Mechanism of high-temperature superconductivity in a striped Hubbard model

E. Arrigoni,[1,2] E. Fradkin,[3] and S. A. Kivelson[4]
[1]*Institut für Theoretische Physik, Technische Universität Graz, Petersgasse 16, A-8010 Graz, Austria*
[2]*Institut für Theoretische Physik und Astrophysik, Universität Würzburg, am Hubland, 97074 Würzburg, Germany*
[3]*Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080, USA*
[4]*Department of Physics, University of California at Los Angeles, 405 Hilgard Avenue, Los Angeles, California 90095, USA*
(Received 24 September 2003; revised manuscript received 26 November 2003; published 25 June 2004)

It is shown, using asymptotically exact methods, that the two-dimensional repulsive Hubbard model with strongly modulated interactions exhibits "high-temperature superconductivity." Specifically, the explicit modulation, which has the same symmetry as period 4 bond-centered stripes, breaks the system into an alternating array of more and less heavily hole-doped, nearly decoupled two-leg ladders. It is shown that this system exhibits a pairing scale determined by the spin gap of the undoped two-leg ladder, and a phase ordering temperature proportional to a low positive power of the interladder coupling.

DOI: 10.1103/PhysRevB.69.214519 PACS number(s): 74.20.Mn, 71.10.Li, 71.10.Pm

Much has been written concerning the mechanism of high-temperature superconductivity (HTC) since the discovery[1] of the cuprate superconductors in 1986, and indeed even before that. However, what is meant by "the mechanism" is rarely defined, and clearly evokes different images for different authors. The BCS mechanism, in which pairing is a consequence of a weak induced *attraction* produced by the exchange of phonons between well-defined quasiparticles, is not only consistent with a remarkable number of experimental facts in conventional superconductors, it is also of well-established theoretical validity in simple models. Because it is a weak coupling theory, even the mean-field (MF) estimate of T_c (which is exponentially small, $T_c \propto \exp(-1/g)$ where g is the induced attraction) is known to be quantitatively reliable.[2] However, there are many well-known reasons to believe that the BCS mechanism always leads to low T_c, as recently reviewed in Ref. 3.

An alternative idea, which has been the focus of much of the theoretical effort in the field, is that in a doped Mott insulator, HTC arises directly from the repulsive interactions between electrons. However, even as a point of principle, the validity of a mechanism of this sort has not been well established for any simple model.

In this article, we demonstrate the existence of a "high-temperature, superconducting" phase of the Hubbard and $t-J$ models on a square lattice with periodically modulated parameters[4] [see Eq. (6)]. In particular, we show that a period 2 modulation can produce superconductivity with a relatively low T_c in a restricted doping range, while a period 4 modulation produces higher critical temperatures on a broader range of doping. Specifically, we consider a caricature of a stripe ordered state consisting of a quasi-one-dimensional array of two-leg Hubbard ladders weakly coupled to each other with a hopping matrix element δt. For a range of electron densities per site, $\langle n\rangle \equiv 1-x$, it has been well established [5,7,6,8–10] that the two-leg ladder exhibits a Luther–Emery (LE) liquid[11] phase, with a large spingap $\Delta_s \sim J/2$, and a divergent superconducting (SC) susceptibility for $T \ll \Delta_s$

$$\chi_{\rm SC}(T) \sim \Delta_s / T^{2-K^{-1}}, \tag{1}$$

where K is the charge Luttinger parameter and T is the temperature. This sounds like a promising start. However, a nonzero T_c is impossible in one dimension (1D), so that to have a chance of a high transition temperature, interladder couplings must be taken into account. If all the ladders are equivalent (a caricature of a period 2 stripe ordered or column state[12,13]), we shall see that this coupling leads to a SC state in a restricted range of small x with rather low T_c. For more substantial values of x, it inevitably leads to an insulating, incommensurate charge density wave (CDW) state with (in units in which the lattice constant is $a=1$) an ordering wave number $P=2\pi x$. (It is customary to call this the $4k_{\rm F}$ CDW since, despite the fact that there is a spin gap and hence no Fermi surface whatsoever, $P=4k_{\rm F}$, where $k_{\rm F}$ is the Fermi momentum of a 1D noninteracting electron gas at the same electron density.) That the SC transition is so easily preempted by CDW order follows from the fact that the CDW susceptibility of the LE liquid diverges as

$$\chi_{\rm CDW}(P,T) \sim \Delta_s / T^{2-K}. \tag{2}$$

Under most circumstances for repulsive interactions, $K<1$, and hence $\chi_{\rm CDW}$ of Eq. (2) is more strongly divergent than $\chi_{\rm SC}$ of Eq. (1). However, if we consider an alternating array of A- and B-type ladders (with different electron affinities) then the tendency to CDW order is greatly suppressed due to the mismatch between ordering vectors P_A and P_B on neighboring ladders. [14,15] We shall show that, as long as the exponent inequalities

$$2 > K_A^{-1} + K_B^{-1} - K_A; \quad 2 > K_A^{-1} + K_B^{-1} - K_B \tag{3}$$

are satisfied, the SC instability wins out. [f the Luttinger parameter is the same for both ladders, these inequalities reduce to $K > K_c \equiv (\sqrt{3}-1) \approx 0.8$.]

