanishes is discussed through a nonlinear realization of the $O(N)$ symmetry, as well as in the $1/N$ and $4-d$ expansions.

I. INTRODUCTION

Universality of scaling behavior in critical phenomena applies only to systems with a given number N of internal degrees of freedom. This is manifest in recent works which use the ϵ expansion technique developed by Wilson and Fisher.[1] This method provides systematic corrections to mean-field theory by a perturbation expansion in $\epsilon=4-d$, where d is the dimension of space. Both critical exponents[1] and the scaling equation of state[2] exhibit explicitly a dependence on N.

In this paper ϵ is not assumed to be small, but may take any value between zero and two. The approximation now lies in the assumption that all quantities may be expanded in power of N^{-1} for N large. The motivation lies in the result of Stanley[3] that the limit $N\to\infty$ of a classical Heisenberg ferromagnet, in which each "spin" has N components, is identical to the exactly soluble spherical model of Berlin and Kac.[4] More recently a simple diagrammatic approach has been presented in a field-theoretical framework by Wilson.[5] This method gives both Stanley's result and systematic corrections in powers of N^{-1}. It is here applied to the calculation of critical exponents and of the equation of state of a magnetic system, to order $1/N$.

The numerical agreement of this expansion with the behavior of an ordinary magnetic system where $N=3$ is not expected to be particularly satisfactory. In fact, the ϵ expansion results seem to indicate that the asymptotic region in N requires at least $N\gtrsim 8$.

Therefore, the aim of this $1/N$ expansion is rather to give theoretical information which the ϵ expansion is not able to provide. In particular, our interest was to study the behavior of the system near the coexistence curve, i.e., below the critical temperature when the applied magnetic field H tends to zero. In this region there are two different characteristic lengths associated with transverse and longitudinal magnetic susceptibilities. It is not clear that the ϵ expansion in which the coupling constant is fixed to induce the expected scaling only in the longitudinal correlation length, does not break down in the vicinity of the coexistence curve. A related problem is the appearance of infrared divergences[2] in the ϵ expansion of the longitudinal susceptibility in the form of powers of $\epsilon \ln H$. Perturbation theory always produces infrared logarithmic singularities which are in fact generated by the expansion of a power behavior in H, which, as explained in Sec. IV, is difficult to reconstruct. New information is obtained about this region and about the nature of the divergences as H tends to zero. The results are in agreement with the predictions based on nonlinear realizations of the $O(N)$ symmetry.

The outline of the paper is as follows. Section II contains the notation, describes the perturbation theory and derives the simplest leading terms. In Sec. III the $1/N$ corrections to critical exponents and to the equation of state are calculated. Section IV is concerned with the vicinity of the coexistence curve. The relevance of the nonlinear realization is discussed.

II. DESCRIPTION OF MODEL

As in previous calculations in the ϵ expansion,[1,2] we use a local $(\phi^2)^2$ interaction in terms of N scalar fields $\phi_i(x)$ [ϕ^2 means $\sum_1^N \phi_i^2(x)$]. In the presence of a constant external field H, conventionally in the direction of the first axis, the Hamiltonian is

$$\frac{\mathcal{H}}{kT}=\int d^dx\left(\frac{1}{2}\sum_1^N(\nabla\phi_i)^2+\tfrac{1}{2}r_0\phi^2+\frac{u_0}{4!}(\phi^2)^2-H\phi_1\right), \tag{1}$$

where r_0 and u_0 are two constants, and r_0 depends linearly on temperature. The perturbation theory of this model has been discussed in Ref. 2. For completeness let us recall briefly that the field $\phi_1(x)$ is translated by its expectation value, the magnetization M, and is thus replaced by a field $L(x)$ with zero expectation value. Also "mass" counter terms, which are different for the first mode along H and the $(N-1)$ transverse ones, are added so that to all orders in u_0, the propagators at zero momentum (i.e., the longitudinal and transverse magnetic susceptibilities) are

$$r_L^{-1}=\int d^dx[\langle\phi_1(x)\phi_1(0)\rangle-M^2], \tag{2}$$

$$r_T^{-1}\delta_{ij}=\int d^dx\,\langle\phi_i(x)\phi_j(0)\rangle,\quad 2\le i,j\le N. \tag{3}$$

