Statistics of heat transfer in mesoscopic circuits

M. Kindermann[1,*] and S. Pilgram[2]

[1]*Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands*

[2]*Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland*

(Received 17 December 2003; revised manuscript received 12 February 2004; published 30 April 2004)

A method to calculate the statistics of energy exchange between quantum systems is presented. The generating function of this statistics is expressed through a Keldysh path integral. The method is first applied to the problem of heat dissipation from a biased mesoscopic conductor into the adjacent reservoirs. We then consider energy dissipation in an electrical circuit around a mesoscopic conductor. We derive the conditions under which measurements of the fluctuations of heat dissipation can be used to investigate higher-order cumulants of the charge counting statistics of a mesoscopic conductor.

DOI: 10.1103/PhysRevB.69.155334 PACS number(s): 72.10.−d, 72.70.+m, 73.23.−b, 05.40.−a

INTRODUCTION

During the second decade of mesoscopic physics there has been an increasing interest in thermal phenomena. Several experiments succeeded in measuring properties of heat transport in mesoscopic samples: Experimental tools based on the Coulomb blockade to measure accurately local temperatures in mesoscopic samples have been established.[1,2] It has been demonstrated that the heat conductance through one-dimensional phonon modes of a microbridge is quantized at low temperatures.[3] The universal heat conductance per mode is given by $\pi^2k^2T/3h$. The same universal heat quantum is also found for carriers other than bosons (see Ref. 4 and references therein). Thermopower was experimentally investigated for a quantum dot in the Coulomb blockade regime[5,6] and for multiwalled carbon nanotubes.[7] The Andreev reflection process is ineffective for heat transport and has been employed to measure thermopower in Andreev interferometers. The thermopower depends on the magnetic flux and shows geometry dependent time inversion properties.[8]

All these experiments have in common that they investigate mean properties averaged over time. In electrical transport, however, noise measurements have become a very useful tool to study properties of non-equilibrium systems which are hidden in measurements of the mean current (for a review see Ref. 9). A recent milestone was the first successful experimental investigation of higher-order current correlators by Reulet *et al.*[10] It revealed an unexpected temperature dependence that was explained by accounting for an external measurement circuit.[11] The calculation of average currents and current fluctuations can be unified within the concept of full counting statistics that has been introduced into mesoscopic physics a decade ago. It was shown by Levitov and Lesovik that coherent charge transfer through a two-terminal conductor can be seen as probabilistic process defined through a set of transmission probabilities.[12] Thereafter different works investigated the statistics of other measurable quantities in mesoscopic structures such as voltage,[13] momentum,[14] and charge inside a mesoscopic volume.[15]

In this publication we address the statistics of fluctuations of energy exchange. We investigate the statistics of energy flow into a subvolume of a quantum system. In the first part of the paper, we write the generating function of this statistics in a Keldysh-representation.[16] The "counting field" that generates moments of transferred energy enters as a gauge field that shifts the time evolution of the studied subvolume relative to that of the rest of the system.

In Sec. II we apply our approach to a simple example which nicely demonstrates the basic features of energy transfer: We consider the energy dissipation from a biased mesoscopic conductor into one of its reservoirs. The problem has been addressed before for a conductor at zero temperature.[17] In Ref. 18 it was shown that dissipation happens symmetrically, that is, each reservoir dissipates half of the energy. We find that this is true only on average and we determine fluctuations around this rule. In contrast to the statistics of charge transfer through such a conductor, the transfer of heat is not quantized. In Sec. III we derive general relations for the statistics of energy exchange between a quantum system and a linear environment. Moments of the exchanged energy are expressed in terms of correlators of the variable that couples the quantum system to its environment. We apply these results to a mesoscopic conductor which is embedded in a macroscopic measurement circuit in Sec. IV. The energy-transfer statistics is related to correlators of the current through the mesoscopic conductor. One expects that the second moment of energy dissipated in a series resistor depends on the fourth moment of current fluctuations in the mesoscopic conductor. This suggests to use a measurement of heat fluctuations as a probe to study higher-order current correlations in mesoscopic conductors. We quantify this expectation and derive the conditions under which higher-order current correlators can be extracted from an energy measurement.

I. APPROACH

We study the statistics of the exchange of energy between two subvolumes V and $\overline{V}$ of a quantum system. We assume that no degrees of freedom leave or enter V and that V is coupled to $\overline{V}$ only through one degree of freedom j_V. j_V shall be coupled to the variable $X_{\overline{V}}$ in $\overline{V}$, see Fig. 1. A generalization of our approach to more than one coupling variable is straightforward. Our model Hamiltonian reads

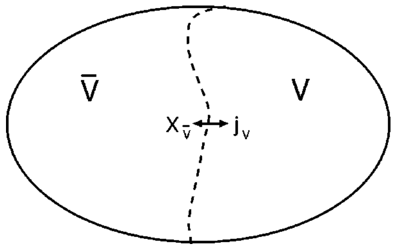

FIG. 1. Quantum system divided into two subvolumes V and $\bar{V}$ that are coupled via the variables j_V and $X_{\bar{V}}$.

$$H = H_V + H_{\bar{V}} + j_V X_{\bar{V}} \tag{1}$$

with $[j_V, X_{\bar{V}}] = [j_V, H_{\bar{V}}] = [H_V, X_{\bar{V}}] = [H_V, H_{\bar{V}}] = 0$.

To calculate the statistics of the flow of energy into V in a period of time $[0,\tau]$ we assume it to be in an eigenstate $|E_V\rangle$ of H_V at time $t=0$ and calculate moments of H_V at time $t=\tau$, defining the generating function

$$\bar{\mathcal{Z}}(\xi) = \langle e^{i\xi H_V/2} e^{iH\tau} e^{-i\xi H_V} e^{-iH\tau} e^{i\xi H_V/2} \rangle. \tag{2}$$

(We have set $\hbar = 1$.) The average is taken over the initial state $|E_V\rangle$. The exponential $\exp\{-i\xi H_V\}$ generates moments of the energy in V at time τ. Through the exponentials $\exp\{i\xi H_V/2\}|E_V\rangle = \exp\{i\xi E_V/2\}|E_V\rangle$ and $\langle E_V|\exp\{i\xi H_V/2\}$ the initial energy E_V in V is subtracted from that such that $\bar{\mathcal{Z}}$ generates moments of $\triangle H_V = e^{iH\tau} H_V e^{-iH\tau} - E_V$, the flux of energy into V during $[0,\tau]$,

$$\langle (\triangle H_V)^p \rangle = i^p \frac{\partial^p}{\partial \xi^p} \bar{\mathcal{Z}}(\xi)\big|_{\xi=0}. \tag{3}$$