Under these circumstances, the SC (Kosterlitz–Thouless) transition temperature can be reliably estimated by treating the 1D fluctuations exactly, but the interladder Josephson coupling $\mathcal{J}$ in MF approximation[17,16]

$$T_c \sim \Delta_s \left(\frac{\mathcal{J}}{\tilde{W}} \right)^{\alpha}; \quad \alpha = \frac{2K_A K_B}{[4K_A K_B - K_A - K_B]}, \tag{4}$$

where $\mathcal{J}$ is an effective coupling and $\tilde{W}$ is a microscopic energy which we will discuss in detail below; typically, we find $\mathcal{J} \sim \delta t^2/J$ and $\tilde{W} \sim J$. Although T_c is small for small $\mathcal{J}$, it is only power-law small. In fact, typically $\alpha \sim 1$. Because of the MF character of this estimate for T_c, one expects this to be an upper bound to the actual T_c. One also generally expects T_c to be somewhat suppressed by phase fluctuations, but typically by no more than a factor of 2. Indeed, a perturbative renormalization-group treatment for small $\mathcal{J}$ yields the same power-law dependence as Eq. (4), suggesting that this expression is asymptotically exact for $\mathcal{J} \ll \tilde{W}$. This fact is supported in Appendix A, where the accuracy of *interchain* MF estimates is discussed for related models.

Since we expect T_c to be smooth function of $\delta t/\mathcal{J}$, it is reasonable to extrapolate Eq. (4) to the case in which δt is a substantial fraction of $\mathcal{J}$. This suggests a maximum T_c of order Δ_s, and so can easily account for relatively high transition temperatures.[18,19] This is in contrast to the case of an exponentially small T_c as obtained, for example, in a BCS-like mechanism.

I. THE STRIPED HUBBARD MODEL

While the results obtained in this paper are quite robust in the sense that they apply for a broad range of microscopic interactions, to establish their validity it is useful to consider an explicit model. The model we study is the striped Hubbard model

$$H = -\sum_{\langle \vec{r},\vec{r}' \rangle,\sigma} t_{\vec{r},\vec{r}'} [c^{\dagger}_{\vec{r},\sigma} c_{\vec{r}',\sigma} + \text{H.C.}] + \sum_{\vec{r},\sigma} [\epsilon_{\vec{r}} c^{\dagger}_{\vec{r},\sigma} c_{\vec{r},\sigma} + (U/2) c^{\dagger}_{\vec{r},\sigma} c^{\dagger}_{\vec{r},-\sigma} c_{\vec{r},-\sigma} c_{\vec{r},\sigma}], \tag{5}$$

where $\langle \vec{r},\vec{r}' \rangle$ designates nearest-neighbor sites, $c^{\dagger}_{\vec{r},\sigma}$ creates an electron on site $\vec{r}$ with spin polarization $\sigma = \pm 1$ and satisfies canonical anticommutation relations, and $U > 0$ is the repulsion between two electrons on the same site. In the limit of strong repulsions, $U \gg t_{\vec{r},\vec{r}'}$, this model reduces approximately to the corresponding $t-J$ model, which operates in the subspace of no double occupied sites, but with an exchange coupling, $J_{\vec{r},\vec{r}'} = 4|t_{\vec{r},\vec{r}'}|^2/U$ between neighboring spins. Our results only depend on the low-energy physics of the ladder and, thus, apply equally to the $t-J$ and Hubbard models.

In the translationally invariant Hubbard model, $t_{\vec{r},\vec{r}'} = t$ and $\epsilon_{\vec{r}} = 0$. The striped version of this model is still translationally invariant along the stripe direction (which we take to be the y axis), so $t_{\vec{r},\vec{r}+\hat{y}} = t$. However, perpendicular to the stripes the hopping matrix takes on alternately large and small values: $t_{\vec{r},\vec{r}+\hat{x}} = t'$ for $r_x =$ even, and $t_{\vec{r},\vec{r}+\hat{x}} = \delta t \ll t' \sim t$ for $r_x =$ odd. This defines a "period 2 striped Hubbard model," as shown in Fig. 1. For the "period 4 striped Hubbard model," we include a modulated site energy, $\epsilon_{\vec{r}} = \sqrt{2}\epsilon \cos[\pi r_x/2 - \pi/4]$, which has site energies ϵ and $-\epsilon$, respectively, on every other two-leg ladder, with $\epsilon \gg \delta t$.

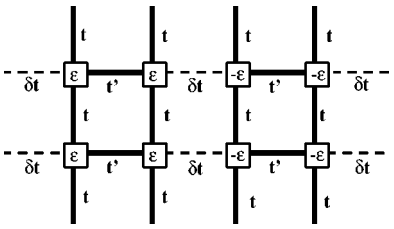

FIG. 1. Schematic representation of the striped Hubbard model analyzed in this paper.

II. ISOLATED TWO-LEG LADDER

For $\delta t = 0$, the model breaks up into a series of disconnected two-leg ladders. Considerable analytic and numerical effort has gone into studying the properties of two-leg $t-J$ and Hubbard ladders, and much is known about them. For $x = 0$, the undoped two-leg ladder has a unique, fully gapped state, referred to as C0S0 in the notation of Ref. 9, meaning 0 gapless charge and 0 gapless spin modes. In the large U limit, the magnitude of the spin gap of the undoped [6,20] ladder is $\Delta_s \approx J/2$. Then, for a substantial range of x $(0 < x < x_c)$ the ladder exhibits a LE or C1S0 phase, with a spin gap that drops smoothly [21] with increasing x, and vanishes at a critical value of the doping, $x = x_c$. [This particular LE liquid is known[5–9] to have "d-wave-like" SC correlations, in the sense that the pair-field operator has opposite signs along the edge of the ladder (y direction) and on the rungs (x-direction).] For $x > x_c$, the numerical results are scarce, nor is there uniform agreement concerning the number of phases; there may[9] or may not[10] be narrow ranges of C2S1 and C2S2 phases for x slightly larger than x_c. At any rate, for x large enough, $x_c \leqslant x_c' < x < 1$, the ladder manifestly enters a Luttinger liquid C1S1 phase, and finally, a trivial C0S0 phase when $x = 1$ $(\langle n \rangle = 0)$.