The Hamiltonian is then split into a free part

$$\frac{\mathcal{H}_0}{kT}=\frac{1}{2}\int d^dx\left((\nabla L)^2+r_LL^2+\sum_2^N[(\nabla\phi_i)^2+r_T\phi_i^2]\right) \tag{4}$$

and a perturbation

$$\frac{\mathcal{H}_1}{kT}=\int d^dx\Big[\tfrac{1}{2}[r_0-r_L+\tfrac{1}{2}(u_0M^2)]L^2+\tfrac{1}{2}[r_0-r_T+\tfrac{1}{6}(u_0M^2)]\sum_2^N\phi_i^2+\frac{u_0}{4!}\Big(L^2+\sum_2^N\phi_i^2\Big)^2$$
$$+\frac{u_0}{3!}ML\Big(L^2+\sum_2^N\phi_i^2\Big)\{[r_0+\tfrac{1}{6}(u_0M^2)]M-H\}L\Big], \tag{5}$$

where

$$L(x)=\phi_1(x)-M. \tag{6}$$

Then the equation

$$\langle L(x)\rangle=0 \tag{7}$$

is expanded in powers of the interaction and use is made of the relations

$$r_L=\frac{\partial H}{\partial M},\quad r_T=\frac{H}{M}. \tag{8}$$

The first of these follows from the definition (2); the second is a result of rotation invariance and is also derived as a Ward identity in Ref. 2.

The large-N limit of field-theoretical models has been discussed extensively by Wilson.[5] The main feature is that, at a given order in u_0, a power of N may be generated by each closed loop and therefore dominant graphs are those with the largest number of bubbles. To compensate for this power of N, u_0 is considered to be of order $1/N$. For example, the leading corrections in order $1/N$ in zero field above T_c to the self-energy operator are given by the sum of the chain of bubbles of Fig. 1. Such sums always give rise to simple geometric series.

A. Limit of Infinite N

When N is strictly infinite we expect to obtain the results of the spherical model[3,4] in the scaling region. It is extremely simple to show this.

First, from Fig. 1 it is clear that the correction to the propagator, in zero field above T_c, is of order $1/N$. Therefore, the field has canonical dimensions or, equivalently, the critical exponent η vanishes.

The only diagram which contributes to Eq. (7), is the closed loop of Fig. 2, and after subtraction at the critical point where r_L and r_T vanish, we obtain

$$\frac{H}{M}=t+\tfrac{1}{6}(u_0M^2)-\tfrac{1}{6}(u_0N)\frac{\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}}\times\frac{\pi}{\sin\frac{1}{2}\pi\epsilon}\left(\frac{H}{M}\right)^{1-\epsilon/2}, \tag{9}$$

where $t=r_0-r_{0c}$ is proportional to the reduced temperature $(T-T_c)/T_c$. In the scaling region, with obvious normalizations for temperature and

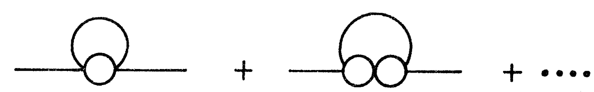

FIG. 1. Diagrams contributing to η at order $1/N$.

fields, this yields the equation of state

$$H/M^{\delta} = (t/M^{1/\beta} + 1)^{\gamma} \tag{10}$$

and the critical exponents

$$\beta = \tfrac{1}{2}, \quad \gamma = 1/(1 - \tfrac{1}{2}\epsilon), \quad \delta = 1 + 2/(1 - \tfrac{1}{2}\epsilon). \tag{11}$$

It is easy to verify that this is precisely how the Berlin and Kac solution to the spherical model behaves in this region.

Let us note and postpone and discussion on the meaning of this result to Sec. IV, that in the vicinity of the coexistence curve the magnetic susceptibility $\chi = r_L^{-1}$ diverges like $H^{-\epsilon/2}$ since from Eqs. (8) and (10), we obtain

$$r_L^{-1} = \tfrac{1}{2}(1 - \tfrac{1}{2}\epsilon) M^{-2} r_T^{-\epsilon/2}. \tag{12}$$

III. 1/N CORRECTIONS

It is useful to understand first the critical behavior in zero external field. The calculation of η, which characterizes the correlation function at $T = T_c$ according to

$$\int d^d x\, e^{i\vec{q}\cdot\vec{x}} \langle \phi_i(x)\phi_i(0)\rangle_{H=0,T=T_c} \propto q^{\eta-2},$$

has been performed by Wilson[5] by consideration of the diagrams of Fig. 1. The result reads

$$\eta = \frac{1}{N}\epsilon^2 \frac{\sin\frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(1-\frac{1}{2}\epsilon)\Gamma(3-\frac{1}{2}\epsilon)}. \tag{13}$$

The critical exponent γ governs the divergence of the magnetic susceptibility when $T \to T_c^+$:

$$r_L\,|_{H=0} = r_T\,|_{H=0} \propto (T - T_c)^{\gamma}. \tag{14}$$

The $1/N$ correction to the value (11) of γ has been obtained by Abe[6] who keeps one more term in Stanley's approach,[3] and also by the same method as used here by Ma.[7] Nevertheless, this perturbative calculation of γ is presented here since it simplifies the treatment of the equation of state.