A generalization of $\bar{\mathcal{Z}}$ that generates correlators of the energy flux at finite frequency is the functional

$$\mathcal{Z}[\vec{\xi}] = \langle \overleftarrow{T} e^{i\int dt [H - \dot{\xi}^-(t) H_V]} \overrightarrow{T} e^{-i\int dt [H - \dot{\xi}^+(t) H_V]} \rangle. \tag{4}$$

The symbols $\overrightarrow{T}(\overleftarrow{T})$ denote (inverse) time ordering and we have collected the two source functions into a vector in a "Keldysh space," $\vec{\xi} = (\xi^+, \xi^-)$. $\mathcal{Z}$ correlates the energy flux at different times,

$$\left\langle \prod_{q=1}^{p} \dot{H}_V(t_q) \right\rangle = \left(\frac{i}{2}\right)^p \prod_{q=1}^{p} \left[\frac{\delta}{\delta \xi^+(t_q)} - \frac{\delta}{\delta \xi^-(t_q)} \right] \mathcal{Z}\Big|_{\xi=0}. \tag{5}$$

$\mathcal{Z}$ arises naturally as one models a linear detector that measures the energy flux $\dot{H}_V$ into V.[16] The detector read-off r is then a linear functional of $\dot{H}_V$,

$$\langle r(\tau) \rangle = \left\langle \int dt\, s(\tau - t) \dot{H}_V(t) \right\rangle, \tag{6}$$

where the response function $s(t)$ is causal, $s(t)=0$ for $t<0$, and it depends on the internal dynamics of the detector. Moments of the detector read-off are then generated by $\mathcal{Z}$ according to

$$\langle r^p(\tau) \rangle = \left(\frac{i}{2}\right)^p \frac{\partial^p}{\partial \xi^p} \mathcal{Z}\left[\left(\begin{array}{c} \xi s(\tau - t) \\ -\xi s(\tau - t) \end{array} \right)\right]\Bigg|_{\xi=0}. \tag{7}$$

Evidently, Eq. (4) reduces to Eq. (2) for the measurement of the time integrated energy flow during $[0,\tau]$, if $s(t) = \theta_\tau(t) \equiv \theta(t) - \theta(t-\tau)$ [with the step function $\theta(t)=1$ for $t>0$ and $\theta(t)=0$ for $t \leq 0$].

In the interaction picture with respect to the uncoupled problem $H_0 = H_V + H_{\bar{V}}$, $j_V(t) = \exp\{iH_V t\} j_V \exp\{-iH_V t\}$ and $X_{\bar{V}}(t) = \exp\{iH_{\bar{V}} t\} X_{\bar{V}} \exp\{-iH_{\bar{V}} t\}$, $\mathcal{Z}$ takes the form

$$\mathcal{Z}[\vec{\xi}] = \langle \overleftarrow{T} e^{i\int dt [X_{\bar{V}}(t) j_V(t) - \dot{\xi}^-(t) H_V]} \overrightarrow{T} e^{-i\int dt [X_{\bar{V}}(t) j_V(t) - \dot{\xi}^+(t) H_V]} \rangle. \tag{8}$$

We rewrite Eq. (8) by breaking the time-ordered product up into a large number of time-development exponentials for infinitesimal time steps ϵ. Applying to each one of them the identity

$$e^{-iH_V \xi(t)} e^{-i\epsilon j_V(t) X_{\bar{V}}(t)} e^{iH_V \xi(t)} = e^{-i\epsilon j_V[t - \xi(t)] X_{\bar{V}}(t)} \tag{9}$$

we find that

$$\mathcal{Z}[\vec{\xi}] = \langle \overleftarrow{T} e^{i\int dt\, j_V[t - \xi^-(t)] X_{\bar{V}}(t)} \overrightarrow{T} e^{-i\int dt\, j_V[t - \xi^+(t)] X_{\bar{V}}(t)} \rangle. \tag{10}$$

The generating functional of the statistics of energy flow into a volume V takes the form of a partition functional of the entire system with shifted time arguments of all variables in V. An analogous structure has been found before for the statistics of transfer of other globally conserved quantities, such as charge[12] and momentum.[14] Also there the generating functional has the structure of Eq. (10). The source term locally shifts the variable conjugated to the measured quantity (the phase for a measurement of charge, position for a momentum measurement, and time for the energy measurement considered here).

In a path integral formulation of Eq. (4) we mark fields that evolve the system forward and backward in time with superscripts $+$ and $-$, respectively, and we collect them in vectors like the source $\vec{\xi}$. We may change integration variables as $j_V^{\pm}(t - \xi^{\pm}(t)) \rightarrow j_V^{\pm}(t)$ in the action corresponding to H_V if $|\dot{\xi}(t)| < 1$ at all times (only then this map is bijective). The path integral takes then a form very similar to Eq. (10),

$$\mathcal{Z}[\vec{\xi}] = \int \mathcal{D}\vec{j}_V \mathcal{D}\vec{X}_{\bar{V}} e^{-i(\mathcal{S}_V[\vec{j}_V] + \mathcal{S}_{\bar{V}}[\vec{X}_{\bar{V}}])} e^{-i\int dt\, \vec{j}_V(t - \vec{\xi}(t)) \tau_3 \vec{X}_{\bar{V}}(t)}. \tag{11}$$

We have introduced actions $\mathcal{S}_V$ and $\mathcal{S}_{\bar{V}}$ for the uncoupled volumes V and $\bar{V}$ that are obtained by integrating the actions corresponding to H_V and $H_{\bar{V}}$ over all variables except j_V and $X_{\bar{V}}$, respectively. τ_3 is the third Pauli matrix and we use the notation $[\vec{j}_V(t - \vec{\xi}(t))]^\alpha = j_V^\alpha(t - \xi^\alpha(t))$ $(\alpha = \{+,-\})$. The assumption $|\dot{\xi}(t)| \leq 1$ has the following physical interpretation. From Eq. (3) and its finite frequency generalization it is seen that one can infer a probability distribution of energy flow by Fourier transforming the generating function $\mathcal{Z}(\xi)$. To resolve energy differences of the order of ΔE in this distribution one needs to know $\mathcal{Z}$ for values $\xi_{\Delta E} \simeq 1/\Delta E$.