For the purposes of the present paper, we will confine ourselves to the range of parameters where both A- and B-type ladders are in the LE phase. The low-energy physics (at all energies less than Δ_s) of the two-leg ladder in the LE phase is contained in the free bosonic Hamiltonian for the collective charge degrees of freedom, given as

$$H = \int dy \frac{v_c}{2} \left[K(\partial_y \theta)^2 + \frac{1}{K} (\partial_y \phi)^2 \right] + \cdots, \tag{6}$$

where ϕ is the CDW phase and θ is the SC phase; these two fields are dual to each other, and so satisfy the canonical commutation relations $[\phi(y'), \partial_y \theta(y)] = i\delta(y - y')$. This effective Hamiltonian is general and physical; the precise x dependence of the spin-gap Δ_s, the charge Luttinger exponent K, the charge velocity v_c, and the chemical potential $\mu(x)$, depends on details such as the values of U/t and t'/t. For certain cases,[5–7] these have been accurately computed in Monte-Carlo studies, and these studies could be straightforwardly extended to other values of the parameters.

The ellipsis in Eq. (6) represent cosine potentials, which we will not explicitly exhibit here, that produce the Mott gap Δ_M at $x = 0$. A consequence of these terms is that for $x \to 0$, the elementary excitations are charge $2e$ solitons that can either be viewed as spinless fermions or hard-core bosons,

with a dispersion relation $E(k)=\Delta_M+\tilde{t}k^2$. One consequence of this is that[5,22] $K\to 2$ and $v_c\to 2\pi\tilde{t}x$ as $x\to 0$. A second consequence is that the renormalized harmonic theory, which retains only the explicitly exhibited terms in Eq. (6), is valid in a range of energies that is small in proportion to the effective Fermi energy, $\tilde{E}_F^{(1D)}=2\pi\tilde{t}x^2$. (An estimate of $\tilde{t}\approx t/2$ can be obtained from the denisty matrix renormalization group (DMRG) study of the $t-J$ ladder with $J/t=1/3$ in Ref. 7.)

For larger x, the numerical studies[5,7,16] generally find that both K and Δ_s drop monotonically with increasing x. By the time $x=x_1\approx 0.1$, K is generally found to be close to 1, and by $x=x_c\approx 0.3$, Δ_s has dropped to values that are indistinguishable from 0, and $K\approx 0.5$. Thus, over most of the entire LE phase, both the SC and the CDW susceptibilities are divergent. However, the SC susceptibility is the more divergent only at rather small values of $x<x_1$.

Although the charge fields exhaust the low-energy degrees of freedom of the LE liquid, when we come to consider the effects of the single-particle hopping perturbation with small coupling constant δt, we need to consider (as virtual intermediate states) high-energy states with the quantum numbers of an electron. Thus, we need to reintroduce gapped fields ϕ_s to represent the spin-degrees of freedom. Since this is standard, [3] we will not belabor the point; the appropriate continuum fermionic fields are

$$\Psi^\dagger_{\pm,\sigma}\sim \exp\{\sqrt{\pi/2}[\theta\pm\phi+\sigma\theta_s\pm\sigma\phi_s]\pm iPy/2\}, \qquad (7)$$

where $\pm$ refer to left- and right-going fermions with momentum near $\pm P/2$, respectively, and $\sigma=\pm 1$ represents the spin polarization. It is important to stress that for strongly interacting problems, such as the present one, there is no simple relation between the original lattice fermions and the continuum fermion fields that describe the "physical" Ψ-fermions of Eq. (7). In particular, what appears as a $2k_F$ CDW expressed in terms of Ψ-fermions, would be considered a $4k_F$ CDW in terms of the original, lattice fermions. In terms of these Ψ-fields, the component of the charge density operator that varies with wave numbers near P is

$$\hat{\rho}_P(y)=\sum_\sigma \Psi^\dagger_{L,\sigma}\Psi_{R,\sigma}\propto \exp[iPy+i\sqrt{2\pi}\phi(y)], \qquad (8)$$

while the singlet pair creation operator

$$\hat{\Phi}(y)=[\Psi^\dagger_{L,\uparrow}\Psi^\dagger_{R,\downarrow}+\Psi^\dagger_{R,\uparrow}\Psi^\dagger_{L,\downarrow}]\propto \exp[i\sqrt{2\pi}\theta], \qquad (9)$$

where in the right-most expressions we have again suppressed the dependence on the spin fields.

Before leaving the single-ladder problem, it is worth mentioning a useful intuitive caricature of its electronic properties. We picture a singlet pair of electrons on neighboring sites as being a hard-core bosonic "dimer." The undoped ladder can be thought of as a Mott insulating state of these dimers, with one dimer per rung of the ladder, that is, "valence bond crystal" with lattice spacing one. To remove one electron from the system, we need to destroy one dimer and remove one electron, leaving behind a single electron with spin 1/2 and charge e. However, when we remove a second electron from the system, we have the choice of either breaking another dimer, thus producing two quasiparticles with the quantum numbers of an electron, or of removing the unpaired electron left behind by the first removal, thus producing a new boson—a missing dimer—with charge $2e$ and spin 0. The persistence of the spin gap upon doping the ladder can thus be interpreted as implying that the energy needed to break a dimer (of order Δ_s) is sufficiently large that one charge $2e$ boson costs less than two charge e quasiparticles. At finite x, the missing dimers can be treated as a dilute gas of hardcore bosons. That the elementary excitations of the undoped ladder can be constructed in this simple manner reflects the fact that this is a confining phase,[23,24] not a spin liquid.

III. INTERLADDER INTERACTIONS

We now address the effect of a small, but nonzero coupling (i.e., single-particle hopping) between ladders, $\delta t>0$. Because of the spin gap, δt is an irrelevant perturbation in the renormalization group sense, and so does not directly affect the thermodynamic state of the system. However, second order processes result in various induced interactions between neighboring ladders. These consist of marginal forward scattering interactions, which are negligible for small δt, and potentially relevant Josephson tunneling and backscattering density–density interactions.

