Two-fluxon dynamics in an annular Josephson junction

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(Received 24 June 2003; revised manuscript received 20 January 2004; published 5 April 2004)

Two-fluxon state in an annular Josephson junction in the presence of external magnetic field is studied analytically, numerically, and experimentally. We obtain an analytical expression for the potential of interaction between the fluxons moving at arbitrary velocities (without the use of the "nonrelativistic" approximation). Treating the fluxons as quasiparticles, we then derive equations of motion for them. Direct simulations of the full extended sine-Gordon equation are in good agreement with results produced by the analytical model, in a relevant parameter region. Experimental data qualitatively agree with the numerical results.

DOI: 10.1103/PhysRevB.69.144502 PACS number(s): 03.75.Lm, 05.45.Yv

I. INTRODUCTION

A magnetic flux quantum (fluxon) in a long Josephson junction (LJJ) is a well-known physical example of a sine-Gordon soliton. Ring-shaped (annular) LJJ's serve as the ideal setting to study the fluxon dynamics, as it is not perturbed by boundary conditions at edges, which is the case for linear LJJ's.[1] Due to the magnetic-flux quantization in a superconducting ring, the number of fluxons initially trapped in an annular junction is conserved. A handy tool, which makes it possible to create an effective spatially periodic potential for a fluxon trapped in the annular LJJ, is external dc magnetic field directed parallel to the ring's plane.[2] If θ is the angular fluxon coordinate along the ring, the effective potential is $U(\theta)\sim H\cos\theta$, where H is the strength of the magnetic field. The minimum of the potential is located at the spot where the fluxon's magnetic moment is directed along the external field. The dynamics of a single fluxon in the spatially periodic potential has attracted a great deal of interest, as shown by many theoretical and experimental works dealing with this subject, see Refs. 3–11 and references therein. The potential for further investigations offered by the annular LJJ's with trapped fluxons is still far from exhaustion, which is attested by the very recent experiments with quantum fluxons in this system at ultralow temperatures—the first ever direct observation of quantum tunneling of solitons.[12]

The objective of the present work is to study, both theoretically and experimentally, dynamics of two fluxons with equal polarities trapped in the annular LJJ. The system is schematically shown in Fig. 1. In the presence of the external magnetic field, the problem is distinguished by the interplay of the above-mentioned effective periodic potential acting on each fluxon and interaction (repulsion) between them. While soliton-soliton interactions are a well-known topic for theoretical analysis,[6] the present setting gives a unique possibility to directly study interactions between solitons in a real physical system under fully controllable conditions, which makes the problem relevant to a much broader context than the LJJ's *per se*. In fact, our approach to the interaction is essentially different in comparison with earlier works, and, in this respect, it may also be of interest to many applications.

The interaction between two fluxons in LJJ's was first studied theoretically by Karpman *et al.* (see Ref. 13 and references therein). Those authors found an analytic expression for the interaction force between fluxons in the case of a small relative velocity, which corresponds to the "nonrelativistic" approximation, when the relative velocity is much smaller than the limit velocity in the LJJ (the Swihart velocity). However, in the situation relevant to the experiment, the latter condition is not met, in the general case. In this paper, we aim to develop an analytical approach to the interaction which will be valid in the general ("relativistic") case, and will make the results interesting in a more general context, as mentioned above. The theory will be based on an asymptotic method for weakly interacting solitons in nonintegrable systems.[6,14,15] The analysis will be followed by direct simulations and presentation of experimental results.

The theoretical model for the annular LJJ in the external magnetic field was proposed in Ref. 2. It is based on the extended (perturbed) sine-Gordon (sG) equation for the superconducting phase difference φ between the electrodes of the junction:

$$\varphi_{xx}-\varphi_{tt}-\sin\varphi=\alpha\varphi_t+\gamma+h\sin(qx). \tag{1}$$

Here x is the coordinate along the ring, which is normalized to the Josephson penetration depth λ_J, the time t is normal-

