Edge modes, edge currents, and gauge invariance in p_x+ip_y superfluids and superconductors

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The excitation spectrum of a two-dimensional p_x+ip_y fermionic superfluid, such as a thin film of ^{3}He-A, includes a gapless Majorana-Weyl fermion which is confined to the boundary by Andreev reflection. There is also a persistent ground-state boundary current which provides a droplet containing N particles with angular momentum $\hbar N/2$. Both of these boundary effects are associated with bulk Chern-Simons effective actions. We show that the gapless edge mode is required for the gauge invariance of the total effective action, but the same is not true of the boundary current.

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I. INTRODUCTION

Read and Green[1] have pointed out close parallels between two-dimensional p_x+ip_y chiral fermionic superfluids (a thin film of ^{3}He-A, for example) and the $\nu=1/2$ Pfaffian quantum Hall state. The many-body ground-state wave function of both systems contains a Pfaffian factor, and both support gapless Majorana-Weyl edge modes.[2] In addition, both systems can host vortex defects with non-Abelian statistics.[3,4] The p_x+ip_y superfluid also possesses an equilibrium edge current that is consistent with an $\hbar$-per-Cooper-pair intrinsic angular momentum.[5–7] This current is similar to that induced by a confining potential at the boundary of a Hall droplet, and occurs because chiral fermionic superfluids manifest an analog of the quantum Hall effect, even in the absence of an external magnetic field.[8]

The principal differences between the Pfaffian Hall state and the chiral superfluid are that, in addition to the Pfaffian, the Hall-state wave function includes a symmetric $\nu=1/2$ Laughlin factor; also the Hall system is spin polarized, while the superfluid retains two active spin components. A less significant difference is that the Hall fluid is incompressible while a neutral superfluid like ^{3}He-A has a gapless acoustic mode. This acoustic mode will, however, be gapped in a charged superfluid state, such as that believed to exist in the layered Sr_2RuO_4 superconductors.[9,10]

In the quantum Hall effect, the lack of low-energy bulk degrees of freedom leads to the dynamics of the boundary and the bulk being intimately connected. Currents flowing in the bulk serve to soak up the anomalies of the edge-mode conservation laws,[11] and, more profoundly, the bulk many-body wave function can be constructed from the conformal blocks of the edge conformal field theory.[3] The existence of the conformally invariant Majorana edge mode suggests that something similar is true for the superfluid.

Because of the interest of quantum systems with non-Abelian statistics, both for their intrinsic appeal as exotic physics and for their potential application to quantum computing,[12,13] there is reason to search for the simplest possible description of their dynamics. Thus we seek an effective action which captures all the essential low-energy degrees of freedom of the system. The construction of such an effective action for the fermionic Pfaffian Hall state is, however, rather complicated.[14] It is therefore worth examining other systems with similar properties.

One such system is the *bosonic* Pfaffian quantum Hall state, which can perhaps be realized in rotating Bose condensates. The bosonic Pfaffian state has a ground-state wave function consisting of the product of the antisymmetric Pfaffian and an antisymmetric $\nu=1$ Laughlin factor, making it even under particle interchange. In this case[14,15] the effective action is an SU(2) Chern-Simons term at level $k=2$. This means that the edge states form representations of an $\mathrm{SU}(2)_2$ current algebra. Indeed, the space of low-energy states is spanned by wave functions consisting of polynomials in the complex coordinates z_i that vanish when any three of the z_i coincide. The generating function for the number of such polynomials is a character of the $\mathrm{SU}(2)_2$ algebra.[16]

The p_x+ip_y superfluid is also relatively easy to analyze because its effective action has been computed by standard gradient expansion methods.[8,17] It is the purpose of this paper to explore some of the mathematical and physical aspects of the resulting expression. We will see that the response of the fluid to a non-Abelian gauge field which couples to the spin degree of freedom is governed by a Chern-Simons term of conventional form, but with a coefficient corresponding to level $k=\frac{1}{2}$. Since invariance under "large" gauge transformations demands that the level k be an integer, this means that the spin action cannot be gauge invariant on its own. Gauge invariance is restored only when we take into account the role of the spin-triplet order parameter in spontaneously breaking of the SU(2) symmetry down to U(1), and then by the subsequent absorption of the residual U(1) gauge dependence by the U(1) axial anomaly of the chiral edge states. The topological origin of the edge modes is therefore illuminated. The response to an Abelian particle-number gauge field is governed by an action which looks superficially Chern-Simons like, but is deficient in that it lacks the terms with time derivatives. In this case, gauge invariance is immediately manifest once we include the U(1) Goldstone mode in the effective action.[8] In particular, no boundary degree of freedom is required. Although this means that there is no explicit coupling of the Abelian gauge field to the edge modes, the Abelian part of the effective action does describe the Hall-like edge *current*, which therefore has a different origin from the edge *modes*. It also reveals an essential difference between the real Hall effect and its field-free analog. In the Hall effect the current is proportional to the applied electric field. In the superfluid, the absence of the $\dot{\mathbf{A}}$ in the action means that the induced current is proportional not to the electric field, but to the gradient of the fluid density. The "Hall current" should therefore be thought of as a two-dimensional analog of the Mermin-Muzikar current,[6] which is due to the intrinsic angular momentum of the fluid.

There have been several recent papers discussing the effective actions for chiral superfluids and superconductors,[18,19] but our point of view differs from these in its interpretation of the "Hall effect," the magnitude and origin of the edge current, and in its emphasis on the role of gauge invariance in establishing the bulk/edge connexion.

In the following section we will discuss the Bogoliubov action for the planar p_x+ip_y spin-triplet superfluid. In Sec. III we will find the eigenfunctions of the corresponding Bogoliubov–de Gennes Hamiltonian in rigid walled containers, and use them to compute the edge-mode spectrum and the magnitude of the persistent edge currents. In Sec. IV we will describe the effective actions governing the response of the fluid to external gauge fields that couple to particle number and to spin. We then show how the existence of the persistent boundary current and gapless edge modes can be deduced from the bulk effective action.

II. BOGOLIUBOV–DE GENNES OPERATOR

The fermionic part of the action describing a BCS superconductor can be written in Nambu two-component formalism as

$$S=\int d^2x\,dt\left(\frac{1}{2}\Psi^\dagger(i\partial_t-\hat{H})\Psi\right). \tag{1}$$

Here,

$$\Psi=\begin{bmatrix}\psi_\alpha\\ \psi^\dagger_\alpha\end{bmatrix},\qquad \Psi^\dagger=[\psi^\dagger_\alpha,\psi_\alpha], \tag{2}$$

are Grassmann fields with a spinor index $\alpha=\uparrow,\downarrow$, and $\hat{H}$ is the Bogoliubov–de Gennes Hamiltonian

$$\hat{H}=\begin{bmatrix}\hat{h} & \hat{\Delta}\\ \hat{\Delta}^\dagger & -\hat{h}^T\end{bmatrix}. \tag{3}$$

The entries in $\hat{H}$ are single-particle operators acting on the tensor product of the position and spinor spaces. The hermiticity of $\hat{H}$ requires that the single-particle Hamiltonian $\hat{h}$ be Hermitian, and Fermi statistics requires that the gap function $\hat{\Delta}$ be skew symmetric, in this combined space.

For a Galilean invariant fluid of particles of mass m, the Hamiltonian is

$$\hat{h}=\left(-\frac{1}{2m}\nabla^2-\epsilon_f\right)I, \tag{4}$$

where I is the identity operator in spin space and ϵ_f the Fermi energy. For a superconductor $\hat{h}$ can be a more general function, $\epsilon(\hat{\mathbf{p}})I$, of $\hat{\mathbf{p}}=-i\boldsymbol{\nabla}$.