The important (possibly relevant) low-energy pieces of these latter interactions are most naturally expressed in terms of the bosonic collective variables defined above:

$$H'=-\sum_j\int dy\{\mathcal{J}\cos[\sqrt{2\pi}(\theta_j-\theta_{j+1})]+\mathcal{V}\cos[(P_j-P_{j+1})y+\sqrt{2\pi}(\phi_j-\phi_{j+1})]\}, \qquad (10)$$

where $P_j=2\pi x_j$, with x_j the concentration of doped holes on ladder j, and ϕ_j and θ_j are the charge field and its dual on each ladder. Here, again, the form of the low-energy interactions between two LE liquids is entirely determined by symmetry considerations, but the magnitude of the Josephson coupling $\mathcal{J}$ and the induced interaction between CDWs ($\mathcal{V}$) must be computed from microscopics; they are renormalized parameters which result from "integrating" out the high-energy degrees of freedom with energies between the bandwidth $W\sim 4t$ and the renormalized cutoff Δ_s, or with wavelengths between a and $\xi_s\equiv v_s/\Delta_s$, where v_s is the spin-wave velocity. Thus, the dimensionless measure of the interladder couplings, which, for instance, enter the expressions for T_c, are $\mathcal{J}/\tilde{W}$ and $\mathcal{V}/\tilde{W}$, where $\tilde{W}=\Delta_s/\xi_s$. (As long as x is not too near x_c, $\Delta_s\sim J$, and hence $\tilde{W}\sim J$.)

Quantitative estimates of $\mathcal{J}$ and $\mathcal{V}$ could certainly be obtained, given the state of DMRG calculations, from studies of four-leg ladders consisting of two weakly coupled two-leg ladders.[25] However, such calculations have not yet been carried out. Fortunately, our qualitative conclusions are not very sensitive to the values of $\mathcal{V}$ and $\mathcal{J}$, which can, in any case, be estimated with reasonable accuracy from bosonization, as discussed in Ref. 16. The subtlety here is that the interladder hopping is expressed in terms of microscopic lattice fermi-

ons, whereas our low-energy theory is expressed in terms of the Ψ-fermions of Eq. (7). However, since these have the same quantum numbers as an electron, and operate on the scale of Δ_s, which is large with respect to δt, there is no reason to expect any large renormalization of the hopping parameters. If we assume that the interladder hopping can be approximated as δt times an operator representing the hopping amplitude for Ψ-fermions, then, from second-order perturbation theory, we obtain

$$\mathcal{J} \approx \mathcal{V} \sim A(\delta t)^2/J, \tag{11}$$

where A is the dimensionless function of Δ_s/J

$$A = J \int dy d\tau |\langle e^{i\sqrt{\pi/2}[[\theta_s(\mathbf{r})-\theta_s(0)+\phi_s(\mathbf{r})+\phi_s(0)]}\rangle_s|^2,$$

where $\mathbf{r}=(y,\tau)$ and τ denotes imaginary time, the expectation value is taken with respect to the spin-fields on the decoupled ladders, and in deriving this expression we have assumed that the charge fields are slowly varying compared to the spin-fields. Simple scaling arguments of the sort discussed in Ref. 16 suggest that $A \sim 1$ as $\Delta_s/J \to 0$. (For further discussion, see Ref. 26.) In any case, as long as x is not too close to x_c, Δ_s is of order of the exchange coupling J, so that it is reasonable to assume $A \sim 1$. The only aspects of this estimate that matter qualitatively for our present purposes is that the two couplings are comparable in size ($\mathcal{J} \sim \mathcal{V}$) and both are small in proportion to δt^2.

IV. RENORMALIZATION-GROUP ANALYSIS AND INTERLADDER MEAN-FIELD THEORY

The effect of these interchain couplings can be deduced from an analysis of the lowest order perturbative renormalization group equations in powers of the couplings $\mathcal{V}$ and $\mathcal{J}$. However, *equivalent* results are obtained from interladder mean-field theory (MFT),[16,17] which is conceptually simpler. These equations are the analog of the BCS gap equations applied to this model, and are expected to give a quantitatively accurate estimate of T_c for small $\delta t/\Delta_s$ for precisely the same reason. A discussion of the accuracy of *interchain* MFT is given in Appendix A. In the present two-dimensional system, T_c should be interpreted as the onset of quasi-long-range order; that is, as a Kosterlitz–Thouless transition.

To implement this MFT, we need to compute the expectation value $M_j(h_j)=\langle \cos[\sqrt{2\pi}\theta_j]\rangle$ of the pair creation operator on an isolated ladder, where the expectation value is taken with respect to the MF Hamiltonian

$$H_{\mathrm{MF}} = H_j - h_j \int dy \cos[\sqrt{2\pi}\theta_j] \tag{12}$$

in which H_j is the effective Hamiltonian in Eq. (6) with parameters appropriate to ladder j, and h_j represents the mean field due to the neighboring ladders, and so satisfies the self-consistency condition

$$h_j = \mathcal{J}[M_{j+1} + M_{j-1}]. \tag{13}$$

The expression for the MF transition temperature can be expressed in terms of the corresponding susceptibility, $\tilde{\chi}^{(j)}_{\mathrm{SC}} = \partial M_j(h)/\partial h|_{h=0}$, which is related to the SC susceptibility in Eq. (1) by a proportionality constant that depends on the expectation value of the spin fields. In the case in which all the ladders are equivalent, this yields the implicit relation $2\mathcal{J}\tilde{\chi}_{\mathrm{SC}}(T_c)=1$. For an alternating array of A- and B-type ladders, the expression for the SC T_c is easily seen to be

$$(2\mathcal{J})^2 \tilde{\chi}^{(A)}_{\mathrm{SC}}(T_c)\tilde{\chi}^{(B)}_{\mathrm{SC}}(T_c) = 1. \tag{14}$$

Notice that in the case in which the A- and B-type ladders are identical, Eq. (14) reduces properly to the expression for equivalent ladders. The expression for χ_{SC} from Eq. (1) can be used to invert Eq. (14) to obtain the estimate for T_c given in Eq. (4).