The relevant diagrams are the sums of streams of bubbles depicted in Fig. 3. Analytically they give

$$t = r - \tfrac{1}{6}u_0 N \int_q \left(\frac{1}{q^2+r} - \frac{1}{q^2}\right) - \tfrac{1}{6}(u_0 N)\int_q \left(\frac{\tilde{\Sigma}(q,r)}{(q^2+r)^2} - (r=0)\right)$$

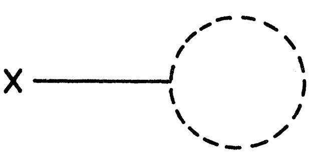

FIG. 2. Dominant contribution to the expectation value of the longitudinal field; solid and dashed lines represent, respectively, longitudinal and transverse propagators.

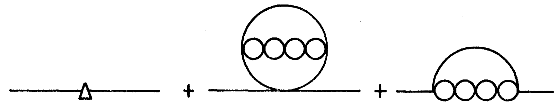

FIG. 3. Mass counter term and typical streams of bubbles for γ.

$$- (\tfrac{1}{3}u_0)\int_q \left(\frac{1}{q^2+r}\,\frac{1}{1+\frac{1}{6}(u_0 N) I(q,r)} - (r=0)\right), \tag{15}$$

where

$$\int_q \equiv [1/(2\pi)^d] \int d^d q,$$

$$I(q,r) = \int_p (\vec{p}^{\,2}+r)^{-1}[(\vec{p}+\vec{q})^2 + r]^{-1}, \tag{16}$$

and

$$\tilde{\Sigma}(q,r) = \Sigma(q,r) - \Sigma(0,r) = -\tfrac{1}{3}u_0 \int_p \{1 + \tfrac{1}{6}(u_0 N) I(p,r)\}^{-1} \times \left(\frac{1}{(\vec{p}+\vec{q})^2+r} - \frac{1}{p^2+r}\right). \tag{17}$$

In the scaling region where r is small the dominant contributions to the right-hand side of Eq. (15) are terms of order $r^{1-\epsilon/2}$ and $(1/N)r^{1-\epsilon/2}\ln r$, and terms in r and $r\ln r$ are negligible.

Then Eq. (15) simplifies to

$$t = \tfrac{1}{6}(u_0 N)\frac{2\pi^{2-\epsilon/2}\pi r^{1-\epsilon/2}}{(2\pi)^{4-\epsilon}\Gamma(2-\frac{1}{2}\epsilon)2\sin(\frac{1}{2}\pi\epsilon)} + \tfrac{1}{3}u_0 \int_p J(p,r)[I^{-1}(p,r) - I^{-1}(p,0)] + \tfrac{1}{3}u_0\int_p I^{-1}(p,0)[J(p,r) - J(p,0)], \tag{18}$$

in which

$$J(p,r) = \int_k (k^2+r)^{-2}\left(\frac{1}{(\vec{k}+\vec{p})^2+r} - \frac{1}{(p^2+r)}\right) = -\frac{1}{2(p^2+r)}\left(2(1-\epsilon)I(p,r) - 3r\frac{\partial I}{\partial r}(p,r)\right). \tag{19}$$

The relevant region of integration in Eq. (18) is $p^2 \gg r$, where

$$I(p,r) \sim \frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}}\frac{\pi}{2\sin\frac{1}{2}\pi\epsilon} \times p^{-\epsilon}\left[(1-\tfrac{1}{2}\epsilon)\frac{\Gamma^2(1-\frac{1}{2}\epsilon)}{\Gamma(2-\epsilon)} - 2\left(\frac{r}{p^2}\right)^{1-\epsilon/2}\right], \quad p^2 \gg r \tag{20}$$

from which follows

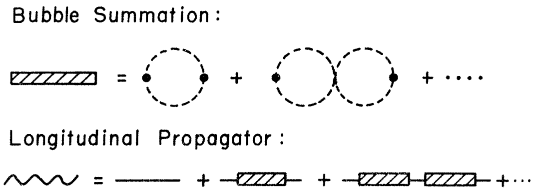

FIG. 4. Definition of the dressed longitudinal propagator.

$$t=\tfrac{1}{6}u_0N\frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)\,(2\pi)^{4-\epsilon}}\times\left(\frac{\pi}{2\sin\frac{1}{2}\pi\epsilon}\,r^{1-\epsilon/2}+\frac{3\Gamma(2-\epsilon)}{N\Gamma^2(1-\frac{1}{2}\epsilon)}\,r^{1-\epsilon/2}\ln r\right)$$

and therefore

$$\gamma=\frac{1}{1-\frac{1}{2}\epsilon}-\frac{3\epsilon}{N}\,\frac{\sin\frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon}\,\frac{\Gamma(2-\epsilon)}{\Gamma^2(2-\frac{1}{2}\epsilon)}\ . \qquad (21)$$

It is easy to verify that the small-ϵ behavior of these critical exponents agrees with the results obtained by the ϵ expansion.