The gap function will usually be a dynamical field, and the total action will contain additional terms that serve to determine its value via a gap equation. We are, however, interested only in the response of the fermions to prescribed changes in the gap function $\hat{\Delta}$, so we will not need to make these terms explicit. Further, we are primarily interested in topological effects, so the precise form of $\hat{\Delta}$ is not significant as long as it produces the required symmetry breaking. For a p_x+ip_y, spin-triplet superfluid, such as ^{3}He-A, we may take

$$\hat{\Delta}=\frac{1}{2}\left(\frac{\Delta}{k_f}\right)e^{i\Phi/2}\{\hat{\Sigma},\hat{P}\}e^{i\Phi/2}. \tag{5}$$

Here, $\{\,,\,\}$ denotes an anticommutator, Δ is the magnitude of the induced gap in the quasiparticle spectrum, and Φ is the overall phase of the order parameter. The spin part $\hat{\Sigma}$ is a symmetric 2×2 matrix with entries

$$\Sigma_{\alpha\beta}=[i(\boldsymbol{\sigma}\cdot\mathbf{d})\sigma_2]_{\alpha\beta}, \tag{6}$$

where $\mathbf{d}$ is a unit vector. The orbital part of the gap function is contained in the operator $\hat{P}$, which we will take to be

$$\hat{P}=-i(\hat{p}_x+i\hat{p}_y)\equiv-(\partial_x+i\partial_y), \tag{7}$$

corresponding to Cooper pairs with their $l=1$ angular momentum vector $\mathbf{l}$ directed in the $+\hat{\mathbf{z}}$ direction, i.e., perpendicular to the plane of the fluid. This orientation for $\mathbf{l}$ ensures that the entire Fermi surface is gapped. We will fix $\mathbf{l}=\hat{\mathbf{z}}$ throughout this paper. We will also take the magnitude of the gap Δ to be a constant. Although this parameter should be determined self-consistently through a gap equation, its magnitude serves only to provide an upper limit for what we mean by "low-energy" degrees of freedom, and so any variation has no role in the following discussion.

When $\hat{\Sigma}$ and Φ are constants, the operator ordering of the spin, phase, and orbital parts of $\hat{\Delta}$ is unimportant. When these quantities vary in space, however, we need the anticommutator of $\hat{\Sigma}$ and $\hat{P}$, and the symmetric distribution of the overall phase $e^{i\Phi}$ about it, to ensure the antisymmetry of $\hat{\Delta}$.

In the sequel, our calculations will be strictly $2+1$ dimensional. They therefore apply to a single stratum of a layered superconductor, or, for a thin film of ^{3}He-A, to the case when only a single transverse momentum mode lies below the Fermi surface. When n transverse modes are occupied but the order parameter remains independent of z, our effective actions should be multiplied by n.

III. EDGE MODES AND BOUNDARY CURRENTS

We now suppose the superfluid to be confined by rigid walls at which the wave function is required to vanish. We also assume, for the duration of this section, that the spin vector $\mathbf{d}$ lies in the $\hat{\mathbf{y}}$ direction, so that $\hat{\Sigma}=iI$ and the two spin components decouple.

A. Rectangular geometry

If we substitute

$$\Psi=\begin{bmatrix} a \\ b \end{bmatrix} e^{ik_x x+ik_y y} \tag{8}$$

with constant a, b, into the Bogoliubov equation, $\hat{H}\Psi=E\Psi$, the eigenvalue condition reduces to

$$\begin{bmatrix} \epsilon(k) & \left(\frac{k}{k_f}\right)e^{i\theta}\Delta \\ \left(\frac{k}{k_f}\right)e^{-i\theta}\Delta & -\epsilon(k) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=E\begin{bmatrix} a \\ b \end{bmatrix}. \tag{9}$$

Here θ is the polar angle such that $k_x=k\cos\theta$, $k_y=k\sin\theta$. The plane-wave eigenstates are therefore

$$\Psi_{E,\mathbf{k}}=e^{i\sigma_3\theta/2}\frac{1}{2\sqrt{E(E+\Delta)}}\begin{bmatrix} E+\epsilon(k)+\Delta \\ E-\epsilon(k)+\Delta \end{bmatrix} e^{ik_x x+ik_y y} \tag{10}$$

with $E=+\sqrt{\epsilon^2(k)+\Delta^2(k^2/k_f^2)}$, and

$$\Psi_{-|E|,\mathbf{k}}=e^{i\sigma_3\theta/2}\frac{1}{2\sqrt{|E|(|E|-\Delta)}}\begin{bmatrix} |E|-\Delta-\epsilon(k) \\ |E|-\Delta+\epsilon(k) \end{bmatrix} e^{ik_x x+ik_y y} \tag{11}$$

with $E=-\sqrt{\epsilon^2(k)+\Delta^2(k^2/k_f^2)}$. We will restrict ourselves to the case of weak coupling, where $\Delta \ll \epsilon_f$, and so all states of interest lie close to the Fermi surface where we can approximate the energy as

$$E=\pm\sqrt{v_f^2(k-k_f)^2+\Delta^2}, \tag{12}$$

where v_f is the Fermi velocity. We have considered only one spin component. There are really two sets of such solutions, one for spin up and one for spin down.

If we expand the quantized field operator $\hat{\Psi}$ in terms of the plane-wave states

$$\hat{\Psi}=\begin{bmatrix} \hat{\psi} \\ \hat{\psi}^{\dagger} \end{bmatrix}=\sum_{E,\mathbf{k}} \hat{a}_{E,\mathbf{k}}\Psi_{E,\mathbf{k}} \tag{13}$$

then, in order for the upper and lower components to reconstruct $\hat{\psi}$ and $\widehat{\psi^{\dagger}}$, respectively, the operators $\hat{a}_{E,\mathbf{k}}$ must obey the reality condition $\hat{a}_{E,\mathbf{k}}=\hat{a}^{\dagger}_{-E,-\mathbf{k}}$.

1. Edge states

We now investigate the effect of a boundary. Suppose, as shown in Fig. 1, that there is a wall at $y=0$, with the fluid lying above it. We seek solutions to the Bogoliubov equation in the form

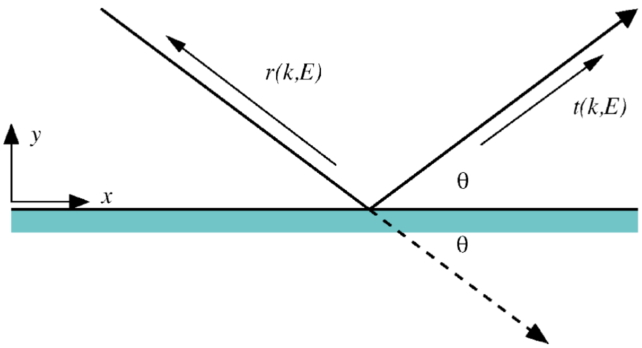

FIG. 1. Geometry of specular and Andreev reflections from the boundary.

$$\Psi=C_+\begin{bmatrix} a_+(y) \\ b_+(y) \end{bmatrix} e^{ik_f x\cos\theta+ik_f y\sin\theta} - C_-\begin{bmatrix} a_-(y) \\ b_-(y) \end{bmatrix} e^{ik_f x\cos\theta-ik_f y\sin\theta}, \tag{14}$$

where now $a_\pm$ and $b_\pm$ are allowed to vary with y—but slowly on the scale of k_f. We impose the boundary condition that $\Psi=0$ at $y=0$. The resultant equations for $a_\pm$ and $b_\pm$ coincide with those derived in the Appendix for the solutions of a one-dimensional Dirac Hamiltonian. The vanishing of the wave function at the wall becomes (because of the minus sign before C_-) the continuity of the Dirac wave function at $x=0$. The incoming particle therefore sees the reflecting boundary only as an abrupt change in the phase of the orbital part of the order parameter, which jumps from $-\theta$ to $+\theta$. The transmitted Dirac wave with amplitude $t(k,E)$ corresponds to those particles that have been specularly reflected at boundary, and the backscattered Dirac wave with amplitude $r(k,E)$ corresponds to those particles that have been Andreev reflected, and so retrace their path. The only significant differences between the two-dimensional geometry and the one-dimensional problem solved in the Appendix are that the coordinate "x" in the Appendix is the distance along the classical trajectory, and that there we have set the Fermi velocity v_f to unity. Thus, expressions appearing in the appendix, such as $e^{-\kappa|x|}$, must here be replaced by $e^{-\kappa|y|/[v_f\sin(\theta)]}$.

On making the appropriate translations from the results in the Appendix, we find that we have a bound state

$$\chi^{\{0\}}(k_x)=\sqrt{\frac{\Delta}{2v_f}}e^{ik_f x\cos\theta}\sin(k_f y\sin\theta)e^{-\Delta y/v_f}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{15}$$

with energy

$$E^{\{0\}}(k_x)=-\Delta\cos\theta=-\Delta(k_x/k_f). \tag{16}$$

The associated edge excitations are therefore *chiral*, or *Weyl*, fermions, having group velocity $\partial E/\partial k_x$ only in the $-\hat{\mathbf{x}}$ direction. This motion is in the opposite sense to the anticlockwise orbital angular momentum of the Cooper pairs.

