Comment on "Dynamic correlations of the spinless Coulomb Luttinger liquid"

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We show that claims [Y. Gindikin and V. A. Sablikov, Phys. Rev. B **65**, 125109 (2002)], concerning the threshold behavior of the spectral function of a Coulomb Luttinger liquid, are based on an inconsistent mathematical analysis. Physical arguments are also presented, indicating that the claimed behavior is unlikely to be correct.

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In a recent paper[1] Gindikin and Sablikov (GS) introduced a new method for analyzing the equations arising in the theory of the Coulomb Luttinger liquid (CLL) and showed that the method predicts apparently previously unnoticed singularities in the charge-density wave (CDW) structure factor S for wave vector q near to $2k_F$ and for low energy ω:

$$S(q,\omega)\sim\frac{e^{-4\beta|\ln\epsilon|^{1/2}}}{\omega\epsilon|\ln\epsilon|^{1/2}}, \tag{1}$$

where $\epsilon\equiv\omega-\omega_{q-2k_F}\to 0^+$. They further claimed (*without showing details*) that when their method is used to calculate the electron spectral function $\rho(q,\omega)$, it led to results in disagreement with our previously published calculations.[2] specifically, they assert that for small q and low energy, $\delta/\omega_q=(\omega-\omega_q)/\omega_q\to 0^+$,

$$\rho(q,\omega)\sim\frac{e^{-A(q)\beta|\ln\delta|^{1/2}}}{\omega\delta|\ln\delta|^{1/2}}, \tag{2}$$

so that $\rho(q,\omega)$ diverges as $\delta\to 0$ at fixed q, whereas we found $\rho(q,\omega)$ rapidly vanishes near such threshold:

$$\rho(q,\omega)\sim\delta^{C|\ln\delta|^{1/2}}, \tag{3}$$

where C is a constant. In Eqs. (1) and (2), wave vector q and frequency ω are measured from Fermi wave vector k_F and Fermi energy E_F, respectively (we set $\hbar\equiv 1$ throughout this Comment); $\omega_q=qv_q\sim v_F\beta^{-1}q\sqrt{-\ln(qd)}$ is the plasmon (boson) mode energy in the long-wavelength limit with d being the characteristic length scale of the system. In Eq. (2), $A(q)\equiv[v_q/v_F-1]^2$ and $\beta\equiv[\pi\hbar v_F/2e^2]^{1/2}$, where v_F is Fermi velocity and v_q is the renormalized plasmon (boson) velocity.[1]

In this Comment we show that although the structure factor $S(q,\omega)$ obtained by GS [Eq. (1)] is in agreement with the results obtained by other more elementary methods,[3] the spectral function quoted by GS [Eq. (2)] is in error: it does not satisfy the equation they derive, whereas our result, Eq. (3), does satisfy their equation. We also adduce physical arguments showing that independent of the mathematical details their proposed form cannot be correct. We further point out that in their paper[1] they misunderstood the effective exponent theory we developed in our earlier work[2] and hence made a wrong statement to the CDW structure factor obtained by the effective exponent theory.

Because GS did not present a detailed derivation of their result for the spectral function $\rho(q,\omega)$ we begin by outlining the derivation here following their method. The analysis proceeds from the standard expression for the electron Green's function in coordinate space[2,3] [following GS (Ref. 1) we restrict to spinless systems and consider only right-moving particles]:

$$G(x,t)=\lim_{\alpha\to 0^+}\frac{e^{ik_Fx}}{2\pi\alpha}\exp\left[-\int_0^{+\infty}\frac{dp}{p}e^{-\alpha p}[\cosh(2\theta_p)-\cosh^2\theta_p\,e^{i(px-pv_pt)}-\sinh^2\theta_p\,e^{-i(px+pv_pt)}]\right], \tag{4}$$

with the exponent parameter θ_p given by

$$e^{-2\theta_p}\equiv g(p)^{-1}=v_p/v_F\sim\beta^{-1}\sqrt{-\ln(pd)}, \tag{5}$$

where the last approximation applies in the limit of small momentum p. The spectral function $\rho(q,\omega)$ is defined by

$$\rho(q,\omega)=[\tilde{\rho}(q,\omega)+\tilde{\rho}(-q,-\omega)]/2\pi, \tag{6}$$

where

$$\tilde{\rho}(q,\omega)=\int_{-\infty}^{\infty}dx\,e^{-iqx}\int_{-\infty}^{\infty}dt\,e^{i\omega t}G(x,t). \tag{7}$$

