Lumped-mass method for the study of band structure in two-dimensional phononic crystals

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A lumped-mass method is introduced to study the propagation of elastic waves in two-dimensional periodic systems. First, it is used to calculate the band structure of an array of Pb columns in an epoxy background. Second, the method is applied to the same array of Pb columns in a soft rubber background. The results are compared with those calculated with the well-known plane-wave expansion formalism, where the advantages of the lumped-mass method are pointed out and analyzed. These advantages make it possible for easy calculations of band structures of phononic crystals with interfaces of large contrast of elastic constants as well as units of any shapes.

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I. INTRODUCTION

In recent years, a great deal of work has been devoted to the study of the propagation of classical waves in periodic structures. A famous example is the photonic crystals.[1] For these new crystals, both theoretical predictions and experiments have shown the appearance of frequency gaps for the propagation of electromagnetic waves, which has been used for the achievement of new optical devices.

Quickly, those studies were extended to the propagation of elastic and acoustic waves in periodic structures made of materials with different elastic properties, which have been named phononic crystals (PCs)[2–7] by analogy with the photonic crystals. The emphasis was laid on the existence of complete acoustic/elastic band gaps (ABG) within which sound and vibrations are all forbidden. These new materials can be of real interest, since a large contrast between the elastic parameters is allowed. For example, systems composed of very soft rubber[8,9] (with elastic constant of five orders lower than common solids) are most likely to obtain the low-frequency gaps with a structure of small dimension, which can lead to promising applications as a low-frequency vibration/noise insulation. Besides, their properties in the transmission bands have been used to build refractive devices such as lenses and acoustic interferometers.[10] On the other hand, more sophisticated combinations such as fluids infiltrated in a drilled solid[11,12] or solid-solid systems[13,14] have been demonstrated to produce a full phononic band gap for ultrasounds.

Several theoretical methods have already been developed in order to study the elastic response of PCs. Mostly, the calculations are based on the plane-wave expansion (PWE) method, in which the wave equations are solved in the Fourier space.[15] Nevertheless, PCs involving media with a large contrast in their elastic properties are not easy to treat with PWE because a large number of plane waves is required to obtain reliable band structures, and unphysical flat frequency bands can appear. Other methods such as the variational method (VM),[9] multiple scattering theory (MST)[16–19] or the finite difference time domain (FDTD) algorithms[20–22] overcome those difficulties in a certain extent.

In this work, we present a lumped-mass (LM) method that offers a new procedure to compute the band structure of PCs. As a numerical example of its strength, we have analyzed two kinds of two-dimensional (2D) structures, which consist of same array of Pb columns while one in epoxy and the other in very soft rubber backgrounds. It will be shown that the LM can improve the PWE technique because of its faster convergence and lower computational times, especially for the Pb soft-rubber combination. In regard to the algorithms based on MST and FDTD, these techniques are more suitable to treat the transmission/reflectance problem in finite structures and their comparison with the lumped-mass method is out of the scope of the present work. However, some works[22,23] in dealing with those techniques indicate that the calculations of the phononic band structure present several drawbacks from the computational point of view. Thus, the technique based on MST can only handle specific arrays of sphere (3D) or cylinder (2D) and requires a large number of terms in the multipole expansion as well as complicated deductions. As to the technique based on FDTD, which needs discretization techniques both in the time and space domain, it requires a very strong reduction of the discrete time parameter in order to ensure the stability in the numerical calculation in dealing with structures having interfaces of large contrast of elastic constants. Thus the calculation time will be prolonged heavily in dealing with these hard cases. While the LM method in the present paper, which requires only the discretization of the space domain, is insensitive to the sharp variation of elastic constants on the interfaces inside the phononic crystals.

The paper is organized as follows. In Sec. II we describe the main ingredients of the LM, which is applied to an array of Pb columns in epoxy background in Sec. III, and to the same array in a soft rubber background in Sec. IV. Comparisons with the results obtained with the PWE are also performed in the two sections. Finally in Sec. V we summarize the work.

II. THE LUMPED-MASS METHOD

Figure 1 shows the cross section of representative 2D phononic crystals, where the z direction is vertical to the paper. The fuscous parts are the cylinders inserted in the

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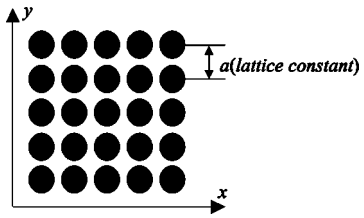

FIG. 1. The cross section of representative two-dimensional phononic crystals.

hosting material, parallel with the z axis. The distance a is the lattice constant.