A. Equation of State

A nonzero external field H is now present and is to be related to the magnetization and the temperature through Eqs. (7) and (8). In order to determine which diagrams contribute to the $1/N$ terms, one has to keep in mind that the spontaneous magnetization is of order $\sqrt{N}$, as can be seen from the lowest order Eq. (9). This observation has the immediate effect of inducing a zeroth-order modification of the longitudinal propagator, as shown in Fig. 4, and the result reads

$$[q^2+r_L-\tilde{\Sigma}_L(q,r_T)]^{-1}\ ,$$

where

$$\tilde{\Sigma}_L(q,r_T)=-\tfrac{1}{3}u_0M^2\{[1+(\tfrac{1}{6}u_0N)I(q,r_T)]^{-1}-[1+(\tfrac{1}{6}u_0N)I(0,r_T)]^{-1}\}\ . \qquad (22)$$

In terms of this modified propagator, the $1/N$ contributions to the equation of state are shown in Fig. 5. All resummations lead to geometric series, but "end effects" should not be overlooked to get the correct weights: Figs. 5(a) and 5(b) have to be counted separately, whereas Figs. 5(d)–5(f) may be simply combined. The result may be written

$$0=t-\frac{H}{M}+\tfrac{1}{6}u_0M^2+\tfrac{1}{6}u_0N\int_q\left(\frac{1}{q^2+r_T}-\frac{1}{q^2}\right)+\tfrac{1}{6}u_0\int_q\left(\frac{1}{q^2+r_L-\tilde{\Sigma}_L(q,r_T)}-\frac{1}{q^2}\right)$$
$$+\frac{u_0}{3}\int_q\left(\frac{[1+\frac{1}{6}u_0NI(q,r_T)]^{-1}}{q^2+r_L-\tilde{\Sigma}_L(q,r_T)}-\frac{[1+(\frac{1}{6}u_0N)I(q,0)]^{-1}}{q^2}\right)$$
$$+\frac{u_0^3M^2N}{54}\int_{q,k}(q^2+r_T)^{-2}[k^2+r_L-\tilde{\Sigma}_L(k,r_T)]^{-1}[1+(\tfrac{1}{6}u_0N)I(k,r_T)]^{-2}\{[(\vec{k}+\vec{q})^2+r_T]^{-1}-(k^2+r_T)^{-1}\}$$
$$-\frac{Nu_0^2}{18}\int_{q,k}\left[(q^2+r_T)^{-2}[1+(\tfrac{1}{6}u_0N)I(k,r_T)]^{-1}\left(\frac{1}{(\vec{k}+\vec{q})^2+r_T}-\frac{1}{k^2+r_T}\right)-(r_T=0)\right]\ . \qquad (23)$$

Let us now simplify this equation. First, the diagram of Fig. 5(c) recombines with the zeroth term of Fig. 2 to generate $r_T^{1/\gamma}$ as comparison with Eq. (15) indicates. Then the dominant contribution comes from the region where the momenta are small, but much bigger than $r_T^{1/2}$. There, $|u_0NI(k,r_T)|$ is much bigger than one. Furthermore, r_L has to be eliminated, but since it appears only in diagrams of order $1/N$, it may be replaced by the zeroth-order expression

$$r_L=(2M^2/N)I^{-1}(0,r_T)+r_T\ .$$

Finally, when all terms which produce powers of r_T higher than $1-\frac{1}{2}\epsilon$ are neglected, we obtain

$$\frac{H}{M}=t+\tfrac{1}{6}u_0M^2-(\tfrac{1}{12}u_0N)\frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)\,(2\pi)^{4-\epsilon}}\,\frac{\pi}{\sin\frac{1}{2}\pi\epsilon}\,r_T^{1/\gamma}$$

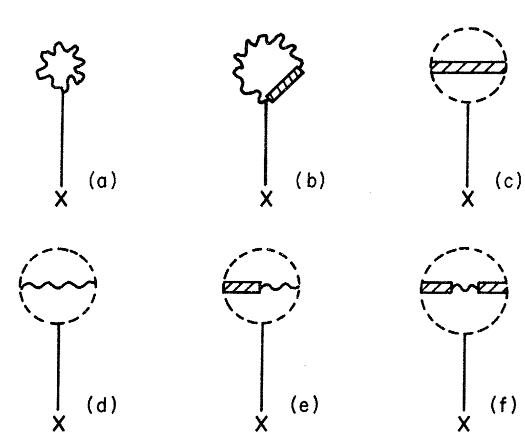

FIG. 5. $1/N$ corrections to the expectation value of the longitudinal field. The notation is as in Figs. 2 and 4.