These edge modes have a topological character. All that is required for them to exist is that the fluid density fall to zero at the boundary.[2] Indeed, the edge-mode spectrum (16) for our rigid wall boundary coincides with that obtained by Read and Green[1] in the opposite extreme of a very soft wall.

The contribution of the edge modes to the field operator $\hat{\Psi}$ is

$$\sum_{k_x} \hat{a}_{k_x}^{\{0\}} \chi_{k_x}^{\{0\}}(x-ct), \tag{17}$$

where $c=-\Delta/k_f$, and, in order for the upper and lower components to reconstruct $\hat{\psi}$ and $\hat{\psi}^\dagger$, respectively, we must have $\hat{a}_{k_x}^{\{0\}}=(\hat{a}_{-k_x}^{\{0\}})^\dagger$. The edge-mode field

$$\hat{\Psi}^{\{0\}}(x-ct)=\int_{-k_f}^{k_f}\frac{dk}{2\pi}\hat{a}_{k_x}^{\{0\}}\chi_{k_x}^{\{0\}}(x-ct) \tag{18}$$

is therefore *real*, $\hat{\Psi}^{\{0\}}(x)=(\hat{\Psi}^{\{0\}}(x))^\dagger$, or *Majorana*.

It is difficult for anything interact with a Majorana-Weyl particle. A Weyl particle can interact only via currents, and no current operator can be constructed from a single Majorana field. In our superfluid there are, however, *two* Majorana-Weyl fields: one for spin up and one for spin down. Out of these two real fields we can construct one complex Weyl field:

$$\hat{\Psi}_c=\frac{1}{\sqrt{2}}(\hat{\Psi}_\uparrow^{\{0\}}+i\hat{\Psi}_\downarrow^{\{0\}}),$$

$$\hat{\Psi}_c^\dagger=\frac{1}{\sqrt{2}}(\hat{\Psi}_\uparrow^{\{0\}}-i\hat{\Psi}_\downarrow^{\{0\}}), \tag{19}$$

and from this we can construct a unique U(1) current operator

$$\hat{\Psi}_c^\dagger\hat{\Psi}_c=\frac{i}{2}(\hat{\Psi}_\uparrow^{\{0\}}\hat{\Psi}_\downarrow^{\{0\}}-\hat{\Psi}_\downarrow^{\{0\}}\hat{\Psi}_\uparrow^{\{0\}})=\frac{1}{2}\hat{\Psi}^{\{0\}}\sigma_2\hat{\Psi}^{\{0\}}. \tag{20}$$

The edge modes may therefore interact with the σ_2 component of a spin-coupled gauge field.

2. Boundary current

The doubling of the number of degrees of freedom in the Bogoliubov formalism requires us, when computing ground-state expectation values of an operator, to sum the contributions of all occupied (negative energy) states, but then divide by 2 to prevent overcounting—this division by 2 being equivalent to imposing the reality condition on the $\hat{a}_{E,\mathbf{k}}$. The mass current carried by the state

$$\Psi=\begin{bmatrix} a \\ b \end{bmatrix} e^{ik_f x\cos\theta+ik_f y\sin\theta}, \tag{21}$$

where a, b are slowly varying compared to k_f is therefore

$$\mathbf{j}=\tfrac{1}{2}(|a|^2+|b|^2)(k_f\cos\theta,k_f\sin\theta). \tag{22}$$

The edge modes with $k_x>0$ have negative energy, and so are occupied in the ground state where they make a positive contribution to the boundary momentum density. For a Galilean invariant system, the momentum density is also the mass current. The edge modes therefore tend to produce a boundary current that flows in the same sense as the Cooper-pair rotation. The bound states, however, are not the only contribution to this boundary current. The balance is provided by the phase-shifted scattering states, which (as they do in the theory of fractional charge[23,24]) partially cancel the contribution of a bound state when it has negative energy and is occupied, and partially make up for the absence of its contribution when it has positive energy and is unoccupied.

The current in the direction perpendicular to the wall will cancel between the incoming and outgoing waves,[26] but the $\hat{\mathbf{x}}$ components will add. Using the results from the Appendix, we find that the net mass current running near the boundary is (for a single spin component)

$$\begin{aligned}\int_0^\infty j_x dy&=\int_0^1\frac{d(k_f\cos\theta)}{2\pi}\left(\frac{\theta}{2\pi}\right)k_f\cos\theta\\&\quad+\int_{-1}^0\frac{d(k_f\cos\theta)}{2\pi}\left(\frac{\theta}{2\pi}-\frac{1}{2}\right)k_f\cos\theta\\&=2\int_0^1\frac{d(k_f\cos\theta)}{2\pi}\,\frac{\theta}{2\pi}k_f\cos\theta=\frac{k_f^2}{16\pi}=\frac{1}{4}\rho,\end{aligned} \tag{23}$$

where, in our weak-coupling approximation, $\rho=k_f^2/4\pi$ is the number density per spin component. If this current flows at the edge of a disk-shaped region of radius R, it provides angular momentum

$$\mathbf{L}=2\pi R^2\left(\frac{\rho}{4}\right)\hat{\mathbf{z}}=\frac{N}{2}\hat{\mathbf{z}}. \tag{24}$$

This result agrees with that of Kita,[7] and Volovik[20] and is exactly the angular momentum we would expect of a fluid of tightly bound Cooper pairs, each pair having orbital angular momentum of $\hbar\hat{\mathbf{z}}$. That the same result holds in the weak coupling limit, where only the particles near the Fermi surface are affected by the pairing, is more surprising, and it is often referred to as the "angular momentum paradox" (see Kita Ref. 7 for a recent review of this).

Our result for the boundary current differs from that of Furusaki *et al.*,[19] because they consider only the the bound state contributions. The boundary current arising from the bound states alone is (again for a single spin component)

$$\frac{1}{2}\int_0^1\frac{d(k_f\cos\theta)}{2\pi}k_f\cos\theta=\frac{k_f^2}{8\pi}=\frac{\epsilon_f m}{4\pi}. \tag{25}$$

After multiplying by 2 to take into account the two spin components, this coincides with Eq. (2.8) of Furusaki *et al.*,[19] and differs from the actual persistent boundary current by a factor of 2.

We have computed the boundary current only in the weak coupling limit, but the result that it precisely accounts for the $\hbar$-per-pair angular momentum should remain valid even as the coupling is increased. This is because, provided the ground state evolves adiabatically, its total angular momentum cannot change as we alter parameters in the Hamiltonian. Adiabatic evolution may fail if there is spectral flow, and such flow does occur and change the angular momentum when we compute the angular momentum of some vortex configurations in bulk ^{3}He-*A*,[21] and also when we consider the *dynamical* intrinsic angular momentum density,[22] but in the present case spectral flow can only occur through the gapless edge states and the $E^{\{0\}}(k_x)=-E^{\{0\}}(-k_x)$ Majorana symmetry prevents the spectrum moving *en masse* with respect to the chemical potential. For the same reason, the total boundary current should also not be affected by local deviations of the order parameter away from its bulk form. The "fractional charge" interpretation[23–25] of the current ensures this, because the total fractional charge depends only on the asymptotics of the order parameter, and is unaffected by local variations.

B. Disk geometry

When the fluid has disk geometry it is convenient to use polar coordinates r, θ, and to work with angular momentum eigenstates $\propto e^{il\theta}$. For example, we have

$$-\nabla^2 J_l(kr)e^{il\theta}=k^2 J_l(kr)e^{il\theta}, \tag{26}$$

where $J_l(kr)$ is a Bessel function. In polar coordinates, the orbital part of the gap operator $\hat{P}=-(\partial_x+i\partial_y)$ becomes

$$\hat{P}=-e^{i\theta}\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\theta}\right), \tag{27}$$

and has a particularly simple action on the Laplace eigenfunctions,

$$\hat{P}e^{il\theta}J_l(kr)=ke^{i(l+1)\theta}J_{l+1}(kr). \tag{28}$$

The adjoint operator

$$\hat{P}^\dagger=-\left(-\frac{1}{r}\frac{\partial}{\partial r}r+\frac{i}{r}\frac{\partial}{\partial\theta}\right)e^{-i\theta}=-e^{i\theta}\left(-\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\theta}\right), \tag{29}$$

similarly acts to reduce the angular momentum eigenvalue

$$\hat{P}^\dagger e^{il\theta}J_l(kr)=ke^{i(l-1)\theta}J_{l-1}(kr). \tag{30}$$

We look for eigenstates in the form

$$\psi=\begin{bmatrix} aJ_{l+1}(kr)e^{i(l+1)\theta} \\ bJ_l(kr)e^{il\theta}\end{bmatrix}. \tag{31}$$