By assuming that elastic wave propagation is along the direction that is parallel to the x-y plane, we have the equations in a homogeneous medium:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \tag{1}$$

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}. \tag{2}$$

Given $\boldsymbol{\sigma}_{xy}=\{\sigma_x,\sigma_y,\tau_{xy}\}^T$; $\boldsymbol{\varepsilon}_{xy}=\{\varepsilon_x,\varepsilon_y,\gamma_{xy}\}^T$ $\boldsymbol{\sigma}_z=\{\tau_{xz},\tau_{yz}\}^T$; $\boldsymbol{\varepsilon}_z=\{\gamma_{xz},\gamma_{yz}\}^T$, Eqs. (1) and (2) form

$$\boldsymbol{\sigma}_{xy}=\mathbf{D}_{xy}\boldsymbol{\varepsilon}_{xy}: \quad \boldsymbol{\sigma}_z=\mathbf{D}_z\boldsymbol{\varepsilon}_z, \tag{3}$$

where σ_x, σ_y, τ_{xy}, τ_{xz}, τ_{yz} are normal and shearing stresses and ε_x, ε_y, γ_{xy}, γ_{xz}, γ_{yz} are the corresponding strains; λ and μ are Lamé constants of the medium.

The idea of the lumped-mass method is to simplify the continuous system to the discrete one. Thus, the density of the medium is concentrated into particles at discrete points and the elastic constant is treated as simple linear elasticity between the adjacent particles.

As shown in Fig. 2(a), the density in a unit of PCs is concentrated on $N\times N$ particles in a square array, where p and q are the periodic number. For the infinite periodic structure, p and q are arbitrary integers.

Figure 2(b) illustrates a rectangle cell composed by four adjacent particles in Fig. 2(a), where the displacements are given by u_i, ν_i, w_i ($i=1\cdots4$ for the of the particles $1\cdots4$, respectively).

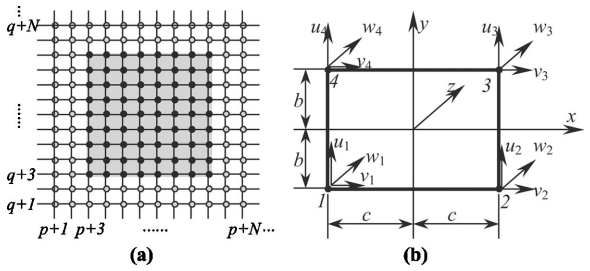

FIG. 2. (a) Lumped masses in a unit of the 2D PCs; (b) a cell composed by four adjacent particles.

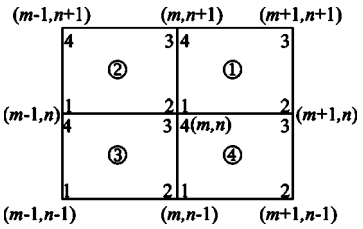

FIG. 3. The four cells adjacent to a certain node (m,n) in Fig. 2(a).

First, the propagation of vibrations along the z axis is considered. As the density is concentrated on the nodes, the displacements along the z axis in the cell can be written in a double linear form:

$$w=\alpha_1+\alpha_2x+\alpha_3y+\alpha_4xy=\sum_{i=1}^{4} N_iw_i, \tag{4}$$

where $N_i=(1/4bc)(c+\xi_ix)(b+\eta_iy)$, $\xi_i=x_i/c$; $\eta_i=y_i/b$; $i=1\cdots4$ for the nodes $1\cdots4$, respectively.

Thus, the strains in the cell are

$$\boldsymbol{\varepsilon}_z=\mathbf{B}\boldsymbol{\Delta}_z, \tag{5}$$

where $\mathbf{B}=(1/4bc)\{\xi_i(b+\eta_iy),\eta_i(c+\xi_ix)\}^T$, $\varepsilon=[\partial w/\partial x,\partial w/\partial y]^T$ are the strains, and $\boldsymbol{\Delta}_z=[w_1,w_2,w_3,w_4]^T$ are the displacements of particles $1\cdots4$ along the z axis.