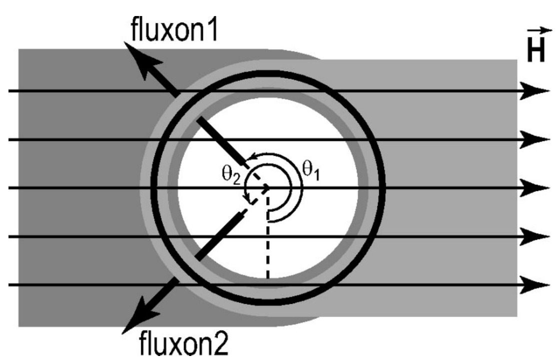

FIG. 1. The schematic view of an annular Josephson junction with two trapped fluxons; the magnetic field $\vec{H}$ is applied in the plane of the junction. Josephson tunnel barrier is shown by thick black line; in gray are shown superconducting electrodes, which are extended in the junction plane in order to feed the bias current.

ized to the inverse plasma frequency ω_0^{-1}, α is a coefficient of the dissipation due to the quasiparticle tunneling across the junction, and γ is the bias current density, normalized to the critical current density j_c of the junction. As commonly accepted, we assume that the bias current is uniformly distributed along the ring. Further, $q \equiv 2\pi/L$, where L is the normalized circumference of the junction and h is the strength of the external magnetic field H, normalized by a sample-specific geometric factor.[2,8] If N fluxons are trapped in the ring, Eq. (1) is supplemented by the boundary condition

$$\varphi(L+x,t) = \varphi(x,t) + 2\pi N, \tag{2a}$$

$$\varphi_x(L+x,t) = \varphi_x(x,t). \tag{2b}$$

The paper is organized as following. The single-fluxon dynamics in the annular LJJ is reviewed in Sec. II A. In Sec. II B, the derivation of an effective force of interaction between two fluxons, valid in the general (relativistic) case, is presented. Results of numerical calculations are displayed in Sec. III. In Sec. III A, we analytically consider a special case of an ostensible resonance in the two-fluxon system. In Sec. IV, we present experimental results for two fluxons trapped in an annular LJJ.

II. THEORY

A. The basic model

In our theoretical approach, we assume, as usual, that the fluxons are well separated from each other, $|\xi_1 - \xi_2| \gg 1$, where $\xi_{1,2}$ are coordinates of their centers. In this case the two-soliton state may be represented by a linear combination of two single-soliton solutions:

$$\varphi = \varphi_1 + \varphi_2, \tag{3}$$

where

$$\varphi_n = 4\arctan\left[\exp\left(-\frac{x-\xi_n(t)}{\sqrt{1-\dot{\xi}_n^2}}\right)\right] + 2\pi(n-1), \tag{4}$$

($n=1,2$) is the single-soliton solution of the unperturbed sG equation, and $\dot{\xi}_n$ is its velocity. The last term in Eq. (4) is a shift of the background phase which is necessary to describe a two-fluxon configuration.

Before discussing the interaction of the fluxons, we briefly recall known results for a single fluxon trapped in an annular LJJ. This fluxon may be considered as a quasiparticle obeying the well-known equation of motion[3,16]

$$\frac{d}{dt}\left(\frac{\dot{\xi}}{\sqrt{1-\dot{\xi}^2}}\right) + \frac{\alpha\dot{\xi}}{\sqrt{1-\dot{\xi}^2}} + \frac{\pi h}{4}\text{sech}\left(\frac{\pi^2}{L}\right)\sin(q\xi) = \frac{\pi\gamma}{4}, \tag{5}$$

which is equivalent to the equation of motion for a relativistic pendulum in a lossy medium under the action of a constant torque. This equation has solutions of two types. The first type gives rise to $|\xi(t)|$ growing indefinitely. It describes progressive motion ("rotation") of the fluxon around the ring, with a nonzero mean value of the velocity $\dot{\xi}$. Solutions of the second type correspond to small oscillations of the fluxon around the minimum of the effective potential with the frequency

$$\omega_0 = \sqrt{\frac{\pi^2}{2L}}\left[h^2\text{sech}^2\left(\frac{\pi^2}{L}\right) - \gamma^2\right]^{1/2}, \tag{6}$$

the average velocity being zero. This state exists if $|\gamma|$ is below the critical value,

$$\gamma_c^{(1)} = h\,\text{sech}(\pi^2/L). \tag{7}$$

In the presence of dissipation ($\alpha \neq 0$), the oscillations are damped, and in the stationary state the fluxon is at rest. On the other hand, the progressive motion remains possible if the dissipation is not too strong. The fluxon moves with the average velocity, which, in the first approximation, is given by the McLaughlin-Scott formula[3]

$$\langle\dot{\xi}_0\rangle = \left[1 + \left(\frac{4\alpha}{\pi\gamma}\right)^2\right]^{-1/2}. \tag{8}$$

Equation (8) determines the normalized current-voltage characteristics of the junction with a single trapped fluxon.