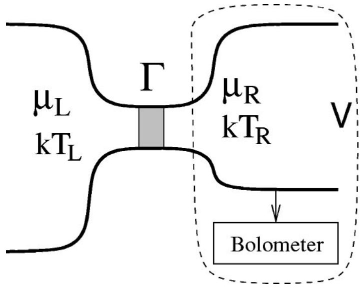

FIG. 2. Geometry considered in this section: A bolometer is attached to one reservoir of a biased mesoscopic conductor. We calculate the statistics of heat dissipated into the bolometer.

The condition $|\dot{\xi}(t)|<1$ means then that the energy detector smears out the measurement over a time $\Delta t > \xi_{\Delta E} \simeq 1/\Delta E$, that is, it does not attempt to measure the energy faster than it is allowed to by the uncertainty relation.

II. HEAT DISSIPATION STATISTICS IN TUNNEL JUNCTIONS

We first illustrate our method with a particularly simple example: we calculate the statistics of heat dissipated from a voltage biased mesoscopic conductor into a bolometer attached to its right contact (see Fig. 2). We assume ideal reservoirs, that is, all energy transferred from one reservoir to the other by scattering electrons is subsequently released as heat by relaxation mechanisms and can be measured by a bolometer. We address the zero frequency limit only, such that the time of measurement exceeds the energy relaxation time in the reservoir. The resulting statistics is then independent of this relaxation time.

We express the action of a simple connector in circuit theory by the Keldysh Green's functions $G_{L,R}$ of the adjacent reservoirs[19,20]

$$S_{\rm con} = \frac{i}{2}\sum_n \,\mathrm{Tr}\ln\left[1+\frac{1}{4}\Gamma_n(\{G_L,G_R^\xi\}-2)\right], \qquad (12)$$

where the Γ_n denote energy independent transmission probabilities and the brackets $\{,\}$ anticommutation. The trace includes integration over time. In the following, we address the zero frequency limit only. In this case, the time integration in Eq. (12) is conveniently rewritten as energy integration. As formulated in Eq. (10), we obtain the statistics of heat dissipation into the right contact by shifting all observables of the right contact in time by ξ_+ on the upper Keldysh contour and by ξ_- on the lower contour. The diagonal elements of G_R in the Keldysh space are left unchanged by this transformation which can be cast in the following form (we introduce the difference $\xi=\xi_+-\xi_-$):

$$G_R^\xi = e^{i\xi(\epsilon-\mu_R)\tau_3/2} G_R^0(\epsilon) e^{-i\xi(\epsilon-\mu_R)\tau_3/2},$$

$$G_j^0(\epsilon) = \begin{pmatrix} 1-2f_j & 2f_j \\ 2(1-f_j) & 2f_j-1 \end{pmatrix}. \qquad (13)$$

The time shift introduces a rotation of the Green's function G_R analogous to charge counting statistics.[19] $f_j=f_j(\epsilon)$ denotes the energy dependent Fermi function of contact $j=L,R$ that depends on temperature T_j and electrochemical potential μ_j. Evaluating Eq. (12) in the zero-frequency limit we find the cumulant generating function[24]

$$\ln\mathcal{Z}[\xi] = \frac{\tau}{2\pi}\int d\epsilon \sum_n \ln\{1+\Gamma_n f_L(1-f_R)(e^{-i\xi(\epsilon-\mu_R)}-1) + \Gamma_n f_R(1-f_L)(e^{i\xi(\epsilon-\mu_R)}-1)\}. \qquad (14)$$

Cumulants of dissipated energy can now be calculated easily by taking derivatives with respect to ξ. The mean dissipation without applied voltage ($\mu_j=0$) is for instance given by

$$\langle \Delta H_V\rangle = i\left.\frac{\partial \ln\mathcal{Z}}{\partial\xi}\right|_{\xi=0} = \frac{\pi\tau}{12}\sum_n \Gamma_n\{(kT_L)^2-(kT_R)^2\}. \qquad (15)$$

For almost equal temperatures $T_L\to T_R$ we recover the mesoscopic version of the Wiedemann-Franz law for thermal conductance.[21,22]

We note that Eq. (14) is similar to the statistics of charge transfer:[12] Charge statistics is recovered by substituting $\xi(\epsilon-\mu_R)\mapsto\chi e$ where e is the elementary charge and the field χ generates cumulants of charge transfer. As for charge statistics, there is a classical probabilistic interpretation of Eq. (14): An electron of total energy ϵ in channel n passes the mesoscopic conductor with probability Γ_n. It then dissipates the energy $\epsilon-\mu_R$ in the right contact. It is the energy integration in Eq. (14) which makes the statistics of heat transfer interesting. Unlike in the binomial charge statistics the exponent $i\xi(\epsilon-\mu_R)$ assigns energy weights to each electron. Therefore, energy transfer is not quantized.

We now turn to a specific example: the tunnel junction with $\Gamma_n\ll 1$. In this case, the logarithm of Eq. (14) can be expanded and the energy integration involves only elementary integrals (for equal temperatures $T_L=T_R=T$ on both sides of the junction). Defining $\mu=\mu_L-\mu_R$ we find the following result for the characteristic function:

$$\ln\mathcal{Z}[\xi] = G\tau\coth\frac{\mu}{2kT}\left(kT\frac{\sin(\xi\mu)}{\sinh(\pi\xi kT)/\pi}-\mu\right) - iG\tau\frac{1-\cos(\xi\mu)}{\sinh(\pi\xi kT)/\pi}. \qquad (16)$$

We introduced the conductance of the junction $G=\Sigma\Gamma_n/2\pi$. For comparison, we also give the statistics of charge transfer[23]

$$\ln\mathcal{Z}[\chi] = G\tau\mu\coth\frac{\mu}{2kT}[\cos(\chi e)-1] - iG\mu\sin(\chi e). \qquad (17)$$

We observe several differences: Whereas $\mathcal{Z}[\chi]$ is strictly periodic, the characteristic function of heat dissipation $\mathcal{Z}[\xi]$ shows damped oscillations. The lacking periodicity is due to the unquantized transfer of energy. For charge transfer, the real part of $\mathcal{Z}[\chi]$ is even in the applied voltage μ in contrast to an odd imaginary part. Odd cumulants therefore change sign under voltage inversion. This is not the case for energy,