Inserting Eq. (5) into Eq. (3) yields the expression of corresponding stresses,

$$\boldsymbol{\sigma}_z=\mathbf{D}_z\boldsymbol{\varepsilon}_z=\mathbf{D}_z\mathbf{B}\boldsymbol{\Delta}_z. \tag{6}$$

With the virtual work principle in analytical mechanics,[24] we obtained

$$\mathbf{F}_z=\mathbf{K}_z\boldsymbol{\Delta}_z, \tag{7}$$

where $\mathbf{F}_z=\{F_1,F_2,F_3,F_4\}^T$ are the forces on the nodes along the z axis, $\mathbf{K}_z=[k_{ij}]$ are the stiffness matrices that can be written as

$$\mathbf{K}_z=\int_c\int \mathbf{B}^T\mathbf{D}_z\mathbf{B}t\,dx\,dy=[k_{ij}], \tag{8}$$

where $i,j=1\cdots4$ for node $1\cdots4$, respectively, t is the thickness on the z direction, can be arbitrary, and

$$k_{ij}=\frac{\mu}{4}\left[\frac{b}{c}\xi_i\xi_j\left(1+\frac{\eta_i\eta_j}{3}\right)+\frac{b}{c}\eta_i\eta_j\left(1+\frac{\xi_i\xi_j}{3}\right)\right]. \tag{9}$$

As the inertial loads caused by the density of the medium are body force, it can be concentrated on the nodes averagely as particle $m_{m,n}$ in the lumped-mass method.

Figure 3 illustrates the four cells adjacent to a certain node (m,n). When the stiffness matrices of the four cells are calculated and given by $[k^1_{i,j}]$, $[k^2_{i,j}]$, $[k^3_{i,j}]$, and $[k^4_{i,j}]$, $i,j=1\cdots4$, the dynamics equation on node (m,n) can be written as

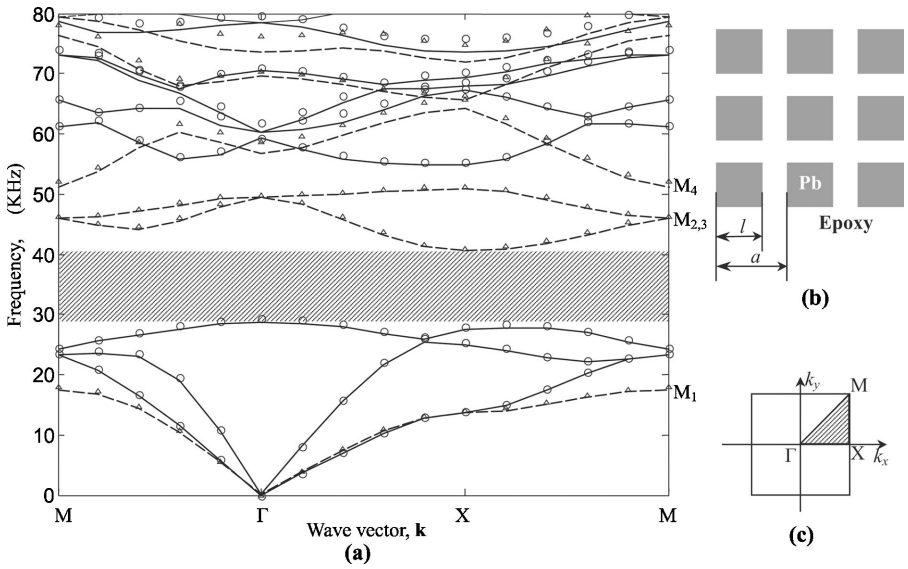

FIG. 4. (a) Phononic band structure along the edge of the irreducible Brillouin zone for a square lattice of Pb columns with a square cross section embedded in epoxy. The filling fraction is 0.36. The continuous (dashed) lines represent the in-plane (out-of-plane) modes computed with the lumped-mass method. The symbols represent the results obtained by using the plane-wave expansion method. The shadowed region defines the complete gap. (b) The cross section of the phononic crystals. (c) The corresponding irreducible Brillouin zone (shown as shadowed regions).

$$m_{m,n}\frac{\partial^2 w}{\partial t^2} = f_{m,n} = k_{31}^1 w_{m-1,n-1} + (k_{32}^3 + k_{41}^4) w_{m,n-1} + k_{42}^4 w_{m+1,n-1} + (k_{12}^1 + k_{43}^4) w_{m+1,n} + k_{13}^1 w_{m+1,n+1} + (k_{14}^1 + k_{23}^2) w_{m,n+1} + k_{24}^2 w_{m-1,n+1} + (k_{21}^2 + k_{34}^3) w_{m-1,n} + (k_{11}^1 + k_{22}^2 + k_{33}^3 + k_{44}^4) w_{m,n}, \quad (10)$$

where $m_{m,n}$ is the mass of particle on node (m,n) in Fig. 3.