The analyticity of $G(x,t)$ in the lower-half complex plane of t implies that $\tilde{\rho}(q,\omega)$ is nonzero only when $\omega\geqslant|qv_q|$ and $\tilde{\rho}(-q,-\omega)$ is nonzero only for $\omega\leqslant-|qv_q|$. We specialize now to positive q and ω and consider only $\tilde{\rho}(q,\omega)$ on the right-hand side of Eq. (6).

The key insight of GS is that an integral equation for $\tilde{\rho}(q,\omega)$ can be obtained by integrating the right-hand side of Eq. (7) by parts in the time variable and using Eq. (4), yielding

$$\omega\tilde{\rho}(q,\omega)=\int_0^{+\infty}dQ[v_Q\cosh^2\theta_Q\tilde{\rho}(q-Q,\omega-Qv_Q)+v_Q\sinh^2\theta_Q\tilde{\rho}(q+Q,\omega-Qv_Q)]. \tag{8}$$

The constraint that $\tilde{\rho}(q,\omega)$ is nonzero only when $\omega \geqslant |qv_q|$ means that for very small $\delta=\omega-\omega_q$, $\tilde{\rho}(q-Q,\omega-Qv_Q)$ is nonvanishing only for Q very near zero or very near q, while $\tilde{\rho}(q+Q,\omega-Qv_Q)$ is nonvanishing only for Q very near zero. After changing variables to highlight these regimes and neglecting Q when compared to q, we find

$$\omega\tilde{\rho}(q,\omega)=2v_q\cosh^2\theta_q\int_0^{Q^*}dQ\tilde{\rho}(Q,\omega-qv_q)+\int_0^{Q^*}dQ\,v_Q\cosh(2\theta_Q)\tilde{\rho}(q,\omega-Qv_Q), \quad (9)$$

with Q^* given by

$$Q^*v_{Q^*}=\omega-qv_q\equiv\delta. \quad (10)$$

As noted by GS a further change of variables is convenient to highlight the possibility of a divergence as $\omega\rightarrow qv_q$. Defining

$$f_q(\omega-qv_q)=\omega\tilde{\rho}(q,\omega), \quad (11)$$

we have

$$f_q(\delta)=2v_q\cosh^2\theta_q\int_0^{Q^*}\frac{dQ}{\delta}f_Q(\delta-Qv_Q)+\int_0^{Q^*}dQ\,\frac{v_Q\cosh(2\theta_Q)}{\delta+qv_q-Qv_Q}f_q(\delta-Qv_Q). \quad (12)$$

GS assert that Eq. (2), which implies

$$f_q(\delta)\propto\frac{\exp[-\beta A(q)\sqrt{-\ln(\delta/E_0)}]}{\delta\sqrt{-\ln(\delta/E_0)}} \quad (13)$$

solves Eq. (12) in the limit of $\delta/qv_q\rightarrow 0$, where $E_0=v_F\beta/d$ is an energy scale. However, simple substitution shows that it does not. In the first term on the right-hand side of Eq. (12), the restriction $Q<Q^*\sim(\delta/E_0d)\ln^{1/2}(E_0/\delta)$ means that this term is $\sim e^{-\beta A(Q^*)\sqrt{-\ln(\delta/E_0)}}$, so much less than $f_q(\delta)$ on the left-hand side [recall $A(Q^*)$ diverges as $Q^*\rightarrow 0$ near the threshold, $\delta\rightarrow 0$, while q is fixed]. From the second term we obtain factors of qv_q in the denominator, inconsistent with the ansatz of Eq. (13). On the other hand, substitution also shows that our solution, Eq. (3), does satisfy Eq. (8) asymptotically as $\delta/qv_q\rightarrow 0$. Therefore one of the main conclusion in GS's paper that the spectral function of a Coulomb Luttinger liquid has a divergent threshold in the low-energy regime [see Eq. (2)] is in correct.