The Bogoliubov equation $\hat{H}\Psi=E\Psi$ then reduces to

$$\begin{bmatrix}\epsilon(k) & (k/k_f)\Delta \\ (k/k_f)\Delta & -\epsilon(k)\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=E\begin{bmatrix}a\\ b\end{bmatrix}. \tag{32}$$

The eigenstates are therefore

$$\Psi_{E,l}(r,\theta)=\frac{1}{2\sqrt{E(E+\Delta)}}\begin{bmatrix}(E+\epsilon+\Delta)e^{i(l+1)\theta}J_{l+1}(kr)\\ (E-\epsilon+\Delta)e^{il\theta}J_l(kr)\end{bmatrix}, \tag{33}$$

where the energy eigenvalue is

$$E=+\sqrt{\epsilon^2(k)+(k/k_f)^2\Delta^2} \tag{34}$$

and

$$\Psi_{-|E|,l}(r,\theta)=\frac{1}{2\sqrt{|E|(|E|-\Delta)}}\times\begin{bmatrix}(|E|-\Delta-\epsilon)e^{i(l+1)\theta}J_{l+1}(kr)\\ (|E|-\Delta+\epsilon|E|)e^{il\theta}J_l(kr)\end{bmatrix} \tag{35}$$

with energy

$$E=-\sqrt{\epsilon^2(k)+(k/k_f)^2\Delta^2}. \tag{36}$$

As before, we are interested in momenta near the Fermi surface where the energy can be approximated by

$$E=\pm\sqrt{v_f^2(k-k_f)^2+\Delta^2}. \tag{37}$$

When we confine the fluid by imposing rigid wall boundary conditions at $r=R$, the eigenstates will be a linear combination

$$\Psi_l=C_+\begin{bmatrix}a_+J_{l+1}(k_+r)e^{i(l+1)\theta}\\ b_+J_l(k_+r)e^{il\theta}\end{bmatrix}-C_-\begin{bmatrix}a_-J_{l+1}(k_-r)e^{i(l+1)\theta}\\ b_-J_l(k_-r)e^{il\theta}\end{bmatrix}, \tag{38}$$

of unconfined states with slightly different momentum $k_\pm=k_f\pm k$, but a common energy

$$E=\pm\sqrt{v_f^2k^2+\Delta^2}. \tag{39}$$

To examine the consequences of the condition that $\Psi=0$ at $r=R$ we use the WKB approximation for the Bessel function,

$$J_l(kr)\approx\sqrt{\frac{2}{\pi kx(r)}}\sin[kx(r)-l\theta(r)-\pi/4], \quad r\gg b. \tag{40}$$

Here $x(r)$ and $\theta(r)$ are functions of r, defined in terms of the semiclassical impact parameter, $b=l/k$, by $x=r\sin\theta$ and $b=r\cos\theta$. As illustrated in Fig. 2, the parameter x has the physical interpretation of being the distance along the straight-line semiclassical trajectory. This approximation is quite accurate once r exceeds b by more than a few percent, and is therefore reliable except for a few large values of l which correspond to classical trajectories grazing the boundary. Using the WKB approximation and the explicit form of the coefficients $a_\pm$ and $b_\pm$ we we end up with exactly the same equations for the bound state and S matrix that we found in the planar boundary case.

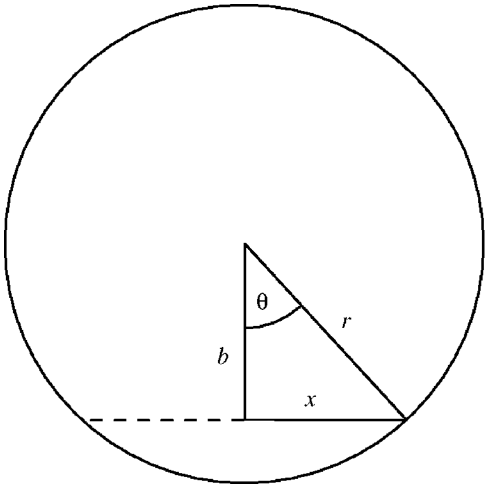

FIG. 2. The geometry of the WKB approximation to the Bessel function.

There is some advantage of working with a circular container, however. With a finite length boundary, the set of edge modes becomes discrete—being labeled by the integer l. The angle θ between the semiclassical trajectory and the boundary is $\theta(R)$, and $l=k_f R\cos\theta(R)$. In terms of l, therefore, the bound state has energy

$$E^{\{0\}}(l)\approx-\left(\frac{l}{l_{max}}\right)\Delta, \tag{41}$$

where $l_{max}=k_f R$. The WKB approximation is not quite good enough to distinguish between l and $l+\frac{1}{2}$ in this expression, but on general grounds we know that if $(u,v)^T$ is an eigenstate of the Bogoliubov Hamiltonian with energy E, then $(v^*,u^*)^T$ is also an eigenstate with energy $-E$. Under this transformation $\Psi_l\to\Psi_{-(l+1)}$, and so the correct equation must be

$$E^{\{0\}}(l)=-\left(\frac{l+1/2}{l_{max}}\right)\Delta. \tag{42}$$

There is therefore no exact zero mode. If, however, the bulk fluid were to contain an odd number of vortices, and thus the phase of the order parameter wind an odd number of times as we encircle the boundary, then the l's appearing in the upper and lower components of Ψ would differ by an *even* number. In this case there *will* be a zero-energy edge state. Since each vortex has a zero mode in its core,[4] there will be an unpaired zero-energy core mode which can pair with the zero-energy edge state and so preserve the even dimension of the total Bogoliubov-particle Hilbert space.

This dependence of the edge-mode spectrum on the number of the bulk vortex excitations is reminiscent of what happens in the quantum Hall effect. There the Hilbert space sector of the edge conformal field theory also depends on the number and type of vortex quasiparticles that are present in the bulk.

IV. EFFECTIVE ACTIONS

We now discuss the effective action governing the low-energy dynamics of the superfluid. We will obtain the action by examining the response of the fluid to external gauge fields that couple to the particle number and spin of the fluid.

A. Particle-number symmetry

We begin by gauging the U(1) symmetry corresponding to particle-number conservation. We will again hold the $\mathbf{d}$ vector fixed in the $\hat{\mathbf{y}}$ direction so we can treat each spin component separately. We minimally couple the particle-number current to an Abelian gauge field $(A_0,\mathbf{A})$, where A_0 is the time component and $\mathbf{A}\equiv(A_1,A_2)$ are the in-plane components of the externally imposed field. This requires the replacement

$$i\partial_t-\hat{H}\to i\partial_t-\hat{H}(A,\Phi), \tag{43}$$

where

$$\hat{H}(A,\Phi)=\begin{bmatrix} -\frac{1}{2m}(\boldsymbol{\nabla}-i\mathbf{A})^2-A_0, & i\left(\frac{\Delta}{k_f}\right)e^{i\Phi/2}\hat{P}e^{i\Phi/2} \\ -i\left(\frac{\Delta}{k_f}\right)e^{-i\Phi/2}\hat{P}^\dagger e^{-i\Phi/2}, & \frac{1}{2m}(\boldsymbol{\nabla}+i\mathbf{A})^2+A_0 \end{bmatrix}. \tag{44}$$

The resulting action

$$S(A,\Phi,\Psi,\Psi^\dagger)=\int d^2x\,dt\,\Psi^\dagger(i\partial_t-\hat{H}(A,\Phi))\Psi \tag{45}$$

is then invariant under the local gauge transformation

$$\begin{bmatrix}\psi\\ \psi^\dagger\end{bmatrix}\to\begin{bmatrix}e^{i\phi}\psi,\\ e^{-i\phi}\psi^\dagger\end{bmatrix}, \tag{46}$$

provided we simultaneously transform

$$\Phi\to\Phi+2\phi,$$

$$\mathbf{A}\to\mathbf{A}+\boldsymbol{\nabla}\phi,$$

$$A_0\to A_0+\partial_t\phi. \tag{47}$$

Because the Grassmann measure in the path integral is left invariant by the transformation (46), the effective action

$$iS_{\rm eff}^{\rm num}(A,\Phi)=\ln\left\{\int d[\Psi]d[\Psi^\dagger]\exp[iS(A,\Phi,\Psi,\Psi^\dagger)]\right\}$$

$$=\frac{1}{2}\ln \mathrm{Det}[i\partial_t-\hat{H}(A,\Phi)] \tag{48}$$

will be invariant under the transformation (47). If we compute Eq. (48) to second order in the gauge field and its gradients, we find,[8,18] for a single spin component,

$$S_{\rm eff}^{\rm num}(A,\Phi)=\int d^2x\,dt\left\{\frac{\rho_0}{2m}\left[\frac{1}{c_s^2}\left(\frac{\partial\Phi/2}{\partial t}-A_0\right)^2-(\nabla\Phi/2-\mathbf{A})^2\right]-\sigma_{xy}\left(\frac{\partial\Phi/2}{\partial t}-A_0\right)(\boldsymbol{\nabla}\times\mathbf{A})_z-\rho_0\left(\frac{\partial\Phi/2}{\partial t}-A_0\right)\right\}. \tag{49}$$