$$+\frac{2}{3}\frac{u_0 M^2}{N}\int_k \frac{J(k,\, r_T)I^{-2}(k,\, r_T)}{k^2+r_T+(2M^2/N)I^{-1}(k,\, r_T)}$$

$$+\tfrac{1}{6}u_0\int_k\left(\frac{1}{q^2+r_T+(2M^2/N)I^{-1}(q,\, r_T)}-\frac{1}{q^2}\right). \qquad (24)$$

The critical exponents β and δ are easily obtained from this equation. Below T_c, when the applied fields vanish, but not the magnetization, we set $r_T=0$ and obtain

$$0=t+\tfrac{1}{6}u_0M^2+\frac{2}{N}\epsilon(3-2\epsilon)\frac{\sin\frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon}$$

$$\times\frac{\Gamma(2-\epsilon)}{\Gamma^2(2-\frac{1}{2}\epsilon)}(\tfrac{1}{6}r_0M^2)\ln M\,. \qquad (25)$$

If we identify this result with the definition $-t \propto M^{1/\beta}$, we obtain

$$\beta=\frac{1}{2}-\frac{\epsilon(3-2\epsilon)}{2N}\frac{\sin\frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon}\frac{\Gamma(2-\epsilon)}{\Gamma^2(2-\frac{1}{2}\epsilon)}\quad . \qquad (26)$$

Similarly, at T_c, $(t=0)$, δ may be obtained by relating M to r_T. Since the integrals which appear in Eq. (24) have a factor $1/N$, M^2 may be substituted into them by its lowest-order expression

$$M^2=\frac{N2\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}}\frac{\pi}{2\sin\frac{1}{2}\pi\epsilon}r_T^{1-\epsilon/2}\,.$$

Using the relation (19) and the asymptotic expansion (20) for $I(k,\,r)$ one obtains

$$M^2\propto\left(\frac{H}{M}\right)^{1-\epsilon/2}\left(1+\frac{\epsilon^2}{2N}\frac{\sin\frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon}\frac{\Gamma(2-\epsilon)}{\Gamma^2(1-\frac{1}{2}\epsilon)}\frac{1}{1-\frac{1}{2}\epsilon}\ln\frac{H}{M}\right).$$

With the usual definition $H\propto M^\delta$ at $t=0$ this yields

$$\delta=1+\frac{2}{1-\frac{1}{2}\epsilon}-\frac{\epsilon^2}{N}\frac{\sin\frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi\epsilon}\frac{\Gamma(2-\epsilon)}{\Gamma^2(2-\frac{1}{2}\epsilon)}\frac{1}{1-\frac{1}{2}\epsilon}\,. \qquad (27)$$

It is now easy to obtain the equation of state in scaling form.[8] When the integrals in Eq. (24) are subtracted at $r_T=0$, $\frac{1}{6}u_0M^2$ is simply replaced by $\frac{1}{6}u_0M^{1/\beta}$, with β given in Eq. (26). That is all that is required to produce the scaling variables

$$x=\frac{t}{M^{1/\beta}}\,,\qquad y=\frac{H}{M^\delta} \qquad (28)$$

in the relation

$$y^{1/\gamma}=x+1+(1/N)[g(x)-(x+1)g(0)]\,,$$

where

$$g(x)=2\int_k\frac{1}{k^2}\left(2k^2J(k,\,1)I^{-1}(k,\,1)-1-(x+1)\Gamma(2-\tfrac{1}{2}\epsilon)\frac{(2\pi)^{4-\epsilon}}{(2\pi)^{2-\epsilon/2}}\frac{\sin\frac{1}{2}\pi\epsilon}{\pi}I(k,\,1)\right)$$

$$\times\left((k^2+1)I(k,\,1)+\frac{1}{x+1}\frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}}\frac{\pi}{\sin\frac{1}{2}\pi\epsilon}\right)^{-1}$$

$$-2\int_k\frac{1}{k^2}[2k^2J(k,0)I^{-1}(k,0)-1]\left(k^2I(k,0)+\frac{1}{x+1}\frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}}\frac{\pi}{\sin\frac{1}{2}\pi\epsilon}\right)^{-1}\,. \qquad (29)$$

Note that (i) the scaling laws which determine β and δ in terms of γ and η and the dimension of space[9] are satisfied, and (ii) the small-ϵ behavior of the critical exponents coincides with previous results.[1,2]

IV. BEHAVIOR NEAR COEXISTENCE CURVE

It has been shown in a previous work in the ϵ-expansion approach, that the equation of state is well defined in the vicinity of the coexistence curve.[2] More specifically, in terms of the scaling variables defined in (28), in this region one has the structure

$$y\sim(x+1)[1+A\epsilon\ln(x+1)+B\epsilon^2\ln^2(x+1)$$

$$+C\epsilon^2\ln(x+1)]\,,\quad x\to-1\,. \qquad (30)$$

One does not know *a priori* the form to which these logarithms should be exponentiated. Assume, and it will be argued later that it is indeed the case, that in the vicinity of $x=-1$, the expression (30) exponentiates as $y=O[(x+1)^p]$, with p greater than one. Then both specific heats at constant magnetization and at constant field remain finite at the coexistence curve. Furthermore, the derivative of C_H with respect to H is also finite, as a simple thermodynamic argument shows.[10]