The MF equations for the CDW order are obtained similarly. The expression for the transition temperature for CDW order with wave vector P is

$$(2\mathcal{V})^2 \tilde{\chi}^{(A)}_{\mathrm{CDW}}(P,T_c)\tilde{\chi}^{(B)}_{\mathrm{CDW}}(P,T_c) = 1, \tag{15}$$

where the notation is the obvious extension of that used in the SC case. The best ordering vector is that which maximizes T_c. For $P=P_A$, $\chi^{(A)}_{\mathrm{CDW}}(P_A,T)$ diverges with decreasing temperature as in Eq. (2), but $\chi^{(B)}_{\mathrm{CDW}}(P_A,T)$ saturates to a finite, low-temperature value when $T \sim v_c|P_A - P_B|$. Thus, even if $\chi^{(A)}_{\mathrm{CDW}}(P_A,T)$ diverges more strongly with decreasing temperature than $\chi^{(A)}_{\mathrm{SC}}$, there are two divergent susceptibilities in the expression for the SC T_c, and only one for the CDW T_c; as long as the inequalities in Eq. (3) are satisfied, the SC transition preempts the CDW transition!

V. THE $x \to 0$ LIMIT

Since $K \to 2$ as $x \to 0$, there is necessarily a regime of small x in which the SC susceptibility on the isolated ladder is more divergent than the CDW susceptibility. Here, in the presence of weak interladder coupling, even the period 2 striped Hubbard model (i.e.,with $\epsilon=0$) is superconducting. However, care must be taken in this limit, since, as mentioned above, the range of energies over which H in Eq. (6) is applicable vanishes in proportion to x^2. Fortunately, a complementary treatment of the problem, which takes into account the additional terms, the ellipsis in Eq. (6), can be employed in this limit. The small-x problem can be mapped onto a problem of dilute, hard-core charge $2e$ bosons (with concentration x per rung) with an anisotropic dispersion $E(\vec{k})=\tilde{t}k_y^2 - \mathcal{J}\cos[2k_x]$. (The "2" reflects the ladder periodicity.) Consequently, for small x,

$$T_c \approx 2\pi \sqrt{2\mathcal{J}\tilde{t}} x F(x) \sim |\delta t| x, \tag{16}$$

where $F(x) \sim 1/\ln\ln(1/x)$ is never far from 1, and the logarithm reflects[27] the fact $d=2$ is the marginal dimension for Bose condensation. (This result is not substantially different for the period 4 striped Hubbard model, as long as ϵ is not too large.) There is a complicated issue of order of limits when both δt and x are small; roughly, we expect that T_c will be determined by whichever expression, Eq. (14) or Eq. (16), gives the higher T_c, but with the understanding that χ_{SC} must be computed taking into account the terms represented by the

ellipsis in Eq. (6), which cause the susceptibility to vanish as $x \to 0$.

The period 2 striped Hubbard or $t-J$ model indeed has a SC phase at small x, because this phase is confined to rather small $x \lesssim 0.1$, where T_c is small in proportion to both δt and x. Moreover, this may still not be enough to establish a mechanism of HTC. The situation looks even worse when the effects of weak disorder[28] are considered: when the disorder strength is greater than the intraladder energy scale $E_F = 2\pi\tilde{t}x^2$, it is unlikely that any sort of SC coherence will survive.

For an array of alternating ladders, the range of x for which superconductivity dominates is much extended. This means that the maximum T_c is much greater, and the superconductivity much more robust to disorder[25] for the period 4 than the period 2 striped Hubbard model.

VI. OPTIMAL DEGREE OF INHOMOGENEITY FOR SUPERCONDUCTIVITY

Here, we have established that in a strongly striped Hubbard model, superconductivity is produced directly by the repulsive interactions between electrons. The resulting T_c is proportional to a positive power of $\delta t/t$, and thus rises as the stripe order becomes less strong. It is thus natural to ask: Is the stripe order introduced in the present paper simply a calculational crutch that permits us to obtain well-controlled results, or is inhomogeneity essential to the mechanism of HTC, as has been suggested[3,25,29–31] in several previous studies?

The answer to this question turns on the issue of whether or not the uniform Hubbard model, and its strong coupling relative the $t-J$ model, by themselves support HTC. This question has been the focus of much theoretical research since the discovery of superconductivity in the cuprates. To this date, this is not a settled issue. Nor is it the purpose of the present paper to review this extensive literature. Variational calculations have been interpreted both as giving evidence in support of[32] and against[33] superconductivity in Hubbard and $t-J$-type models. There is also considerable evidence, from several numerical techniques and high-temperature expansions, that the canonical $t-J$ and Hubbard models on a square lattice most likely *do not* support HTC; instead, they show clear evidence for other types of order that compete with superconductivity.[3,34]

Assuming that the uniform model does not support HTC, it follows from the arguments given in the previous sections that there is an optimal degree of inhomogeneity (an optimal degree of stripe order) for a strongly correlated system to exhibit superconductivity. Probably, this occurs when $\delta t \sim \Delta_s$. An analogous result was established[25] recently in the weakly interacting limit of the four-leg ladder (itself a caricature of a single unit cell of the present model). We should also note that there is nothing essential about having period 4. In fact, the longer the period, the more the CDW instability is suppressed and the larger the range of superconductivity.

VII. RELATION TO SUPERCONDUCTIVITY IN THE CUPRATES

While the main purpose of the present paper was to establish, as a point of principle, that the striped Hubbard model analyzed here exhibits HTC, a few comments are in order concerning the more general implications of the present results for the mechanism of superconductivity in the cuprates.