Here, $\sigma_{xy}=1/8\pi$, the parameter c_s is the speed of sound, and ρ_0 the equilibrium number density. In the weak coupling limit the coefficient σ_{xy} is proportional to a topological winding number.[8] The other quantities depend on the details of the fluid. For a $(2+1)$-dimensional Galilean invariant system of particles with mass m we have $\rho_0=(m/2\pi)\epsilon_f$ and $c_s=v_f/2$. The action (49) is manifestly invariant under the transformation (47).

The term with coefficient σ_{xy} contains a Chern-Simons-like part

$$\sigma_{xy}\int d^2x\,dt\,\epsilon_{0ij}A_0\partial_iA_j. \tag{50}$$

This is not a complete Chern-Simons action, however, because there is no $\epsilon_{i0j}A_i\,\partial_tA_j$ term. It does, nonetheless, imply the existence of a Hall-like response to the external field. We find for the particle-number current

$$\mathbf{j}_{\rm num}\equiv\frac{\delta S_{\rm eff}^{\rm num}}{\delta\mathbf{A}}=\frac{\rho_0}{m}(\boldsymbol{\nabla}\,\Phi/2-e\mathbf{A})+\sigma_{xy}(\hat{\mathbf{z}}\times\nabla)\left(\frac{\partial\Phi/2}{\partial t}-A_0\right)$$
$$=\rho_0\mathbf{v}_s+\sigma_{xy}(\hat{\mathbf{z}}\times\boldsymbol{\nabla})\left(\frac{\partial\Phi/2}{\partial t}-A_0\right). \tag{51}$$

Although the term with σ_{xy} contains a gradient of A_0, it is not equal to $\sigma_{xy}(\mathbf{E}\times\hat{\mathbf{z}})$, where $\mathbf{E}=\boldsymbol{\nabla}A_0-\dot{\mathbf{A}}$, as it would be in the Hall effect. (Observe that $\boldsymbol{\nabla}\dot{\Phi}/2$ cannot be $\dot{\mathbf{A}}$ in disguise, because the former is necessarily curl-free.) We note, however, that the combination $(\partial_t\Phi/2-A_0)$ occurs in the the expression for the density

$$\rho\equiv\frac{\delta S_{\rm eff}^{\rm num}}{\delta A_0}=\rho_0-\frac{\rho_0}{mc_s^2}\left(\frac{\partial\Phi/2}{\partial t}-A_0\right)+\sigma_{xy}\boldsymbol{\nabla}\times\mathbf{A}. \tag{52}$$

Consequently, it seems preferable to write

$$\mathbf{j}_{\rm num}=\rho_0\mathbf{v}_s-\frac{1}{4m}(\hat{\mathbf{z}}\times\boldsymbol{\nabla})(\rho-\sigma_{xy}B_z), \tag{53}$$

where $B_z=(\boldsymbol{\nabla}\times\mathbf{A})_z$, and so recognize that the "Hall" current depends on the external field primarily through its effect in modifying the density. The natural analogy is then with the bound current $\mathbf{j}_{\rm bound}=\boldsymbol{\nabla}\times\mathbf{M}$ in a magnet with varying magnetization $\mathbf{M}$. In the superfluid, the magnetic-moment density $\mathbf{M}$ is replaced by the intrinsic angular momentum density $\frac{1}{2}\hbar(\rho-\sigma_{xy}B_z)\hat{\mathbf{z}}$. The $\sigma_{xy}B_z$ term is presumably present because a diamagnetic response to the external field will reduce the *kinetic* angular momentum of a Cooper pair even while leaving its *canonical* angular momentum fixed at $\hbar$. In the absence of an external B_z field, we therefore have a mass flux

$$\mathbf{j}=m\,\mathbf{j}_{\rm num}=\tfrac{1}{4}\boldsymbol{\nabla}\times\rho\hat{\mathbf{z}}, \tag{54}$$

and this is a planar analog of the Mermin-Muzikar current.[6] When ρ goes to zero slowly at a boundary we can use this Mermin-Muzikar expression to compute the equilibrium boundary current. The resulting boundary momentum density coincides with that we computed for a rigid wall in Sec. III, and again provides an $\hbar$-per-Cooper-pair total angular momentum for the fluid.

The agreement between the rigid-wall calculation of the boundary current and the gradient expansion when expressed in terms of the density, coupled with the physical interpretation in terms of the $\hbar$-per-Cooper-pair intrinsic angular momentum of the fluid, leads us to conjecture that the nontopological corrections to σ_{xy} that arise as we move away from weak coupling[8,18] conspire with corrections to the compressibility in such a manner that the total mass current is always proportional to the change in density, rather than to the external force that causes the change.

B. Spin-rotation symmetry

The fields in

$$S=\int d^2x\,dt[\Psi^\dagger(i\partial_t-\hat{H})\Psi] \tag{55}$$

can also be coupled to an SU(2) gauge field which acts on the spin indices. To do this we replace the derivatives in S by covariant derivatives

$$\partial_\mu\to\partial_\mu+\mathcal{A}_\mu, \tag{56}$$

where

$$\mathcal{A}_\mu=i\sigma_a\mathcal{A}_\mu^a \tag{57}$$

is an externally imposed gauge field. Under a local gauge transformation the Fermi fields transform as

$$\Psi=\begin{bmatrix}\psi\\ \psi^\dagger\end{bmatrix}\to\begin{bmatrix}U\psi\\ U^*\psi^\dagger\end{bmatrix},\qquad \Psi^\dagger=[\psi^\dagger,\psi]\to[\psi^\dagger U^{-1},\psi U^T], \tag{58}$$

where $U\in{\rm SU}(2)$. The covariant derivatives transform as

$$\partial_\mu+\mathcal{A}_\mu\to U^{-1}(\partial_\mu+\mathcal{A}_\mu)U=\partial_\mu+(U^{-1}\mathcal{A}_\mu U+U^{-1}\partial_\mu U). \tag{59}$$

When considering how the transformation act in the $-\hat{h}^T$ entry in $\hat{H}$, we need to recognize that derivative operators appearing there are the transpose $\partial_\mu^T=-\partial_\mu$ of those in $\hat{h}$, and so the covariant derivatives will be also be the transpose $\partial_\mu^T+\mathcal{A}_\mu^T=-\partial_\mu+\mathcal{A}_\mu^T$. Using $U^*=(U^{-1})^T=(U^T)^{-1}$, we have

$$\begin{aligned} U^T(\partial_\mu^T+\mathcal{A}_\mu^T)(U^T)^{-1} &= -\partial_\mu + U^T\mathcal{A}_\mu^T(U^T)^{-1} \\ &\quad - U^T(U^T)^{-1}\partial_\mu U^T(U^T)^{-1} \\ &= -\partial_\mu + (U^{-1}\mathcal{A}_\mu U + U^{-1}\partial_\mu U)^T \\ &= [\partial_\mu + (U^{-1}\mathcal{A}_\mu U + U^{-1}\partial_\mu U)]^T, \end{aligned} \tag{60}$$

and the effect of the transformation is consistent for both entries:

$$\mathcal{A}_\mu \to A_\mu^U \equiv U^{-1}\mathcal{A}_\mu U + U^{-1}\partial_\mu U. \tag{61}$$

Note that $(\mathcal{A}^U)^V = \mathcal{A}^{UV}$. For the off diagonal terms we have

$$U^{-1}\hat{\Sigma} U^* = U^{-1}[i(\boldsymbol{\sigma}\cdot\mathbf{d})\sigma_2]U^* = [iU^{-1}(\boldsymbol{\sigma}\cdot\mathbf{d})U\sigma_2]. \tag{62}$$