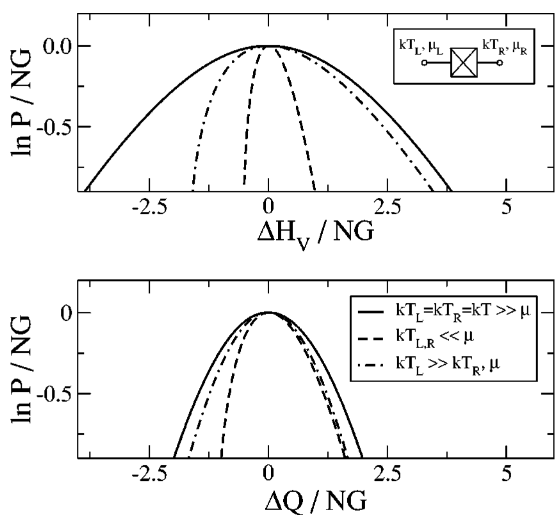

FIG. 3. Comparison of energy dissipation $\Delta H_V = H_V - \langle H_V \rangle$ (upper panel) and charge $\Delta Q = Q - \langle Q \rangle$ statistics (lower panel) for a tunnel contact in equilibrium and nonequilibrium due to an applied bias or temperature difference. The energy is normalized to temperature kT_L or bias $\mu = \mu_L - \mu_R$, respectively. The product of $N = \mu\tau/2\pi, kT_L\tau/2\pi$, and dimensionless conductance G is assumed to be large.

$\mathcal{Z}[\xi]$ does not depend on the sign of the applied voltage. Heat dissipation takes place in both reservoirs symmetrically regardless of the current direction! Odd cumulants of charge transfer [the imaginary part of Eq. (17)] do not depend on temperature.[23] In contrast, we find that the asymmetry of the distribution of dissipated heat does depend on temperature [see the imaginary part of Eq. (16)].

Another interesting case is the statistics in the presence of a temperature gradient only. Analytical results are available for $T_L \gg T_R$. We find

$$\begin{aligned}\ln \mathcal{Z}[\xi] &= 2G\tau kT_L \left\{ \frac{i\pi}{\sinh(\pi kT_L \xi)} + \ln(2) - \beta(-ikT_L\xi) \right\} \\ &= G\tau kT_L \left\{ -i\frac{\pi^2}{6} kT_L\xi - \frac{3\zeta(3)}{2}(kT_L\xi)^2 \right. \\ &\quad \left. + i\frac{7\pi^4}{360}(kT_L\xi)^3 + \cdots \right\} \end{aligned} \tag{18}$$

for energy transport and

$$\ln \mathcal{Z}[\chi] = 2G\tau kT_L \ln(2)[\cos(\chi e) - 1] \tag{19}$$

for charge transport [we introduced the β-function $\beta(iz) = \sum_{n=0}^{\infty}(-1)^n/(iz+n)$]. Figure 3 illustrates several limiting cases: On a log scale, it compares the charge- and energy-transfer statistics in equilibrium, and in nonequilibrium due to an applied voltage [see Eqs. (16) and (17)] and due to an applied temperature gradient [see Eqs. (18) and (19)].

In general, Eq. (14) has to be evaluated numerically. Figure 4 shows the probability distribution of dissipated heat for barriers with various transparencies $\Gamma_n = \Gamma$ in the high-voltage regime $\mu \gg kT$. Noise disappears in the ballistic limit

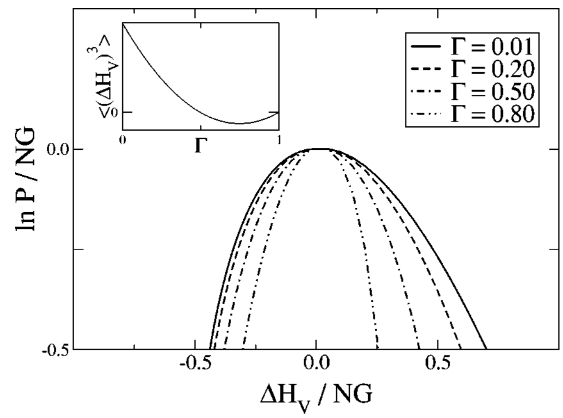

FIG. 4. Probability Distribution of dissipated energy into one reservoir of a biased mesoscopic conductor ($\mu \gg kT$). The distribution is broader for tunneling junctions ($\Gamma = 0.01$) than for open point contacts ($\Gamma = 0.80$). It is clearly seen that the third cumulant of the distribution changes sign. The inset shows the third cumulant $\langle(\Delta H_V)^3\rangle$ as a function of the transparency Γ. (G is the dimensionless conductance and $N = \mu\tau/2\pi$, we assume NG to be large).

$\Gamma = 1$. It is clearly visible that the third cumulant of the distribution changes its sign as a function of Γ. We find that

$$\langle(\Delta H_V)^3\rangle = G\mu^4(1-\Gamma)(1-2\Gamma)/4 \tag{20}$$

(see also the inset of Fig. 4).

In this entire section, we assumed that the energy dissipation in the right reservoir is measured by an ideal bolometer which does not act back on the mesoscopic contact. A realistic bolometer is characterized by a finite thermal conductance. Energy fluctuations in the right reservoir are then converted into temperature fluctuations which modulate the noise intensity of the mesoscopic contact. This backaction is similar to the backaction from a nonideal current meter.[11]

III. ENERGY FLOW INTO A LINEAR MEDIUM

We use now Eq. (11) to calculate the statistics of energy flow from a system S into a linear medium. An example is the energy that S emits as electromagnetic radiation. The electromagnetic field is then the linear medium into which energy flows. This energy flow may for example be measured with a photodetector. The distribution of the number of photons emitted by a source current j_S can be expressed in terms of correlators of j_S.[26] In Ref. 26 this relation has been established perturbatively in a weak coupling of the photodetector to the electromagnetic field. In Ref. 27 it has been used to calculate the fluctuations in the number of photons emitted by a mesoscopic conductor. Equation (11) allows us to obtain these results nonperturbatively. As a second application of our method we calculate the fluctuations in the amount of heat that is dissipated in a macroscopic electrical circuit around a mesoscopic conductor. Such fluctuations could be measured with a bolometer.