We can get a series of similar equations on each node in one unit of the phononic crystals. With the Bloch theorem,[25] the traveling wave solution is assumed as

$$w_{p+m,q+n} = C_{m,n} e^{i[(p+m)(a/N)k_x + (q+n)a/N\, k_y - \omega t]}, \quad (11)$$

where k_x, k_y are the wave numbers on the x and y axis, ω is the wave frequency, and $C_{m,n}$ is the wave amplitude of particle $m_{m,n}$.

Inserting Eq. (11) into Eq. (10) yields $N \times N$ complex equations,

$$[-(k_{11}^1 + k_{22}^2 + k_{33}^3 + k_{44}^4)/m_{m,n} - \omega^2] C_{m,n} = [k_{31}^1 C_{m-1,n-1} e^{-i(\gamma_x + \gamma_y)} + (k_{32}^3 + k_{41}^4) C_{m,n-1} e^{-i\gamma_y} + k_{42}^4 C_{m+1,n-1} e^{i(\gamma_x - \gamma_y)} + (k_{12}^1 + k_{43}^4) C_{m+1,n} e^{i\gamma_x} + k_{13}^1 C_{m+1,n+1} e^{i(\gamma_x + \gamma_y)} + (k_{14}^1 + k_{23}^2) C_{m,n+1} e^{i\gamma_y} + k_{24}^2 C_{m-1,n+1} e^{i(\gamma_y - \gamma_x)} + (k_{21}^2 + k_{34}^3) \times C_{m-1,n} e^{-i\gamma_x}]/m_{m,n}, \quad (12)$$

where $\gamma_x = k_x a/N$, $\gamma_y = k_y a/N$, and $m,n = 1,...,N$.

An infinite number of lattices are considered and the following periodic boundary conditions can thus be applied:

$$C_{m,0} = C_{m,N}, \quad (13)$$

$$C_{m,N+1} = C_{m,1}, \quad (14)$$

$$C_{0,n} = C_{N,n}, \quad (15)$$

$$C_{N+1,n} = C_{1,n}. \quad (16)$$

Equation (12) forms with Eqs. (13)–(16) a standard complex eigenvalue problem,

$$[\mathbf{S}_z(\mathbf{k}) - \omega^2 \mathbf{I}]\mathbf{C} = 0. \quad (17)$$

Similarly, the traveling wave solution of x-y vibration modes forms another standard complex eigenvalue problem,

$$[\mathbf{S}_{xy}(\mathbf{k}) - \omega^2 \mathbf{I}]\mathbf{A} = 0, \quad (18)$$

where $\mathbf{k} = \{k_x, k_y\}$ is the wave vector.

Equations (17)–(18) can be solved to construct the band structure of wave frequencies ω for known wave vector $\mathbf{k}$.

It is not necessary to solve Eqs. (17)–(18) for all values of $\mathbf{k}$. Due to the periodicity all propagating modes are captured by restricting the wave vector to the irreducible Brillouin zone, as shown later in Fig. 4(c).[25]

III. RESULT FOR Pb COLUMNS IN EPOXY

Here, as a test example of the LM, we consider a square lattice of Pb columns with a cross section of a square in epoxy. Figure 4(a) shows the dispersion relations of the elastic modes corresponding to a crystal having a filling fraction $f_h = (l/a)^2 = 0.36$, where l is the length of square side of the inclusion and a is the lattice constant. The elastic parameters employed in the calculations were $\rho_{\mathrm{Pb}} = 11\,600\ \mathrm{kg\,m^{-3}}$, $\rho_{\mathrm{epo}} = 1180\ \mathrm{kg\,m^{-3}}$, $\lambda_{\mathrm{Pb}} = 4.23 \times 10^{10}$, $\lambda_{\mathrm{epo}} = 4.43 \times 10^{9}$, $\mu_{\mathrm{Pb}} = 1.49 \times 10^{10}$, $\mu_{\mathrm{epo}} = 1.59 \times 10^{9}$. The modes propagating in the plane (full lines) as well as those propagating along z direction (dashed lines) were computed with the lumped-mass method by employing 400 nodes. The results obtained with the PWE method by using a set of 441 PW are represented by circular (in-plane modes) and triangular (out-of-plane modes) symbols. A good agreement is found between the two methods. A gap separates the dispersion curves of in-plane modes as well of the pure transverse ones. A com-