The net result is that the gauged action is invariant under the transformation (58), provided we simultaneously transform

$$(\mathbf{d}\cdot\boldsymbol{\sigma}) \to U(\boldsymbol{\sigma}\cdot\mathbf{d})U^{-1},$$

$$\mathcal{A}_\mu \to \mathcal{A}_\mu^{U^{-1}} = U\mathcal{A}_\mu U^{-1} + U\partial_\mu U^{-1}. \tag{63}$$

Volovik and Yakovenko computed the low-energy effective action for the case of a $\mathbf{d}$ vector fixed to lie in the $\hat{\mathbf{y}}$ direction.[17] They found this to be Chern-Simons term

$$S_{\text{eff}}^{\text{spin}}(\mathbf{d}=\hat{\mathbf{y}},\mathcal{A}) = \frac{1}{8\pi}\int \text{tr}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right). \tag{64}$$

Here we are using use differential-form notation in which $\mathcal{A} \equiv i\sigma_a \mathcal{A}_\mu^a dx^\mu$ is a matrix valued one-form. For any compact simple gauge group G, the Chern-Simons action is defined to be

$$C(\mathcal{A}) = \frac{1}{4\pi}\int_\Omega \text{tr}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right), \tag{65}$$

where $\mathcal{A} \equiv i\lambda_a \mathcal{A}_\mu^a dx^\mu$ is a Lie(G)-algebra-valued one-form, and, as is customary, we normalize the trace and the Hermitian generators λ_a by $\text{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. When such a term is to appear in a functional integral,

$$Z = \int d[\mathcal{A}]e^{ikC(\mathcal{A})}, \tag{66}$$

then coefficient k must be quantized. This is because under a gauge transformation

$$\mathcal{A} \to \mathcal{A}^g \equiv g^{-1}\mathcal{A}g + g^{-1}dg \tag{67}$$

we have

$$\begin{aligned} C(\mathcal{A}) \to C(\mathcal{A}^g) &= C(\mathcal{A}) - \frac{1}{12\pi}\int_\Omega \text{tr}[(g^{-1}dg)^3] \\ &\quad - \frac{1}{4\pi}\int_{\partial\Omega} \text{tr}(dgg^{-1}\mathcal{A}). \end{aligned} \tag{68}$$

Here, Ω is the region occupied by the fluid and $\partial\Omega$ its boundary. The last term is zero when Ω is closed manifold, or if we restrict the gauge transformation to those constant on the boundary. The second term

$$\mathcal{W}(g) = \frac{1}{12\pi}\int_\Omega \text{tr}[(g^{-1}dg)^3] \tag{69}$$

has no reason to vanish, however. It is the pull-back to Ω of an element of $H_{DR}^3(G,\mathbf{Z})$, the third de-Rham cohomology group of G. It can be nonzero whenever the gauge transformation $g(x)$ maps Ω into a homologically nontrivial three-manifold in G. In particular, when Ω is S^3 and the group G is SU(2), then

$$\frac{1}{12\pi}\int_\Omega \text{tr}[(g^{-1}dg)^3] = 2\pi n, \tag{70}$$

where n is the degree of the map from $S^3 \to \text{SU}(2) \simeq S^3$. In this case k has to be an integer so that the gauge ambiguity in $C(\mathcal{A})$ is $2\pi kn$ and $\exp[ikC(\mathcal{A})]$ is well defined.

The Chern-Simons action found by Volovik and Yakovenko has a coefficient corresponding to $k=1/2$, and so violates the quantization condition on k. It cannot be gauge invariant on its own. The complete effective action will be gauge invariant, of course, but we must include additional degrees of freedom to make this manifest. One of these is the direction of the vector $\mathbf{d}$, which we must therefore allow to vary. We parametrize $\mathbf{d}$ in terms of a group element $V \in \text{SU}(2)$ by setting

$$(\mathbf{d}\cdot\boldsymbol{\sigma}) = V\sigma_2 V^{-1}. \tag{71}$$

Then,

$$S_{\text{eff}}^{\text{spin}}(\mathbf{d},\mathcal{A}) = S_{\text{eff}}^{\text{spin}}(\hat{\mathbf{y}},\mathcal{A}^V) = \tfrac{1}{2}C(\mathcal{A}^V). \tag{72}$$

This expression is clearly invariant under the simultaneous replacement $\mathcal{A} \to \mathcal{A}^U$ and $V \to U^{-1}V$. This is good, but not perfect. The problem is that V is not unique. The vector $\mathbf{d}$ is more correctly parametrized by elements of the coset SU(2)/U(1), since we can replace V by $Ve^{i\sigma_2\phi}$ without changing $\mathbf{d}$. This substitution does affect $C(\mathcal{A}^V)$, however. Compensating for the effects of $e^{i\sigma_2\phi}$ requires yet another degree of freedom. We will write this as $W = e^{i\sigma_2\chi}$. A completely gauge invariant action is then

$$S_{\text{eff}}^{\text{spin}}(\mathbf{d},\mathcal{A},\chi) = \tfrac{1}{2}C(\mathcal{A}^{VW}). \tag{73}$$

This is manifestly invariant under the simultaneous replacement

$$\mathcal{A} \to \mathcal{A}^U,$$

$$V \to U^{-1}Ve^{i\sigma_2\phi},$$

$$\chi \to \chi - \phi. \tag{74}$$

What is the physical interpretation of this extra field χ? This question is easiest to answer if we set $V=I$ on the boundary, so that $\mathbf{d}=\hat{\mathbf{y}}$ there. Then, using the Polyakov-Wiegmann identity (68) we can write

$$S_{\rm eff}^{\rm spin}(\mathbf{d},\mathcal{A},\chi)=\frac{1}{2}C(\mathcal{A}^{VW})=\frac{1}{2}C(A) - \frac{1}{24\pi}\int_{\Omega}\mathrm{tr}[(V^{-1}dV)^3] - \frac{1}{8\pi}\int_{\partial\Omega}\mathrm{tr}\{dWW^{-1}\mathcal{A}\}. \tag{75}$$

The second term

$$\frac{1}{2}\mathcal{W}(V)=\frac{1}{24\pi}\int_{\Omega}\mathrm{tr}[(V^{-1}dV)^3] \tag{76}$$

is precisely the Hopf index found by Volovik and Yakovenko. On a closed manifold, or when $\mathbf{d}$ is fixed at the boundary, it is equal to $n\pi$ where n labels the homotopy class of the map $\mathbf{d}:S^3\to S^2$. The Berry phase provided by this term makes a skyrmion soliton in the $\mathbf{d}$ field into a fermion. The third term

$$-\frac{1}{8\pi}\int_{\partial\Omega}\mathrm{tr}\{dWW^{-1}\mathcal{A}\} \tag{77}$$

is equal to

$$-\frac{i}{8\pi}\int_{\partial\Omega}\mathrm{tr}\{d\chi\sigma_2\mathcal{A}\}=\frac{1}{4\pi}\int dxdt\{\mathcal{A}_0^2\partial_x\chi-\mathcal{A}_x^2\partial_t\chi\}, \tag{78}$$

and represents the coupling of the χ field current to the σ_2 component of $\mathcal{A}$. This interaction takes place only on the boundary, which we have taken to be the x axis as in Sec. III A. This suggests that the χ field is the bosonized form of the complex Weyl fermion Ψ_c that we constructed out of the two Majorana-Weyl edge modes in Sec. III A. As we noted there, Ψ_c naturally couples to the σ_2 component of $\mathcal{A}$.