Firstly, the explicit striped inhomogeneities introduced here are a caricature of the spontaneous symmetry breaking in a charge striped phase. However, the model possesses a large spingap, so that it does not contain any of the physics of low-energy incommensurate spin fluctuations, which are the principle experimental signatures to date of stripe correlations in the cuprates. Secondly, although the SC state is "*d*-wave-like"in the sense that the order parameter changes sign under rotations by $\pi/2$, since the striped Hamiltonian explicitly breaks this symmetry, there is no precise symmetry distinction between *d*-wave and *s*-wave superconductivity. Thirdly, the SC state is not even truly adiabatically connected to the SC state observed in the cuprates, because the existence of a spin gap implies the absence of gapless "nodal" quasiparticles in the SC state. However, the transition between a nodeless and nodal *d*-wave-like state was studied in Refs. 35 and 36 where it was found to be a MF (Lifshitz) transition with relatively little effect on T_c. Moreover, using the same lines of reasoning employed in that article, it is possible to make compelling (although not entirely rigorous) arguments that upon heavier doping, the present model, too, will exhibit a nodal SC state. We are currently working to obtain a more complete treatment of the phase diagram of the present model.

The present model realizes the idea that the pairing scale, in this case the spin gap, can be inherited from a parent Mott insulating state. Moreover, like the underdoped cuprates, the gap scale in the present model is a decreasing function of increasing x, while the actual superconducting transition occurs at a T_c much smaller than $\Delta_s/2$, and is determined by the phase ordering temperature rather than the pairing scale. Hence, for x not too close to x_c, this model exhibits a pseudogap regime for temperatures between T_c and $T^* \sim \Delta_s/2$, reminiscent of that seen in underdoped cuprates. However, T_c is always bounded from above by Δ_s and so tends to zero as $x \to x_c$.

ACKNOWLEDGMENTS

This work was supported, in part, by NSF Grant nos. DMR 01-10329 at UCLA (S.A.K.), and DMR-01-32990 at the University of Illinois (E.F), and by a Heisenberg grant (AR 324/3-1) fro the DFG (E.A.).

APPENDIX A: ACCURACY OF THE INTERCHAIN MEAN-FIELD THEORY ESTIMATES

In this section, we discuss the accuracy of the interchain MFT. Although no general proof exists (to the best of our knowledge), we believe that it is asymptotically exact in the present case, at least to logarithmic accuracy [as defined in Eq. (A11)]. The latter conclusion also follows, as mentioned in the text, by comparison with perturbative RG calculations.

Quite generally, using an argument based on Griffiths inequalities, one knows that the exact T_c of a general aniso-

tropic ferromagnetic system (not necessarily an Ising model) will obey the bounds $T_c(J_x) \leqslant T_c \leqslant T_c(J_y)$, for $0 < J_x \leqslant J_y$, where $T_c(J)$ is the T_c of an isotropic system of coupling constant J. However, for specific systems, it is possible to establish more precise estimates of T_c.

As our first example, consider the two-dimensional (2D) anisotropic Ising model on a square lattice, with couplings J_x and $J_y \leqslant J_x$ in the x and y directions, respectively. In particular, for the case of the 2D Ising model, it is also known that the exact T_c is the solution to the equation[37]

$$\sinh(2J_x/T_c)\sinh(2J_y/T_c) = 1. \tag{A1}$$

Interchain MFT for the same model gives the familiar expression for the MF transition temperature, T_0, given by

$$2J_y\chi_{1D}(T_0) = 1, \tag{A2}$$

which is analogous to Eq. (14), and where

$$\chi_{1D}(T) = T^{-1}\exp[2J_x/T] \tag{A3}$$

is the susceptibility of the 1D Ising model. In the limit of small J_y/J_x, it thus follows that the ratio

$$\frac{T_0}{T_c} = 1 + \frac{\ln 2}{\ln[J_x/J_y]} + \cdots \tag{A4}$$

tends to 1 as $J_y/J_x \to 0$; that is, the interchain MFT is asymptotically exact without any apologies.

Before leaving the Ising example, it is interesting to see how well the interchain MFT works when extrapolated to the isotropic case $J_x = J_y = J$. It is easy to verify that $T_0 = 3.53J$ and $T_c = 2J/\ln[1+\sqrt{2}]$, so that

$$T_0/T_c = 1.55, \text{ for } J_y/J_x = 1. \tag{A5}$$

In general, T_0/T_c rises monotonically from 1 for increasing J_y/J_x, but T_0 gives a reasonably good estimate of T_c over the entire range of parameters. [Note, ordinary MFT gives $T_0' = 2(J_x+J_y)$, which is not much worse than interchain MFT in the isotropic limit, but $T_0'/T_c \to \infty$ as $J_y/J_x \to 0$.]

Now, we move to the 2D classical XY model on a square lattice; the case of most direct relevance to the estimates of T_c made in the text. The susceptibility of an isolated chain can easily be seen to be

$$\chi_{1D}(T) = \frac{1}{2T}\left[\frac{I_0(J_x/T) + I_1(J_x/T)}{I_0(J_x/T) - I_1(J_x/T)}\right], \tag{A6}$$

where $I_n(x)$ is a Bessel function. For $J_y/J_x \ll 1$, Eqs. (A3) and (A6) yield the following estimate of the critical temperature:

$$T_0 = 2\sqrt{J_xJ_y}[1 + \mathcal{O}(\sqrt{J_y/J_x})], \tag{A7}$$

while $T_0 = 1.755J$ in the isotropic limit.