A. General relations

In our analysis we divide space into three regions. The system S, a volume V into which the energy flow is measured

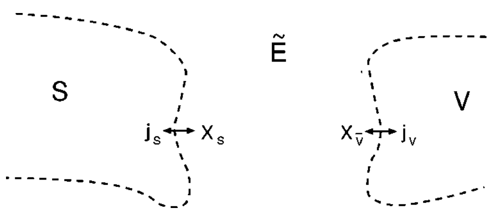

FIG. 5. S is an arbitrary quantum system. The energy flow into the part V of its linear environment is considered in this section. The variables j_S, X_S and j_V, $X_{\bar{V}}$ couple the three constituent volumes.

and a region $\tilde{E}$. We call $E=\tilde{E}+V$ the environment of S. The subspaces are connected via two variables X_S and $X_{\bar{V}}$ in $\tilde{E}$, that couple to j_S in S and j_V in V, respectively (see Fig. 5). With the Hamiltonians H_S, $H_{\tilde{E}}$, and H_V of S, $\tilde{E}$, and V we then have the total Hamiltonian

$$H=H_S+H_{\tilde{E}}+H_V+X_S(j_S-\overline{j_S})+X_{\bar{V}}(j_V-\overline{j_V}), \quad (21)$$

where $\overline{j_V}$ and $\overline{j_S}$ are sources that generate moments of $X_{\bar{V}}$ and X_S. The generating functional for correlators of the energy flow into V takes the form of Eq. (11) with

$$e^{-i\mathcal{S}_{\bar{V}}[\vec{X}_{\bar{V}}]}=\int \mathcal{D}\vec{X}_S\mathcal{D}\vec{j}_S e^{-i(\mathcal{S}_S[\vec{j}_S]+\mathcal{S}_{\tilde{E}}[\vec{X}_S,\vec{X}_{\bar{V}}])}e^{-i\int dt\,\vec{j}_S(t)\tau_3\vec{X}_S(t)} \quad (22)$$

with the actions $\mathcal{S}_S$ and $\mathcal{S}_{\tilde{E}}$ corresponding to H_S and $H_{\tilde{E}}$. $\mathcal{S}_{\tilde{E}}$, being the action of a linear system, is quadratic and depends only on the response functions of $\tilde{E}$ and its temperature T. We characterize the response of E when disconnected from S (corresponding to the Hamiltonian $H-H_S-X_Sj_S$) by four functions,

$$R_{SS}(\omega)=\left.\frac{\partial X_S(\omega)}{\partial\overline{j_S}(\omega)}\right|_{\overline{j_V}}, \quad R_{SV}(\omega)=\left.\frac{\partial X_S(\omega)}{\partial\overline{j_V}(\omega)}\right|_{\overline{j_S}},$$

$$R_{VS}(\omega)=\left.\frac{\partial X_{\bar{V}}(\omega)}{\partial\overline{j_S}(\omega)}\right|_{\overline{j_V}}, \quad R_{VV}(\omega)=\left.\frac{\partial X_{\bar{V}}(\omega)}{\partial\overline{j_V}(\omega)}\right|_{\overline{j_S}}. \quad (23)$$

The volume V is described by the response to a source $\overline{X_V}$ coupling to j_V in the absence of $\tilde{E}$, corresponding to the Hamiltonian $H_V-j_V\overline{X_{\bar{V}}}$,

$$R_{iso}^{-1}(\omega)=\frac{\partial j_V(\omega)}{\partial\overline{X_{\bar{V}}}(\omega)}. \quad (24)$$

The environment's action is determined by the fluctuation dissipation theorem.[16] The action for j_V when V is isolated reads

$$\mathcal{S}_{iso}[\vec{j}_V]=\vec{j}_V\otimes M_{iso}\otimes\vec{j}_V. \quad (25)$$

The matrix multiplication $\otimes$ extends over Keldysh as well as time indices, $[A\otimes B](t,t'')\equiv\int dt'\,A(t,t')B(t',t'')$ and correspondingly for vectors like $\vec{j}_V(t)$. In a steady state $M_{iso}(t,t')$ depends only on the time difference $t-t'$ and in Fourier representation $M(\omega)=\int d(t-t')e^{i\omega(t-t')}M(t-t')$ it reads

$$M_{iso}(\omega)=\tfrac{1}{2}R_{iso}(\omega)\tau_3+i\,\mathrm{Im}\,R_{iso}(\omega)G_0(\omega), \quad (26)$$

$$G_0(\omega)=\begin{pmatrix} N(\omega) & -N(\omega)\\ -N(\omega)-1 & N(\omega)+1\end{pmatrix}, \quad (27)$$

with the Bose-Einstein distribution $N(\omega)=(\exp\{\omega/kT\}-1)^{-1}$. (We define the response functions such that they have negative imaginary part.)

We analyze first the particularly simple case that the energy flow into the entire environment to S is measured, $V=E$. Then we have $X_{\bar{V}}=j_S$, $X_S=j_V$ and $\mathcal{S}_{\bar{V}}=\mathcal{S}_S$. We rewrite Eq. (11) by introducing a coupling matrix

$$\sigma(t,t')=\begin{pmatrix}\delta[t-t'+\xi^+(t')] & 0\\ 0 & \delta[t-t'+\xi^-(t')]\end{pmatrix}, \quad (28)$$

$$\mathcal{Z}[\vec{\xi}]=\int \mathcal{D}\vec{j}_S\mathcal{D}\vec{j}_V e^{-i(\mathcal{S}_S[\vec{j}_S]+\mathcal{S}_{iso}[\vec{j}_V]+\vec{j}_V\otimes\tau_3\sigma\otimes\vec{j}_S)}. \quad (29)$$

The Gaussian integrals over $\vec{j}_V$ in Eq. (29) are easily done, resulting in

$$\mathcal{Z}[\vec{\xi}]=\int \mathcal{D}\vec{j}_S e^{-i(\mathcal{S}_S[\vec{j}_S]+\mathcal{S}_E[\vec{j}_S]+\mathcal{A}_{\vec{\xi}}[\vec{j}_S])} \quad (30)$$

with an action $\mathcal{S}_E$ that describes the influence of the environment on S,

$$\mathcal{S}_E[\vec{j}_S]=-\tfrac{1}{4}\vec{j}_S\tau_3\otimes M_{iso}^{-1}\otimes\tau_3\vec{j}_S, \quad (31)$$

and a source term $\mathcal{A}_{\vec{\xi}}$ that vanishes at $\vec{\xi}=0$,

$$\mathcal{A}_{\vec{\xi}}[\vec{j}_S]=\vec{j}_S\otimes A_{\vec{\xi}}\otimes\vec{j}_S, \quad A_{\vec{\xi}}=-\tfrac{1}{4}\tau_3 a_{\vec{\xi}}\tau_3, \quad (32)$$

$$a_{\vec{\xi}}=\sigma^\dagger\otimes M_{iso}^{-1}\otimes\sigma-M_{iso}^{-1}. \quad (33)$$