Consider the usual relation between the two specific heats

$$\frac{C_H}{T}=\frac{C_M}{T}+\frac{1}{r_L}\left(\frac{\partial H}{\partial t}\right)^2$$

$$=-\int_{M_0}^{M}\frac{\partial^2 H}{\partial t^2}(M',\,t)\,dM'+\frac{1}{r_L}\left(\frac{\partial H}{\partial t}\right)^2+C(M_0,\,t)\,, \qquad (31)$$

where $C(M_0,\,t)$ is a "constant" of integration, irrelevant for our purpose. If the scaling equation of state reads[8]

$$\frac{H}{M^\delta}=f\left(\frac{t}{M^{1/\beta}}\right)\,, \qquad (32)$$

and normalizations are, as previously, such that

$t/M^{1/\beta}=-1$ at the coexistence curve and $H/M^\delta=1$ on the critical isotherm; then from Eq. (8) the dominant part of r_L is

$$r_L \sim (1/\beta)M^{\delta-1}f'(x)\ ,\quad x\to -1\ .$$

Therefore the dominant terms of C_H are, with $\alpha \equiv 2-\beta(\delta+1)$,

$$\frac{C_H}{T}\sim \beta(-t)^{-\alpha}\left[\int^x \frac{du}{u}\,|u|^\alpha f''(u)+f'(x)\right]\ ,\quad x\to -1\ .$$

When x approaches -1, a cancellation occurs, and if

$$f(x)\sim (x+1)^p\ ,\quad x\to -1\ ,$$

C_H has the same behavior.

This means that the energy fluctuations remain finite and not rapidly varying in this region. In particular this indicates that this property holds for the quantity

$$\int d^dx\left(\left\langle \sum_1^N \phi_i^2(x)\sum_1^N\phi_i^2(0)\right\rangle - \left\langle \sum_1^N\phi_i^2\right\rangle^2\right)\ .$$

This is to be contrasted with the fact that in the same region individual fields have infinite fluctuations, since according to Eqs. (2) and (3) they are precisely the divergent susceptibilities r_L^{-1} and r_T^{-1}. This is why it seems reasonable to explore the implications of a nonlinear model in which $\sum_1^N\phi_i^2(x)$ is held constant.[11]

It has been foreseen by Wilson[12] that the use of a nonlinear realization implies that the transverse field has its canonical dimension in this region, and we shall attempt to make this conjecture plausible.[13]

Consider the Hamiltonian in its original form (1). Let us replace the longitudinal variable $\phi_1(x)$ by a new variable $\sigma(x)$ according to the relation

$$\phi_1(x)=\left(m^2+2m\sigma(x)-\sum_2^N\phi_i^2(x)\right)^{1/2}\ , \tag{33}$$

where m is a parameter similar to the magnetization. This expression is to be understood as a power series in $1/m$. As defined, the field $\sigma(x)$ is invariant under $O(N)$ rotations.

Collecting in the transformed Hamiltonian the terms in σ and σ^2, we see that $\sigma(x)$ plays a role similar to $L(x)$ of Eq. (6). Thus, at lowest order in r_0, the equation $\langle\sigma\rangle=0$ implies

$$\frac{H}{m}=r_0+u_0\,\frac{m^2}{6} \tag{34a}$$

and the bare "mass" term of the σ field is

$$r_\sigma=\frac{H}{m}+\frac{u_0M^2}{3}\ . \tag{34b}$$

This means that, at this order one may identify r with r_L, and m with M.

The advantage of this change of variable[14] is that, in the region $r_T\ll r_L$ and for small momenta $\ll\sqrt{r_L}$, the σ field decouples from the other modes since the insertion of an internal σ-line produces a factor r_L^{-1}. The reasons why this argument fails for the longitudinal field in the initial Hamiltonian are twofold. First, in the interaction Hamiltonian (5) there is a coupling $L(x)\phi_i^2(x)$, with a coefficient u_0M which, in this region, is of order $(u_0r_L)^{1/2}$, and the insertion of an L-propagator does not always produce an r_L^{-1} factor. Second, in contrast to $\sigma(x)$, $\phi_1(x)$ plays a role in maintaining the $O(N)$ symmetry.