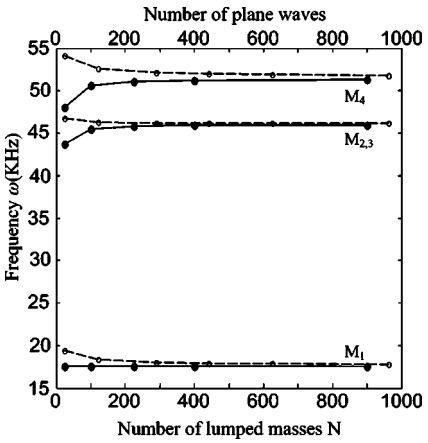

FIG. 5. Convergence of the lower four out-of-plane frequencies for both lumped-mass (solid lines) and plane-wave expansion (dashed lines) methods. They correspond to the points M_1, $M_{2,3}$, and M_4 in Fig. 4. The top axis representing the number of plane-waves employed in the plane-wave expansion calculation is aligned with the number of lumped masses in the bottom axis that produces a matrix having the same dimension as in the lumped-mass method.

plete gap, resulting from the superposition of the two band structures, settles between the fourth and the fifth bands of about 29–41 kHz.

The convergence of the LM and the PWE method are compared in Fig. 5. The behavior of the lower four frequencies of the out-of-plane modes at the M point [points M_1, $M_{2,3}$, and M_4 in Fig. 4(a)] is shown as a function of the number of nodes (from 25 to 900) employed in a unit with the LM method. For the PWE calculations we make a similar study and have changed the number of plane waves, from 25 to 961. The two abscissa scales are the same in such a way that the sizes of matrices are the same in both methods. From Fig. 5, it is noticeable that the three higher frequencies, M_4 and $M_{2,3}$, converge similarly in both methods. However, the LM improves the convergence of lower modes M_1. This improvement can be understood with the well-known Gibbs oscillations at the interfaces, which are brought on by the Fourier expansions in the PWE calculations when a finite number of Fourier components were employed. In an opposite way, the LM method that works in the direct space describes the interfaces directly and correctly, which justifies its faster convergence.

In addition, the LM method converges to the true value from the direction opposite to the PWE, this provides a credible way to estimate the true value, which must be located between the two frequencies calculated by the PWE and LM methods.

IV. RESULT FOR Pb COLUMNS IN RUBBER

Very soft rubber was recently used[8] as the coating of Pb spherical inclusions arranged in a simple cubic lattice in an epoxy host. The very low transverse velocity of the coating layer resulted in a strong resonant band structure with a gap at a frequency of two orders of magnitude lower than the expected one by Bragg scattering. This is significant for the applications of low-frequency sound and the vibration shelter of small dimension.

Here, as another test example of the LM method, we calculated the band structures of same square lattice of Pb columns with a square cross section in a different background of soft rubber. The elastic parameters of this new hosting medium employed in the calculations were $\rho_{\mathrm{rub}} = 1300\ \mathrm{kg\,m^{-3}}$, $\lambda_{\mathrm{rub}} = 6\times10^5$, $\mu_{\mathrm{rub}} = 4\times10^4$, being the same with Refs. 8, 9. Figure 6 presents the band structures corresponding to the phononic crystals. The modes propagating in the plane x-y (full lines) as well as those propagating out of the plane (dashed lines) were computed with the LM method by employing 400 nodes. The results obtained with the PWE method by using a set of 1225 plane waves are represented by a circle (in-plane modes) and triangular (out-of-plane modes) symbols. The elementary agreement between the two methods exists only at the low-frequency bands while bad agreement is found at high-frequency ones.