Physically, we can argue that the low-energy behavior of the spectral function must have a pseudogap structure rather than a divergent singularity. In any realistic system, the long-ranged Coulomb interaction is always screened by the external charge with a finite screening length q_0^{-1}. The low-energy behavior of one-dimensional (1D) electron system under such short-ranged interaction can be also well described by the standard Luttinger liquid theory[4] giving a well-known power-law singularity near the threshold of the spectral function,5 $\rho(q,\omega)\propto(\omega-\omega_q)^{\alpha-1}$. The exponent α can be larger than one if only the interaction strength is strong enough, leading to a cusp threshold similar to a pseudogap structure.[5] If we reduce the screening length of the screened Coulomb interaction, the effective interaction strength for the low-energy behavior should become even stronger, naturally leading to a pseudogap structure (which decays faster than any power law in low-energy regime) as we found in the earlier work.[2] The divergent singularity GS obtained in the low-energy regime indicates, however, a behavior similar to a LL of weak interaction, and therefore cannot be a correct result for a long-range Coulomb interaction in any physical sense. The numerical calculation of a true long-ranged Coulomb interaction[2,6] also confirms the existence of a pseudogap structure in the spectral function, consistent with the results obtained by the renormalization group analysis in Ref. 7.

We now further point out that another comment in the GS paper[1] about the CDW structure factor is also incorrect. They claim that using the effective exponent theory we developed in Ref. 2, one will obtain a cusp threshold in the low-energy regime of $S(q\sim 2k_f,\omega)$. We can use a simple qualitative analysis to show that this statement is incorrect. According to the accepted Luttinger liquid theory for short-ranged interaction,[3] the CDW dynamical structure factor diverges at $\omega=\omega_{q-2k_F}$ with a power of $\alpha_{CDW}=-1+g(0)$. Assuming that the long-ranged Coulomb interaction is screened and becomes short-ranged with a finite screening length q_0^{-1}, the exponent α_{CDW} then becomes $\sim -1+\beta\ln^{-1/2}(1/q_0d)$. Following the spirit of scaling-dependent effective exponent theory,[2] we can simply replace q_0 by the scaling cutoff, Q^* in Eq. (10) (but now using $\epsilon=\omega-\omega_{q-2k_F}\rightarrow 0^+$ instead of δ), and relate such an *effective* exponent to a *logarithmic* derivative of the structure factor near the threshold, i.e.,

$$\alpha_{CDW}^{\rm eff}(\epsilon)\equiv\left.\frac{\partial\ln S(\epsilon)}{\partial\ln\epsilon}\right|_{\epsilon\rightarrow 0^+}=-1+\frac{\beta}{\ln^{1/2}(E_0/\epsilon)}, \quad (14)$$

which leads to the low-energy threshold behavior of the CDW structure factor

$$S(q,\omega)\sim\epsilon^{-1}\exp[-2\beta\ln^{1/2}(E_0/\epsilon)]. \quad (15)$$

Except for unimportant prefactors, the above result is consistent with Eq. (1) obtained by GS and shows a divergent singularity in the vicinity of $q=2k_F$. Therefore GS's criticism[1] that our effective exponent[2] will lead to a cusp threshold structure in the CDW structure factor is also incorrect.

In summary, the methods of GS lead to the threshold behavior of $2k_F$ CDW structure factor of 1D Luttinger liquid equivalent to those obtained from our effective exponent procedure, but the latter applies over a wider energy range whereas the GS method only applies very close to threshold.

The spectral function GS obtained is shown to be based on an inconsistent mathematical assumption and in disagreement with physical arguments and widely accepted results.

Note added. After this comment was submitted for publication, the authors of Ref. 1 submitted an Erratum,[8] where they admitted the mistakes they made in calculating the spectral function [Eq. (2)] in Ref. 1. Their revised result shows the same pseudogap structure in the low-energy threshold, which we obtained by the effective exponent theory in our earlier work.[2]

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