Unlike the Ising case, no exact results exist for the 2D XY model. Extensive Monte-Carlo work has been done on the isotropic 2D XY model, from which we know[38] that the Kosterlitz–Thouless transition occurs at $T_c = 0.89J$, so that in this limit, $T_0/T_c \approx 2$. In the limit of extreme anisotropy, the 2D classical XY model can be mapped onto the familiar 1D quantum XY (rotor) model with Hamiltonian

$$H = \sum_n \left[\frac{L_n^2}{2} - \frac{\lambda}{2}\cos(\theta_{n+1} - \theta_n)\right], \tag{A8}$$

where the coupling constant is $\lambda = 2J_xJ_y/T^2$ (see Ref. 39). The critical value of the coupling of the quantum rotor model (λ_c) has been computed quite accurately using a Padé–Borel resummation of the strong-coupling series.[40,41] Using the notation of these papers, an accurate estimate for the critical coupling to be $\lambda_c = 1.8 \pm 0.5$ is obtained. By carefully inverting this mapping, we get

$$T_c = A\sqrt{J_xJ_y}[1 + \mathcal{O}(\sqrt{J_y/J_x})], \tag{A9}$$

where $A = 1.05 \pm 0.1$. Thus, we see that

$$T_0/T_c \to (2/A), \quad \text{as } J_y/J_x \to 0. \tag{A10}$$

It seems unlikely that the error bars on A are sufficient to be consistent with a limit of 1. The interchain MFT is therefore found to be asymptotically exact only to logarithmic accuracy, that is,

$$\frac{\ln T_0}{\ln T_c} \to 1, \quad \text{as } \frac{J_y}{J_x} \to 0. \tag{A11}$$

This, we believe, is generically true of interchain MFT as applied in the present paper. Nonetheless, in all cases for which the exact answers are known, interchain MFT gives estimates of T_c that are within a factor of 2 of the exact results. This is certainly sufficiently accurate for present purposes.

Finally, it is worth mentioning that the 2D XY model is something of a worst-case example, because 2D is the lower critical dimension and hence fluctuation effects are anomalously large. If we consider an anisotropic three-dimensional XY model with couplings $J_x \geqslant J_y \geqslant J_z$ in the three directions, the MF transition temperature can still be readily computed according to $2(J_y+J_z)\chi_{1D}(T_0) = 1$. Monte-Carlo results exist [42] for the T_c of layered models, $J_y = J_x \equiv J$ for various values of J_z/J_x. For instance,

$$T_c = 1.1J, \quad T_0/T_c = 1.60, \quad \text{for} \quad J_z/J = 0.01;$$

$$T_c = 1.324J, \quad T_0/T_c = 1.41, \quad \text{for } J_z/J = 0.1;$$

$$T_c = 2.2J, \quad T_0/T_c = 1.29, \quad \text{for } J_z/J = 1.0.$$

Clearly, even a very small amount of interplane coupling can be expected to greatly improve the accuracy of our T_c estimates. (Interplane MFT, of course, is still more accurate, as shown in Ref. 42.)

[1] J. G. Bednorz and K. A. Müller, Z. Phys. B: Condens. Matter **64**, 189 (1986).

[2] R. Brout, Phys. Rev. **118**, 1009 (1960).

[3] For a review, with extensive references to the original literature, see E. W. Carlson, V. J. Emery, S. A. Kivelson, and D. Orgad, in "The Physics of Conventional and Unconventional Superconductors" edited by K. H. Bennemann and J. B. Ketterson (Springer-Verlag, in press), arXiv:cond-mat/0206217. In particular, this paper reviews reasons to believe that the BCS mechanism does not produce HTC reviews the numerical searches for superconductivity in the uniform $t-J$ and Hubbard models, reviews the relevant aspects of the theory of the 1D electron gas, especially the LE liquid, and issues concerning the compertition between CDW and SC order in quasi-1D systems.

[4] H. Tsunetsugu, M. Troyer, and T. M. Rice, Phys. Rev. B **51**, 16 456 (1995), studied arrays of two-leg $t-J$ ladders as a way to understand the physics of the translationally invariant 2D system. Although the model they studied nominally conrresponds to the period 2 case we discuss below, and some of their discussion prefigures the present analysis, the questions asked by these authors were quite different. In particular, they did not consider the mechanism of superconductivity in inhomogeneous 2D systems which we discuss here.

[5] S. White, I. Affleck, and D. J. Scalapino, Phys. Rev. B **65**, 165122 (2002). Note that the normalization convention on the fields used in the present paper differs from that of White and coworkers, so that our K is the same as their $2K_{c,+}$.

[6] R. M. Noack, S. R. White, and D. J. Scalapino, Phys. Rev. B **56**, 7162 (1997).

[7] T. Siller, M. Troyer, T. M. Rice, and S. R. White, Phys. Rev. B **63**, 195106 (2001).

[8] For a review of the theory of two-leg ladders, see E. Dagotto and T. M. Rice, Science **271**, 618 (1996); C. Wu, W. V. Liu, and E. Fradkin, Phys. Rev. B **68**, 115104 (2003).

[9] L. Balents and M. P. A. Fisher, Phys. Rev. B **53**, 12 133 (1996); H. J. Schulz, *ibid.* **53**, R2959 (1996).

[10] V. J. Emery, S. A. Kivelson, and O. Zachar, Phys. Rev. B **59**, 15641 (1999).

[11] A. Luther and V. J. Emery, Phys. Rev. Lett. **33**, 589 (1974).

[12] K. Park and S. Sachdev, Phys. Rev. B **65**, 121304 (2002).

[13] M. Vojta and S. Sachdev, Phys. Rev. B **83**, 3916 (1999).

[14] S. A. Kivelson, E. Fradkin, and V. J. Emery, Nature (London) **393**, 550 (1998).

[15] V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, Phys. Rev. Lett. **85**, 2160 (2000).

[16] E. W. Carlson, D. Orgad, S. A. Kivelson, and V. J. Emery, Phys. Rev. B **62**, 3422 (2000).

[17] D. J. Scalapino, Y. Imry, and P. Pincus, Phys. Rev. B **11**, 2042 (1975).