Gauge invariance tells us that the edge modes must exist, but it does not determine their dynamics. We can, however, add a manifestly gauge invariant boundary term that ensures that the χ field propagates unidirectionallly at speed $c=-\Delta/k_f$. This term is[11]

$$-\frac{c}{8\pi}\int dxdt\,\mathrm{tr}\{[W^{-1}(\partial_x+\mathcal{A}_x)W][W^{-1}(\partial_x+c^{-1}\partial_t+\mathcal{A}_x +c^{-1}\mathcal{A}_t)W]\}. \tag{79}$$

Including it, and writing $W=e^{i\sigma_2\chi}$, the effective action becomes

$$S_{\rm eff}^{\rm spin}(\mathbf{d},\mathcal{A},\chi)=\frac{1}{2}C(\mathcal{A})-\frac{1}{2}\mathcal{W}(V)+\frac{c}{4\pi}\int dxdt\{\partial_x\chi[(c^{-1}\partial_t +\partial_x)\chi]\}+\frac{1}{2\pi}\int dxdt\{(c\mathcal{A}_x^2+\mathcal{A}_t^2)\partial_x\chi\} -\frac{1}{8\pi}\int dxdt\,\mathrm{tr}\{(c\mathcal{A}_x+\mathcal{A}_t)\mathcal{A}_x\}. \tag{80}$$

The terms containing χ now constitute the action for a chiral boson interacting with the appropriate chiral component of the gauge field. The entire expression is invariant under gauge transformations U which reduce to the form $e^{i\sigma_2\phi}$ on the boundary, and so maintain $\mathbf{d}=\hat{\mathbf{y}}$ there. We note that the factor of 1/2 in the "level" k is compensated for by a factor 2 coming from $\mathrm{tr}(\sigma_2^2)=2$, so as to give the correct scale for a chiral boson representing a Weyl fermion.

Since the actions of the U(1) and SU(2) gauge groups commute with one another, the complete gauge invariant effective action containing all the low-energy degrees of freedom is the sum

$$S_{\rm eff}^{\rm tot}(A,\Phi,\mathbf{d},\mathcal{A},\chi)=2S_{\rm eff}^{\rm num}(A,\Phi)+S_{\rm eff}^{\rm spin}(\mathbf{d},\mathcal{A},\chi). \tag{81}$$

Here the "2" in front of $S_{\rm eff}^{\rm num}$ comes from the two spin components.

V. CONCLUSIONS

We have investigated the gauge invariance of the low-energy effective action for a $(2+1)$-dimensional chiral superfluid coupled to external gauge fields. When the order parameter completely breaks the gauge group, as in the case of Abelian particle-number symmetry, the effective action becomes manifestly gauge invariant as soon as we include the bulk Goldstone modes among the dynamical fields. When the gauge symmetry is not completely broken, as in the case of spin-rotation symmetry, we found that additional, non-Goldstone, degrees of freedom were required for manifest gauge invariance. These were identified as being the spin-up and spin-down Majorona-Weyl edge fermions, which could be combined to produce a current that soaks up the remaining gauge dependence.

The consequences of the effective actions for spin- and particle-number currents also differ. The former has a true spin-Hall effect, with a dissipationless spin current proportional to the spin "electric" field. The latter, we have argued, has only a "mock" Hall effect, the induced current being proportional to the change in density, and not to the external field causing the change.

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APPENDIX: DIRAC EQUATION

The twisted-mass Dirac equation that results from our two-dimensional problem is a standard illustration of the theory of fractional charge.[23] We review it here so as to make clear the contribution of the extended scattering states to the boundary current.

We consider the one-dimensional Dirac Hamiltonian

$$\hat{H} = -i\tau_3\partial_x + \Delta\tau_1 e^{-i\tau_3\phi(x)} = \begin{bmatrix} -i\partial_x & \Delta e^{i\phi(x)} \\ \Delta e^{-i\phi(x)} & i\partial_x \end{bmatrix}, \tag{A1}$$

where Δ is a constant.

We will compute the extra particle number that is accumulated in the vicinity of $x=0$ when ϕ is discontinuous, jumping abruptly from $\phi=\phi_L$ when $x<0$ to $\phi=\phi_R$ when $x>0$. Suppose the eigenstates of $\hat{H}$ are χ_n with energy E_n. The ground-state number density is

$$\langle \psi^\dagger \psi(x) \rangle = \sum |\chi_n(x)|^2, \tag{A2}$$

where the sum is over occupied states, i.e., those with $E_n<0$. Because the sum of $|\chi_n|^2$ over *all* states is independent of ϕ by completeness, we can equally well write

$$\langle \psi^\dagger \psi(x) \rangle = \text{const.} - \sum_{E_n>0} |\chi_n(x)|^2, \tag{A3}$$

and this form is slightly more convenient. We will show that

$$\begin{aligned} Q &= \int \langle \psi^\dagger \psi(x) \rangle dx \\ &= \frac{1}{2\pi}(\phi_R-\phi_L), \quad 0<(\phi_R-\phi_L)<\pi, \\ &= \frac{1}{2\pi}(\phi_R-\phi_L)-1, \quad \pi<(\phi_R-\phi_L)<2\pi. \end{aligned} \tag{A4}$$

When ϕ is constant we have a continuum of positive and negative energy eigenfunctions

$$\psi^{\phi}_{k,E} = e^{i\sigma_3\phi/2}\frac{1}{2\sqrt{E(E+\Delta)}}\begin{bmatrix} E+k+\Delta \\ E-k+\Delta \end{bmatrix} e^{ikx}, \tag{A5}$$

where $E(k)=\pm\sqrt{k^2+\Delta^2}$. With the discontinuity present, we will have scattering solutions

$$\begin{aligned} \psi &= a_L^{(in)}\psi^{\phi_L}_{k,E} + a_L^{(out)}\psi^{\phi_L}_{-k,E}, \quad x<0, \\ &= a_R^{(in)}\psi^{\phi_R}_{-k,E} + a_R^{(out)}\psi^{\phi_R}_{k,E}, \quad x>0. \end{aligned}$$

The function ψ must be continuous at $x=0$, and from this condition we obtain the S-matrix relation

$$\begin{bmatrix} a_L^{(out)} \\ a_R^{(out)} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix}\begin{bmatrix} a_R^{(in)} \\ a_L^{(in)} \end{bmatrix}, \tag{A6}$$

where

$$\begin{aligned} t(k,E) &= \frac{1}{\cos(\Phi/2)-(iE/k)\sin(\Phi/2)}, \\ r(k,E) &= \frac{i(\Delta/k)\sin(\Phi/2)}{\cos(\Phi/2)-(iE/k)\sin(\Phi/2)}. \end{aligned} \tag{A7}$$

Here Φ is shorthand for $\phi_L-\phi_R$.

In addition to the continuum states, there is also a single bound state

$$\begin{aligned} \psi^{\{0\}} &\propto e^{i\sigma_3\phi_R/2}\begin{bmatrix} E^{\{0\}}+i\kappa+\Delta \\ E^{\{0\}}-i\kappa+\Delta \end{bmatrix} e^{-\kappa x}, \quad x>0 \\ &\propto e^{i\sigma_3\phi_L/2}\begin{bmatrix} E^{\{0\}}-i\kappa+\Delta \\ E^{\{0\}}+i\kappa+\Delta \end{bmatrix} e^{\kappa x}. \quad x<0. \end{aligned}$$

The bound-state energy $E^{\{0\}}$ is also determined by continuity at $x=0$, which requires

$$E^{\{0\}} = \Delta\cos(\Phi/2),$$

$$\kappa = \Delta\sin(\Phi/2).$$

These formulas are valid for $0<\Phi<2\pi$, where κ is positive, and extend outside that interval with period 2π. At $\Phi=0$ there is no bound state. As Φ increases, a bound state peels off the upper continuum. It passes through $E=0$ at $\Phi=\pi$, and merges with the lower continuum as Φ reaches 2π. If Φ increases beyond 2π, the process repeats with another state peeling off the upper continuum. Thus each 2π twist in Φ results in the net transfer of one state from the upper continuum to the lower.

Using the relation between $E^{\{0\}}$ and κ, we can write the normalized bound state as

$$\chi^{\{0\}} = \sqrt{\frac{\kappa}{2}}\begin{bmatrix} e^{i(\phi_L+\phi_R)/4} \\ e^{-i(\phi_L+\phi_R)/4} \end{bmatrix} e^{-\kappa|x|}. \tag{A8}$$

A complete set of states comprises $\chi^{\{0\}}$ together with

$$\chi_{k,E} = \begin{cases} \psi^{\phi_L}_{k,E} + r(k,E)\psi^{\phi_L}_{-k,E}, & x<0, \\ t(k,E)\psi^{\phi_R}_{k,E}, & x>0, \end{cases} \quad k>0. \tag{A9}$$

$$\chi_{k,E} = \begin{cases} t(k,E)\psi^{\phi_L}_{k,E}, & x<0, \\ \psi^{\phi_R}_{k,E} + r(k,E)\psi^{\phi_R}_{-k,E}, & x>0, \end{cases} \quad k<0. \tag{A10}$$

These basis states therefore switch from a wave incident from the left to one incident from the right as k changes sign.