Therefore we are left with a Hamiltonian from which the longitudinal mode disappears:

$$\frac{\mathcal{H}}{kT}=\int d^dx\left\{\frac12\sum_2^N\nabla\phi_i^2+\frac12\left[\nabla\left(m^2-\sum_2^N\phi_i^2\right)^{1/2}\right]^2 - H\left(m^2-\sum_2^N\phi_i^2\right)^{1/2}\right\}. \tag{35}$$

Near the coexistence curve r_T goes to zero, and therefore we are exploring the critical behavior of the Hamiltonian (35). From Wilson's general arguments on "irrelevant variables" in the renormalization-group equations,[15] only interaction terms $(\sum_2^N\phi_i^2)^2$ are relevant for inducing anomalous dimensions. However, in the Hamiltonian (35) this term appears with a factor H and tends to zero. Therefore, field dimensions are expected to be canonical.

Let us now extract a precise statement from these considerations. Consider the transverse susceptibility

$$r_T^{-1}=\int d^dx\,\langle\phi_i(x)\phi_i(0)\rangle\ ,\quad i\geq 2\ .$$

Denote by ξ the transverse correlation length. In the x integration there is a region of order ξ which contributes a term ξ_Φ^{d-2d}, with $d-2d_\Phi=2$ for canonical dimensions. Similarly, let us consider the longitudinal susceptibility

$$r_L^{-1}=\int d^dx[\langle\phi_1(x)\phi_1(0)\rangle-M^2]\ ,$$

which, according to Eq. (33) and to the fact that higher powers of ϕ_i^2 have higher dimensions and are negligible, behaves like

$$r_L^{-1}\sim\int d^dx\ \left\langle\sum_2^N\phi_i^2(x)\sum_2^N\phi_i^2(0)\right\rangle\ .$$

The same argument gives now

$$r_L^{-1}\sim\xi_{\phi^2}^{d-2d}+\text{const}\ ,$$

and since dimensions are canonical $d_{\phi^2}=2d_\phi=d-2$. If ξ is eliminated in favor of r_T, one finally obtains

$$r_L^{-1}\sim\text{const}+r_T^{-\epsilon/2} \tag{36}$$

to all orders in ϵ and in $1/N$. Since r_L is simply related to the derivative of $f(x)$, this result implies that $f'(x)$ vanishes at $x=-1$, as was initially assumed. Of course this argument involves an element of circularity but this picture is strongly supported both by the ϵ and $1/N$ expansions.

Thus, from the equation of state obtained in Ref. 2 by the ϵ expansion, it is straightforward to extract the relation

$$r_L^{-1} \sim 1 - \frac{\epsilon(N-1)}{2(N+8)}\left[\ln(x+1) - (\tfrac{1}{4}\epsilon)\ln^2(x+1)\right] + D\epsilon^2 \ln(x+1), \quad x \to -1 \qquad (37)$$

where D is some constant. From consideration of the $\epsilon \ln(x+1)$ and $\epsilon^2 \ln^2(x+1)$ terms in Eq. (37), one verifies at lowest order in ϵ the structure predicted in (36). However it is not possible to substantiate further the form (36) using the $\epsilon^2 \ln(x+1)$ term of Eq. (37), because it is impossible at this order to decide whether it builds up a new power $r_T^{-\epsilon/2+a\epsilon^2}$, or changes the normalization as in $(1+b\epsilon) r_T^{-\epsilon/2}$. The essence of the problem lies in the existence of two powers of r_T differing by order ϵ, namely, r_T^0 and $r_T^{-\epsilon/2}$. Indeed if a third such power were present nothing could be checked at this order. A renormalization-group argument would rule out this possibility,[16] but another argument in favor of the same conclusion may be found in the $1/N$ expansion.

When N is infinite, it was noted in Sec. II that, to all orders in ϵ, Eq. (36) holds.[17] To obtain the $1/N$ corrections let us return to Eq. (24). When the integrals are subtracted at $r_T=0$ simple algebra leads to the equivalent form

$$0 = \frac{6t}{u_0} + M^{1/\beta} - \tfrac{1}{2}N \frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\frac{1}{2}\epsilon)(2\pi)^{4-\epsilon}} \, \frac{\pi}{\sin\frac{1}{2}\pi\epsilon} \, r_T^{1-\epsilon/2} - 2\int_k \left(J(k, r_T) I^{-1}(k, r_T) \frac{k^2}{k^2 + r_T + (2M^2/N) I^{-1}(k, r_T)} - (r_T = 0) \right). \qquad (38)$$

Using Eq. (19) and the asymptotic expression (20), it is easy to check that the integrals in Eq. (38) behave, for small r_T and fixed M, like

$$(1/N)\left(r_T^{1-\epsilon/2} + \text{const } M^{-\epsilon/(1-\epsilon/2)} r_T\right),$$

and no $(1/N) r_T^{1-\epsilon/2} \ln r_T$ or $(1/N) r_T \ln r_T$ terms are present. This establishes the form (36) to order $1/N$ for all ϵ.