In the general case $V\neq E$, the environment's action is determined by R_{SS},

$$\mathcal{S}_E[\vec{j}_S]=\vec{j}_S\otimes G_{SS}\otimes\vec{j}_S, \quad (34)$$

$$G_{\alpha\gamma}=\tfrac{1}{2}[R_{\alpha\gamma}\tau_3+(R_{\alpha\gamma}-R^*_{\gamma\alpha})G_0], \quad \alpha,\gamma\in\{S,V\}. \quad (35)$$

The source term then takes the form

$$A_{\vec{\xi}}=G_{SV}\otimes\{[G_{VV}+G_{VV}\otimes a_{\vec{\xi}}\otimes G_{VV})]^{-1}-G_{VV}^{-1}\}\otimes G_{VS}. \quad (36)$$

Equation (32) is recovered for $X_{\bar{V}}=j_S$ and $X_S=j_V$, such that $R_{VV}=0$, $R_{SS}=-R_{\mathrm{iso}}^{-1}$, and $R_{VS}=R_{SV}=1$.

B. Zero frequency

For concrete results we focus on zero-frequency correlators of the energy flow, choosing $\vec{\xi}=(1,-1)\,\theta_\tau(t)\xi/2$ [cf. Eq. (7)]. We neglect transient effects, that is, we calculate only terms of leading order in the time τ over which energy is accumulated. We may then work in the discrete Fourier space of τ-periodic functions f, with Fourier coefficients $f_l = \int_0^\tau dt\, e^{i\omega_l t} f(t)/\tau$ and frequencies $\omega_l = 2\pi l/\tau$. In this representation the source term in Eq. (30) reads

$$\mathcal{A}_\xi[\vec{j}_S] = \tau \sum_l \vec{j}_{S,l} A_{\xi,l} \vec{j}_{S,l}. \tag{37}$$

σ is now diagonal in frequency indices,

$$\sigma_l = \begin{pmatrix} e^{-i\omega_l \xi/2} & 0 \\ 0 & e^{i\omega_l \xi/2} \end{pmatrix}, \tag{38}$$

and so is A_ξ. At zero temperature it takes a simple form even in the general case $E \neq V$,

$$A_{\xi,l} = |R_{VS}(\omega_l)|^2 \begin{pmatrix} 0 & 0 \\ i \operatorname{Im} R_{iso}^{-1}(\omega_l)(e^{-i\xi\omega_l} - 1) & 0 \end{pmatrix}. \tag{39}$$

Applied to a linear photodetector this reproduces the distribution of photocounts in response to a source j_S obtained perturbatively in Ref. 26. To see this we substitute $\xi\omega_l \to \xi_n$ in Eqs. (30) with Equation (37) and (39) to obtain the statistics of the number of absorbed quanta n rather than that of the absorbed energy. The further substitution $e^{-i\xi_n} - 1 \to \chi_n$ yields the generating function for factorial moments of n. To compare the resulting factorial moments of the photocount to the formulas obtained in Ref. 26, we write them in the time domain,

$$\begin{aligned} &\langle n(n-1)\cdots(n-p)\rangle \\ &= \left\langle \left[\int_{-\infty}^{\infty} dt'\, dt'' \int_0^\tau d\tau'\, d\tau'' \operatorname{Im} R_{iso}^{-1}(t',t'') R_{VS}(t',\tau') \right.\right. \\ &\quad \left.\left. \times R_{VS}(t'',\tau'') j_S^-(\tau') j_S^+(\tau'') \right]^p \right\rangle_{\mathcal{S}_S + \mathcal{S}_E}. \end{aligned} \tag{40}$$

This is equivalent to Eq. (30) in Ref. 26. The detector sensitivity f there corresponds to our $\operatorname{Im} R_{iso}^{-1}$, the density of absorbing detector modes per unit of frequency. The retarded photon propagator D_{ret} corresponds to our cross-response function iR_{VS}. The time ordering of j_S^+ and j_S^- along the Keldysh contour corresponds to the ''apex'' time order of the sources in Ref. 26. The expectation value in Eq. (40) is taken with respect to the action S_S of the source system and the piece S_E that describes the back action of the environment on S.

IV. LINEAR ELECTRICAL CIRCUITS

Electrical conductors couple to their electromagnetic environment—the circuit that they are embedded in—with the current I_S that flows through them. To calculate the flow of energy into this environment we may therefore apply the formulas obtained in the previous section with $j_S = I_S$, H_S being the Hamiltonian of the conductor. Also in this case energy is transferred by means of photons. These photons may be either detected by a photocounter[27] or with a bolometer, that measures the thermal energy exchanged by means of photons. According to Eq. (40) the nth factorial moment of energy transfer is proportional to the $2n$th moment of fluctuations of the source j_S. This suggests that a measurement of the energy flow and its fluctuations may be a useful tool for measuring higher-order correlators of electrical currents whose measurement poses an experimental challenge. While experimental techniques for the measurement of the variance of electrical current fluctuations are by now well developed, so far only one experiment has been successful in measuring higher-order current correlators. In Ref. 10 the measurement of the third moment of current fluctuations through a tunnel barrier has been reported. Equation (40) suggests that the fourth moment of current fluctuations can be inferred from the variance of the heat produced by them. Since for statistical reasons determining the variance of a quantity is much easier than measuring the fourth cumulative moment, a measurement of heat fluctuations has advantages over the direct charge fluctuation detection.

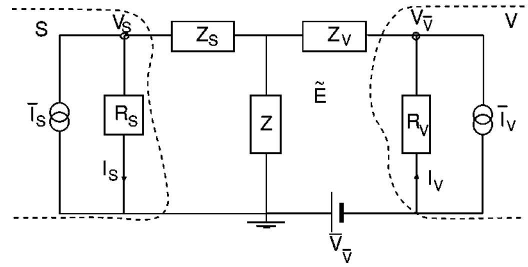

FIG. 6. Electrical circuit analyzed in this section. R_S is the conductor that generates current fluctuations δI_S. These fluctuations can be characterized by a measurement of fluctuations of the heat that is dissipated in the external resistor R_V.