After using $|r|^2+|t|^2=1$, and the explicit form of the free eigenfunctions, we find

$$\begin{aligned} \sum_{E>\Delta} |\chi_E(x)|^2 &= \int_{-\infty}^{\infty}\frac{dk}{2\pi}|\chi_{E,k}(x)|^2 \\ &= \text{const.} + \int_{-\infty}^{\infty}\frac{dk}{2\pi} r(k,E)\left(\frac{\Delta}{k}\right) e^{2ik|x|}. \end{aligned} \tag{A11}$$

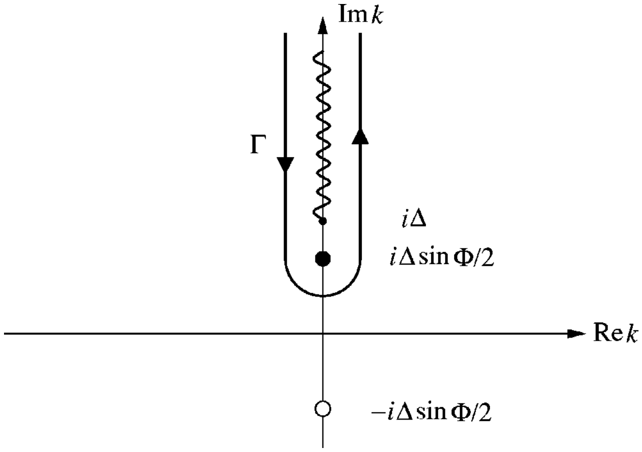

FIG. 3. The contour Γ, showing the cut starting at $ik=\Delta$. When $0<\Phi<\pi$ there is a pole at $k=i\Delta\sin\Phi/2$, and a pole on the second sheet at $k=-i\Delta\sin\Phi/2$.

Here the "const." refers to terms that are independent of $\phi_{R,L}$. We can improve the numerical convergence of the integral by using Jordan's lemma to push the contour of integration Γ into the upper half plane, as shown in Fig. 3.

We observe that the reflection coefficient has a cut running from $k=i\Delta$ to $i\infty$, and that for $0<\Phi<\pi$ there is a pole in the upper half plane at $k=i\Delta\sin\Phi/2$. At $\Phi=\pi$ the pole merges with the cut. For $\pi<\Phi<2\pi$ the pole is apparently returning towards the real axis again, but a more careful investigation shows that it is now on the second sheet, and no longer contributes. (At the same time a pole at $-i\Delta\sin\Phi/2$ has emerged onto the first sheet in the lower half plane. This is below the real axis, however, and also does not contribute.)

The integral in Eq. (A11) then becomes

$$-\Delta\sin\Phi/2e^{-2\Delta|\sin\Phi/2||x|}+\int_{\Delta}^{\infty}\frac{dk}{2\pi}\frac{\Delta^2\sin\Phi}{\kappa^2-\Delta^2\sin^2(\Phi/2)}\frac{1}{\sqrt{\kappa^2-\Delta^2}}\kappa e^{-2\kappa|x|}. \tag{A12}$$

The first term, the pole contribution, is only present if $0<\Phi<\pi$.

There does not seem to be a closed-form expression for the integral in Eq. (A12), but if we first integrate over x to get the total charge, we end up with an elementary integral

$$\int_{\Delta}^{\infty}\frac{dk}{2\pi}\frac{\Delta^2\sin\Phi}{\kappa^2-\Delta^2\sin^2(\Phi/2)}\frac{1}{\sqrt{\kappa^2-\Delta^2}}=\frac{\Phi}{2\pi},\quad -\pi<\Phi<\pi. \tag{A13}$$

This expression extends to a 2π periodic function of Φ, and the integral is therefore discontinuous at odd multiples of π. After we include the pole contribution, which is discontinuous at *all* multiples of π, we find that the total continuum contribution is discontinuous only at $\Phi=0 \pmod{2\pi}$. Thus

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}|\chi_{E,k}(x)|^2=\text{const.}+\frac{\Phi}{2\pi},\quad 0<\Phi<2\pi. \tag{A14}$$

The discontinuity at $\Phi=0 \pmod{2\pi}$ is due to the sudden loss of the bound state from the upper continuum. As the bound state makes its way to the lower continuum, the spectral weight in the upper continuum is gradually recovered.

We can now apply these results to compute

$$Q=\int\sum_{E_n<0}|\psi_n(x)|^2dx. \tag{A15}$$

We observe that our sum over the positive continuum is equal to minus the sum over the negative continuum together with the bound state. The bound state has negative energy, however, if $\pi<\Phi<2\pi \pmod{2\pi}$. If we are computing the ground-state charge, and if $0<\Phi<\pi \pmod{2\pi}$, we should reduce the sum by unity. We also note that the constant in Eq. (A14) is fixed by the requirement that the the accumulated charge be zero when $\phi_L=\phi_R$. Thus

$$\begin{aligned} Q&=\frac{1}{2\pi}(\phi_R-\phi_L),\quad 0<(\phi_R-\phi_L)<\pi,\\ &=\frac{1}{2\pi}(\phi_R-\phi_L)-1,\quad \pi<(\phi_R-\phi_L)<2\pi. \end{aligned} \tag{A16}$$

This result repeats with 2π periodicity.

This expression for Q is consistent with the basic result of Goldstone and Wilczek,[23] that a total charge of

$$\int\frac{1}{2\pi}\partial_x\phi dx=\frac{1}{2\pi}(\phi_R-\phi_L) \tag{A17}$$

is drawn into a region as the phase ϕ is slowly twisted. The difference between Eqs. (A16) and (A17) arises because the latter does not keep track of individual particles that are lost to the reservoir when their energy exceeds the chemical potential.

In a superfluid the role of charge and current is interchanged. The fractional charge, multiplied by k_f and divided by 2, gives the momentum density, or equivalently, the mass current.[25]

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[1] N. Read and D. Green, Phys. Rev. B **61**, 10 267 (2000).
[2] G.E. Volovik, Pis'ma Zh. Éksp. Teor. Fiz. **66**, 492 (1997) [JETP Lett. **66**, 522 (1997)].
[3] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[4] D.A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[5] K. Ishikawa, Prog. Theor. Phys. **57**, 1836 (1977).
[6] N.D. Mermin and P. Muzikar, Phys. Rev. B **21**, 980 (1980).
[7] T. Kita, J. Phys. Soc. Jpn. **67**, 216 (1998).
[8] G.E. Volovik, Phys. Scr. **38**, 321 (1988); Zh. Éksp. Teor. Fiz. **94**, 123 (1988) [Sov. Phys. JETP **67**, 1804 (1988)].

[9] T.M. Rice and M. Sigrist, J. Phys.: Condens. Matter **7**, l643 (1995).
[10] G. Baskaran, Physica B **223-224**, 490 (1996).
[11] M. Stone, Ann. Phys. (N.Y.) **207**, 38 (1991).
[12] A. Yu Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
[13] L.B. Ioffe, M.V. Felgel'man, A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, Nature (London) **415**, 503 (2002).
[14] E. Fradkin, C. Nayak, A.M. Tsvelik, and F. Wilczek, Nucl. Phys. B **516**, 704 (1998).
[15] E. Fradkin, C. Nayak, and K. Schoutens, Nucl. Phys. B **546**, 711 (1999).
[16] A.V. Stoyanovskii and B.L. Feigin, Funct. Anal. Appl. **28**, 55 (1994).
[17] G.E. Volovik and V.M. Yakovenko, J. Phys.: Condens. Matter **1**, 526 (1989).
[18] J. Goryo and K. Ishikawa, Phys. Lett. A **246**, 549 (1998); **260**, 294 (1999).
[19] A. Furusaki, M. Matsumoto, and M. Sigrist, Phys. Rev. B **64**, 054514 (2001).
[20] G.E. Volovik, Pis'ma Zh. Éksp. Teor. Fiz. **55**, 363 (1992) [JETP Lett. **55**, 368 (1992)].
[21] G.E. Volovik, Pis'ma Zh. Éksp. Teor. Fiz. **61**, 935 (1995) [JETP Lett. **61**, 958 (1995)].
[22] G.E. Volovik and V.P. Mineev, Zh. Éksp. Teor. Fiz. **81**, 989 (1981) [Sov. Phys. JETP **54**, 524 (1981)].
[23] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[24] M. Stone, Phys. Rev. B **31**, 6112 (1985).
[25] M. Stone and F. Gaitan, Ann. Phys. (N.Y.) **178**, 89 (1987).
[26] We ignore the cross terms which fluctuate as $\exp i2k_f$, and contribute no net flux.