[18] This mechanism bears a family resemblance to the interlayer tunneling mechanism of S. Chakravarty, A. Sudbø, P. W. Anderson, and S. Strong, Science **261**, 337 (1993), although many aspects are quite different.

[19] This mechanism is also related to the original resonating valence bond ideas of P. W. Anderson, Science **235**, 1196 (1987); S. A. Kivelson, D. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 865 (1987), in that the pairing correlations (and scale) are inherited from the correlated insulator.

[20] S. R. White, R. M. Noack, and D. J. Scalapino, Phys. Rev. Lett. **73**, 886 (1994); S. Chakravarty, *ibid.* **77**, 4446 (1996).

[21] For a restricted range of x, the authors of Ref. 6 show numerical evidence indicating that the spin gap decreases smoothly with increasing x. We are not aware of any published studies that carefully trace the spin gap as a function of x and in particular ones that accurately determine the critical doping x_c, at which it vanishes.

[22] H. J. Schulz, Phys. Rev. B **59**, 2471 (1999).

[23] E. Fradkin and S. A. Kivelson, Mod. Phys. Lett. B **4**, 225 (1990); E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Redwood City, 1991), Chap. 6.

[24] N. Read and S. Sachdev, Nucl. Phys. B **316**, 609 (1989).

[25] E. Arrigoni and S. A. Kivelson, Phys. Rev. B **68**, 180503 (2003).

[26] In deriving the estimates of $\mathcal{J}$ and $\mathcal{V}$, we have made a crude approximation concerning the spectrum of the isolated ladder at energies above Δ_s. Specifically: (1) We have neglected the possibility that there are additional important excitations, requiring extra bosonic fields, which might be vestiges of the two-bands present in weak coupling. For instance, we can define a gap Δ_o to the lowest lying excited state that is odd under exchange of the legs of the ladders. As far as we know, no numerical studies have measured this gap, although this should be straightforward. As long as $\Delta_o \geqslant \Delta_s$, we do not expect the existence of such additional modes will affect the estimates of these parameters qualitatively. However, it would be interesting in future numerical studies to evaluate Δ_o. In particular, if there are intermediate C2S1 and/or C2S2 phases between the C1S0 phase at $x<x_c$ and the C1Sl phase at $x<x_c'$ with $x_c'>x_c$, then Δ_o must vanish as $x \rightarrow x_c^-$, while if there are no such intermediate phases, Δ_o must remain finite at $x=x_c$. (2) We have neglected spin–charge coupling. Although such couplings are irrelevant in the renormalization group sense, this only justifies their neglect at low energies, whereas the states we are integrating out to derive $\mathcal{J}$ are high-energy states. Indeed, it is clear from the intuitive dimer picture that at $x=0$, there are (gapped) elementary excitations that carry both spin and charge. In the conceptually interesting case of x near x_c, where Δ_s/J is small, spin–charge coupling may seriously affect the value of $\mathcal{J}$, but in the typical case, $\Delta_s/J \sim 1$, no interesting renormalizations are possible.

[27] D. S. Fisher and P. C. Hohenberg, Phys. Rev. B **37**, 4936 (1988).

[28] E. Orignac and T. Giamarchi, Phys. Rev. B **56**, 7167 (1997).

[29] S. Chakravarty, M. Gelfand, and S. A. Kivelson, Science **254**, 970 (1991).

[30] V. J. Emery, S. A. Kivelson, and O. Zachar, Phys. Rev. B **56**, 6120 (1997).

[31] A model of an inhomogeneous strongly correlated system with attractive interactions was studied in MFT by I. Martin, G. Ortiz, A. V. Balatsky, and A. R. Bishop, Europhys. Lett. **56**, 849 (2001), and an inhomogeneous $t-J$ model with magnetic anisotropy was studied numerically in small systems by J. Eroles, G. Ortiz, A. V. Balatsky, and A. R. Bishop, Europhys. Lett. **50**, 540 (2000).

[32] S. Sorella, G. B. Martins, F. Becca, C. Gazza, L. Capriotti, A. Parolla and E. Dagotto, Phys. Rev. Lett. **88**, 117002 (2002); S. Sorella, A. Parola, F. Becca, L. Capriotti, C. Gazza, E. Dagotto and G. Martins, *ibid.* **89**, 279703 (2003); M. Calandra and S. Sorella, Phys. Rev. B **61**, R11894 (2000).

[33] S. Zhang, J. Carlson, and J. Gubernatis, Phys. Rev. Lett. **78**, 4486 (1997); T. K. Lee, C. T. Shih, Y. C. Chen, and H. Q. Lin, *ibid.* **89**, 279702 (2003); C. T. Shih, Y. C. Chen, H. Q. Lin, and T. K. Lee, *ibid.* **81**, 1294 (1998).

[34] L. Pryadko, S. A. Kivelson, and O. Zachar, Phys. Rev. Lett. **92**, 067002 (2004).
[35] M. Granath, V. Oganesyan, S. A. Kivelson, E. Fradkin, and V. J. Emery, Phys. Rev. B **87**, 167011 (2001).
[36] S. Sachdev, Physica A **313**, 252 (2002), and references therein.
[37] T. Schultz, D. C. Mattis, and E. Lieb, Rev. Mod. Phys. **36**, 856 (1964).
[38] R. Gupta and C. F. Baillie, Phys. Rev. B **45**, 2883 (1992).
[39] E. Fradkin and L. Susskind, Phys. Rev. D **17**, 2637 (1978).
[40] C. J. Hamer, J. B. Kogut, and L. Susskind, Phys. Rev. Lett. **41**, 1337 (1978).
[41] C. J. Hamer and J. B. Kogut, Phys. Rev. B **20**, 3859 (1979).
[42] E. W. Carlson, S. A. Kivelson, V. J. Emery, and E. Manousakis, Phys. Rev. Lett. **83**, 612 (1999).