To quantify this we apply now the relations obtained in the previous section to the electrical circuit depicted in Fig. 6. We assume all resistors in the circuit to be macroscopic and linear, except R_S, that plays now the role of the system S. Fluctuations of the amount of heat that is dissipated in R_V as measured with a bolometer are by virtue of Eq. (40) related to fluctuations in the electrical current through R_S. For that the volume V should be chosen the resistor R_V. The current I_V flowing through the volume V is coupled to the variable $\chi_{\bar{V}}$ in the rest of the environment $\tilde{E}$. This variable $\chi_{\bar{V}}$ is the time integral of the voltage $V_{\bar{V}}$ over R_V, $\chi_{\bar{V}}(t) = \int^t dt'\, V_{\bar{V}}(t')$. The current I_S through R_S similarly couples to the environment variable $\chi_S(t) = \int^t dt'\, V_S(t')$. Again we introduce external sources $\bar{I}_V$ and $\bar{\chi}_{\bar{V}}$. The circuit Fig. 6 is then mapped onto the general model of Fig. 5 with the choice $X_{\bar{V}} = \chi_{\bar{V}}$, $\bar{X}_{\bar{V}} = \bar{\chi}_{\bar{V}}$, $X_S = \chi_S$, $j_V = I_V$, $\bar{j}_V = \bar{I}_V$, $\bar{j}_S = \bar{I}_S$, and $j_S = I_S$. We need the response functions of the circuit Fig. 6 without the conductor R_S,

$$R_{SS} = \left.\frac{\partial \chi_S}{\partial \bar{I}_S}\right|_{\bar{I}_V} = \frac{1}{i\omega}\left[Z_S + \left(\frac{1}{Z} + \frac{1}{Z_V + R_V} \right)^{-1} \right],$$

$$R_{VS}=R_{SV}=\frac{\partial \chi_S}{\partial \overline{I_V}}\bigg|_{\overline{I_S}}=\frac{1}{i\omega}\,\frac{ZR_V}{Z+Z_V+R_V}, \tag{41}$$

$$R_{VV}=\frac{\partial \chi_{\overline{V}}}{\partial \overline{I_V}}\bigg|_{\overline{I_S}}=\frac{1}{i\omega}\left(\frac{1}{R_V}+\frac{1}{Z_V+Z}\right)^{-1}, \tag{42}$$

$$R_{iso}^{-1}=\frac{\partial I_V}{\partial \chi_{\overline{V}}}\bigg|_{\text{isolated}}=\frac{i\omega}{R_V} \tag{43}$$

(all quantities here are frequency-dependent). Note, that any four-terminal circuit connecting the conductor R_S with R_V can be modeled by the three resistors Z, Z_S, and Z_V. The resulting zero-frequency energy flow statistics takes a particularly simple form in the limit of an infinite Z. Then the current flowing through R_S is directly fed into R_V and the source term in the generating functional (30) is given by Eq. (37) with

$$A_{\xi,l}=-\frac{\mathrm{Re}\,R_V(\omega_l)}{i\omega_l}\times\begin{pmatrix}0 & N(\omega_l)(e^{i\xi\omega_l}-1)\\ [N(\omega_l)+1](e^{-i\xi\omega_l}-1) & 0\end{pmatrix}. \tag{44}$$

This is a generalization of Eq. (39) to finite temperature, allowing for emission of energy quanta from V as well as absorption. The transferred quanta have predominantly energies that are smaller than the inverse of the RC time τ_{RC} of R_V. For finite Z thermal current fluctuations $\langle \delta I^2\rangle_{th}\approx kT/Z$ created in Z mix into the fluctuations of I_S. As a consequence Eq. (44) is then only valid in the regime $(\tau_{RC}kT)^2 \ll |Z/R_V|$.

The formulas for the statistics when the mixing in of thermal fluctuations occurs are more complicated. To make further progress we assume that the frequency dispersion of the measured current correlators is negligible on the scale $1/\tau_{RC}$, $\langle\langle \Pi_{q=1}^{p} I_S(\omega_q)\rangle\rangle \approx 2\pi\delta(\Sigma_{q=1}^{p}\omega_q)C_p$ for $\omega_q \ll 1/\tau_{RC}$ (here $\langle\langle\cdots\rangle\rangle$ denotes irreducible, or cumulant correlators). For a mesoscopic conductor this is satisfied if $1/\tau_{RC}$ is smaller than the voltage applied to the conductor. In this limit we find that

$$\langle \Delta H_V\rangle = 2\tau\int_0^\infty \frac{d\omega}{2\pi}R_{eff}(\omega)C_2 \tag{45}$$

$$\langle\langle(\Delta H_V)^2\rangle\rangle = 4\tau\left\{\left[\int_0^\infty\frac{d\omega}{2\pi}R_{eff}(\omega)\right]^2 C_4+\int_0^\infty\frac{d\omega}{2\pi}\times\left[R_{eff}^2(\omega)C_2^2-\frac{\omega}{2}[2N(\omega)+1]R_{eff}(\omega)C_2\right]\right\} \tag{46}$$

with $R_{eff}=\mathrm{Re}\,R_V|Z/(Z+Z_V+R_V)|^2$. We have assumed that $R_{eff}(0)=0$ and $R_{eff}(\omega)\to 0$ for $\omega>1/\tau_{RC}$ such that the dc component (that will be avoided in experiments) and the contributions from frequencies $\omega>1/\tau_{RC}$ in Eqs. (45) and (46) are negligible. We conclude that the variance of the heat dissipated in R_V depends on the fourth cumulant of current fluctuations in the mesoscopic conductor, as one would expect. There are, however, also contributions from lower-order current correlators. The environment circuit has a finite response time τ_{RC} and therefore effectively averages fluctuations over that time. As a result, the energy fluctuations become dominated by the lowest order current correlator in the limit of a long τ_{RC}, when higher-order current correlators become negligible and the statistics becomes Gaussian. In this limit of "narrow-band detection" the statistics of energy transfer is negative binomial.[27] Only deviations from this encode non-Gaussian current correlations. In order to see them the statistics of charge flow through the conductor during τ_{RC} has to be strongly non-Gaussian. This is the case if $\tau_{RC}\overline{I}/e$, the mean number of transmitted electrons in that period, is small.