In conclusion, we discuss the available evidence on an $H^{-1/2}$ divergence of the susceptibility in three dimensions. Besides the Berlin and Kac model, this behavior is also found in perturbation theory for a system of interacting spin waves[18] but, to our knowledge, there is no experimental evidence for it. However, if the ϵ expansion may be trusted, let us show how difficult it will be to observe this divergence. To this end, we return to Eq. (37) which now may be exponentiated and replaced, in the vicinity of the coexistence curve, by

$$r_L^{-1} \sim 1 + \frac{N-1}{9}(x+1)^{-\epsilon/2}, \quad x \to -1 .$$

For an isotropic ferromagnet, $N=3$, and in three dimensions the diverging term takes over from the constant when $(x+1) \lesssim (2/9)^2$. Since, in this region,

$$y \sim (x+1)^{1/(1-\epsilon/2)},$$

this requires values of $H/M^\delta \lesssim (2/9)^4$; that is to say only 3×10^{-3} of its value on the critical isotherm.

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Mössbauer Scattering in $^{141}_{59}$Pr[†]

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We have completed a series of Mössbauer scattering experiments on $^{141}_{59}$Pr using an axially symmetric backscattering arrangement. From the relative isomer shifts between Pr^{+4} and Pr^{+3} compounds we estimate that the change in the mean-square charge radius between the 145-keV excited state and ground state of the $^{141}_{59}$Pr nucleus is 1.28×10^{-28} cm^2. PrO_2 exhibits ordered magnetic behavior below 15 K. We estimated the hyperfine fields for the electronic ground states of a Pr^{+4} ion in a magnetic cubic crystal and compared the observed values of the effective fields with the calculated fields. We found the best agreement when we assumed that the exchange field was directed toward a nearest-neighbor Pr^{+4} ion. We obtained $(3.1\pm 0.2)\mu_N$ for the magnetic moment of the 145-keV spin-$\frac{7}{2}$ level of $^{141}_{59}$Pr.

I. INTRODUCTION

The nuclear ground state of the stable isotope $^{141}_{59}$Pr has a spin of $\frac{5}{2}$ and the first excited state at 145 keV has a spin of $\frac{7}{2}$ (see Fig. 1). Praseodymium acts chemically as a metal with a valence of +3 or +4. The electron configurations of the Pr^{+4} and Pr^{+3} ions consist of the xenon closed shells plus one $4f$ and two $4f$ electrons, respectively.

Observation of the Mössbauer effect with the 145-keV γ transition in $^{141}_{59}$Pr is difficult, owing to the relatively high energy of the γ ray and the resulting small recoilless fraction for praseodymium compounds. In 1963 Bukarev[2,3] reported unsuccessful attempts to detect the effect. He tried both a conventional transmission geometry and a right-angle scattering arrangement. Using an axially symmetric backscattering geometry, Morrison[4] and Debrunner and Frauenfelder[5] were the first to observe the Mössbauer effect in $^{141}_{59}$Pr. Recently, additional results from scattering experiments on $^{141}_{59}$Pr were reported by Bent *et al.*,[6] Kapfhammer *et al.*,[7] and Groves *et al.*[8]

We have completed a series of resonance scattering experiments on $^{141}_{59}$Pr and have observed the Mössbauer effect in several praseodymium compounds. We used an improved version of the axially symmetric backscattering arrangement of Ref. 4. $^{141}_{59}$Pr is particularly well suited for resonance scattering studies, since the source $^{141}_{58}$Ce has a simple decay scheme with γ's only from the 145-keV level, the level of interest. Consequently, in the backscattering geometry a Ge(Li) detector was able to resolve the elastic peak due to Rayleigh and Mössbauer scattering from inelastic processes such as Compton scattering and the photoelectric effect.

We observed signal-to-background ratios greater than 0.20 for some single-line spectra. From the relative isomer shifts between Pr^{+4} and Pr^{+3} compounds we estimate that the change in the mean-square charge radius between the 145-keV excited state and the ground state of $^{141}_{59}$Pr nucleus is 1.28×10^{-28} cm^2. PrO_2 exhibits magnetic ordering below 15 K. We estimated the hyperfine fields from the electronic ground states of a Pr^{+4} ion in a magnetic cubic crystal and compared the observed values of the effective fields with the calculated fields. We found the best agreement when we assumed that the exchange field was directed toward a nearest-neighbor Pr^{+4} ion. Also, we obtain $(3.1\pm0.2)\mu_N$ for the magnetic moment of the 145-keV spin-$\frac{7}{2}$ level, in agreement with Bent *et al.*[6] and Kapfhammer *et al.*[7]

We have written computer programs for simulating and least-squares analyzing the data from Mössbauer scattering experiments with large, axially symmetric scatterers. We included both scatterer and detector solid-angle effects, and we included the non-Lorentzian terms for partially resolved spectra.[9]