The convergence of the LM has also been analyzed and

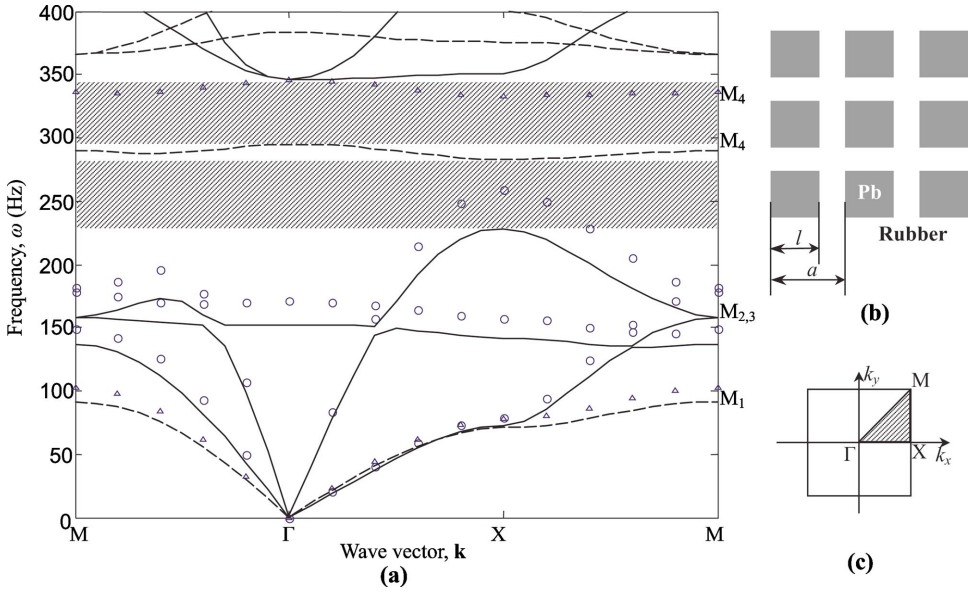

FIG. 6. (a) Phononic band structure along the edge of the irreducible Brillouin zone for the same square lattice of Pb columns with square cross section as in Fig. 4(b), but embedded in soft rubber. The filling fraction is 0.36. The continuous (dashed) lines represent the in-plane (out-of-plane) modes computed with the lumped-mass method. The symbols represent the results obtained by using the plane-wave expansion method. The shadowed region defines the complete gap. (b) The cross section of the phononic crystals. (c) The corresponding irreducible Brillouin zone (shown as shadowed regions).

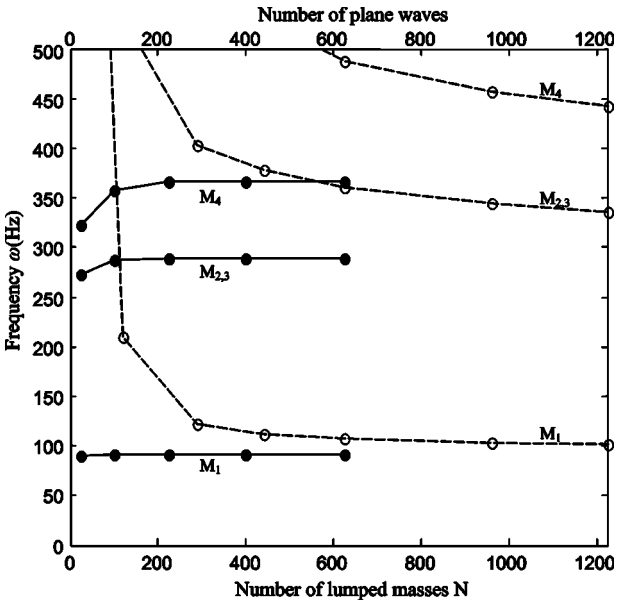

FIG. 7. Study of the convergence of out-of-plane modes for both the plane-wave (dashed lines) and lumped-mass (solid lines) methods. Four modes were analyzed at the M point of the Brillouin zone, denoted by M_1, $M_{2,3}$, and M_4.

compared with the PWE method. The behavior of frequencies at M_1, $M_{2,3}$, and M_4 points is presented (the solid lines in Fig. 7) as a function of the number of nodes employed in the LM method. For the PWE calculations, we made the similar study and have changed the number of plane waves (dashed lines in Fig. 7). It is shown that the LM method has almost the same convergence in comparison with the result obtained in Sec. III, while the PWE method have a very low one, especially at higher frequencies. Four hundred nodes are needed with the LM method in order to guarantee convergence better than 0.3%, while for the PWE method, usable convergence cannot be obtained, even 1225 plane waves are employed. This distinct contrast is due to the high contrast between the elastic parameters of Pb and rubber, which requires us to introduce a large number of Fourier components for the PWE method to approach it, while it can be described directly and correctly with the LM method.

V. CONCLUSION

Based on the discretization of continuous system, we have presented a lumped-mass method that works in the direct space and allows computing the band structures of two-dimensional phononic crystals. Two examples were studied to show its correctness and advantages in comparison with the plane-wave expansion method. We concluded that the lumped-mass method converges faster and its convergence is insensitive to the sharp variation of elastic constants on the interfaces inside the phononic crystals. Especially, the latter advantage is unique in comparison with other work[9] on the improvement of plane-wave expansion methods. Another unique feature of the new method is that it need not deduce the structure factors for every inerratic shape in the lumped-mass method. Thus, it can be used to calculate the band structures of two-dimensional phononic crystals with any unit shapes directly.

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