More concretely, for a measurement of the fourth cumulant C_4 of current fluctuations in R_S one would want the first term in Eq. (46) to be dominant. An estimate of the three summands in Eq. (46) for a tunnel contact with mean current $\overline{I}$ [assuming that $R_{eff}(\omega)\approx R_{eff}$ for $\omega<1/\tau_{RC}$],

$$\langle\langle(\Delta H_V)^2\rangle\rangle \approx \frac{\tau}{\tau_{RC}^2}R_{eff}^2 G_Q e\overline{I}\times\left(1+\frac{\tau_{RC}\overline{I}}{e}+\frac{\tau_{RC}^2 kT\min\{kT,1/\tau_{RC}\}}{G_Q R_{eff}}\right) \tag{47}$$

(with the conductance quantum $G_Q=e^2/2\pi$) confirms this. The variance of fluctuations of the heat flux into R_V is a direct measure for the fourth cumulant of current fluctuations in a tunnel contact if $\tau_{RC}\overline{I}/e\ll 1$ and $\tau_{RC}kT\ll G_Q R_{eff}$. The origin of the first condition has been explained qualitatively above. The second condition ensures that the fourth cumulant is visible on the thermal background. Back action effects of the measuring resistor on the measured tunnel contact (conductance G) are avoided if additionally $R_{SS}G\ll 1$.[16] These requirements are rather restrictive, but can in principle be met in experiments. For a practical implementation, the first condition $\tau_{RC}\overline{I}/e\ll 1$ may be relaxed to $\tau_{RC}\overline{I}/e\simeq 1$ by measuring the voltage dependence of $\langle\langle(\Delta H_V)^2\rangle\rangle$ and extracting the term that is linear in $\overline{I}$.

CONCLUSION

We have presented a theory for the statistics of energy exchange between coupled quantum systems. As an application we have calculated the statistics of energy dissipation into the leads connected to a mesoscopic conductor. General and exact expressions can be obtained for the energy flow from a quantum system into a linear environment. We have used these results to calculate the moments of the heat dissipated in a linear circuit around a mesoscopic conductor. They have been expressed in terms of moments of the current fluctuations produced by the conductor. As one expects, the variance of this dissipated heat depends on the fourth cumulative moment of current fluctuations produced by the conductor. Heat fluctuations may therefore serve as a tool to detect the fourth cumulant, whose direct measurement is difficult. We have analyzed the conditions under which a measurement of heat fluctuations can reveal higher-order cumulants of the charge counting statistics.

ACKNOWLEDGMENTS

We thank Carlo Beenakker, Markus Büttiker, and Yuli Nazarov for valuable discussions. This research was supported by the ‘‘Nederlandse organisatie voor Wetenschappelijk Onderzoek’’ (NWO) and by the ‘‘Stichting voor Fundamenteel Onderzoek der Materie’’ (FOM), by the Swiss National Science Foundation, and by the European Community’s Human Potential Program for Nanoscale Dynamics, Coherence and Computation under Contract No. HPRN-CT-2000-00144.

*Present address: Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139.

[1] J.P. Pekola, K.P. Hirvi, J.P. Kauppinen, and M.A. Paalanen, Phys. Rev. Lett. **73**, 2903 (1994).
[2] J.P. Kauppinen, K.T. Loberg, A.J. Manninen, J.P. Pekola, and R.A. Voutilainen, Rev. Sci. Instrum. **69**, 4166 (1998).
[3] K. Schwab, E.A. Henriksen, J.M. Worlock, and M.L. Roukes, Nature (London) **404**, 974 (2000).
[4] Luis G.C. Rego and G. Kirczenow, Phys. Rev. B **59**, 13 080 (1999).
[5] A.A.M. Staring, L.W. Molenkamp, B.W. Alphenaar, H. van Houten, O.J.A. Buyk, M.A.A. Mabesoone, C.W.J. Beenakker, and C.T. Foxon, Europhys. Lett. **22**, 57 (1993).
[6] A.S. Dzurak, C.G. Smith, C.H.W. Barnes, M. Pepper, L. Martń-Moreno, C.T. Liang, D.A. Ritchie, and G.A.C. Jones, Phys. Rev. B **55**, R10197 (1997).
[7] P. Kim, L. Shi, A. Majumdar, and P.L. McEuen, Phys. Rev. Lett. **87**, 215502 (2001).
[8] J. Eom, C.-J. Chien, and V. Chandrasekhar, Phys. Rev. Lett. **81**, 437 (1998)
[9] Ya.M. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).
[10] B. Reulet, J. Senzier, and D.E. Prober, Phys. Rev. Lett. **91**, 196601 (2003).
[11] C.W.J. Beenakker, M. Kindermann, and Yu.V. Nazarov, Phys. Rev. Lett. **90**, 176802 (2003).
[12] L.S. Levitov and G.B. Lesovik, JETP Lett. **58**, 230 (1993); cond-mat/9401004 (unpublished); L.S. Levitov, H. Lee, and G.B. Lesovik, J. Math. Phys. **37**, 4845 (1996).
[13] M. Kindermann, Yu.V. Nazarov, and C.W.J. Beenakker, Phys. Rev. Lett. **90**, 246805 (2003).
[14] M. Kindermann and C.W.J. Beenakker, Phys. Rev. B **66**, 224106 (2002).
[15] S. Pilgram and M. Büttiker, Phys. Rev. B **67**, 235308 (2003).
[16] M. Kindermann, Yu.V. Nazarov, and C.W.J. Beenakker, Phys. Rev. B **69**, 035336 (2004).
[17] Y. Adamov and B. Muzykantskii, Phys. Rev. B **64**, 245318 (2001); **68**, 155304 (2003).
[18] M. Rokni and Y. Levinson, Phys. Rev. B **52**, 1882 (1995).
[19] W. Belzig and Yu.V. Nazarov, Phys. Rev. Lett. **87**, 067006 (2001); **87**, 197006 (2001).
[20] Yu.V. Nazarov and D.A. Bagrets, Phys. Rev. Lett. **88**, 196801 (2002).
[21] U. Sivan and Y. Imry, Phys. Rev. B **33**, 551 (1986).
[22] Ya.M. Blanter, C. Bruder, R. Fazio, and H. Schoeller, Phys. Rev. B **55**, 4069 (1997).
[23] L.S. Levitov and M. Reznikov, cond-mat/0111057 (unpublished).
[24] This type of energy transfer statistics has been used in Ref. 25 for open point contacts to calculate the charge noise of a chaotic cavity in the hot electron regime.
[25] S. Pilgram, A.N. Jordan, E.V. Sukhorukov, and M. Büttiker, Phys. Rev. Lett. **90**, 206801 (2003).
[26] M. Fleischhauer, J. Phys. A **31**, 453 (1998).
[27] C.W.J. Beenakker and H. Schomerus, Phys. Rev. Lett. **86**, 700 (2001).