In the case of two trapped fluxons, we assume that they are quasiparticles interacting with a certain force (see below), and all the forces in Eq. (5) act on each fluxon separately. In this case, three different dynamical regimes are expected: (i) oscillations of both fluxons [due to dissipation, the oscillations of fluxons in the well are damped, i.e., a zero-voltage state, hence, we call this state "static-static" (S-S) regime], (ii) rotation of both fluxons ("R-R" regime), and (iii) rotation of one fluxon and oscillations of the other one ("R-O" regime). Note that in the R-O regime oscillations take place even in the presence of dissipation, because the fluxon whose average velocity is zero is periodically excited by collisions with the rotating one.

B. The interaction force

In order to find the interaction force between two fluxons, we use the center-of-mass reference frame (C frame). In this frame, the solitons move with velocities $\pm u$. It follows from the Lorentz transformation that

$$u = \frac{1 - \dot{\xi}_1\dot{\xi}_2 - \sqrt{1-\dot{\xi}_1^2}\sqrt{1-\dot{\xi}_2^2}}{\dot{\xi}_1 - \dot{\xi}_2}. \tag{9}$$

In the subsequent calculation, we set $\xi_1 < \xi_2$ for the definiteness' sake.

To calculate an effective potential of the interaction between the fluxons, which will then produce the interaction force, we start with the Hamiltonian of the unperturbed sG equation in the infinitely long system:

$$H = \int_{-\infty}^{\infty} dx\left(\frac{1}{2}\varphi_t^2 + \frac{1}{2}\varphi_x^2 + 1 - \cos\varphi\right). \tag{10}$$

For the calculation of the Hamiltonian (10), we divide the space into two parts. The left (right) part occupies the space from $-\infty$ $(+\infty)$ to the midpoint between the two fluxons, $a \equiv (\xi_1+\xi_2)/2$. We perform the actual calculation for the left part only, as for the right part the calculation is just a mirror image.

In the left part, we substitute the solution as the linear combination (3), where φ_2 is considered as a small perturbation, once the two fluxons are assumed to be well separated. Then, the Hamiltonian is written, in the first approximation, as

$$H_{\rm left}=H_1+\delta H_{\rm left,int}, \tag{11}$$

where $H_1=8/\sqrt{1-u^2}$ is the Hamiltonian of the unperturbed sG soliton, and the interaction term is of the first order with respect to the weak field φ_2,

$$\delta H_{\rm left,int}=\int_{-\infty}^{a} dx[(\varphi_1)_x(\varphi_2)_x+(\varphi_1)_t(\varphi_2)_t+\varphi_2\sin\varphi_1]. \tag{12}$$

Substituting the expression valid for a moving soliton, $(\varphi_n)_t=-\dot{\xi}_n(\varphi_n)_x$ and, integrating by parts, we obtain

$$\delta H_{\rm left,int}=(1-u^2)(\varphi_1)_x\varphi_2|_{-\infty}^{a} + \int_{-\infty}^{a} dx[-(\varphi_1)_{xx}+(\varphi_1)_{tt}+\sin\varphi_1]\varphi_2. \tag{13}$$

The integral term in Eq. (13) is zero as the bracketed expression is the sine-Gordon equation (this way of nullifying the integral terms is known in the general analysis of the interaction between separated solitons[15]). The contribution to the first term in Eq. (13) at the left limit, $x=-\infty$, is zero too because both $(\varphi_1)_x$ and φ_2 decay exponentially at infinity. In order to calculate the contribution from the upper limit, we use the asymptotic forms of $(\varphi_1)_x$ and φ_2 valid at large values of x (it is also a known point in the general analysis of the soliton-soliton interactions[15]):

$$(\varphi_{1,x})_{\rm asymp}=-\frac{4}{\sqrt{1-u^2}}\exp\left(-\frac{x-\xi_1}{\sqrt{1-u^2}}\right),$$

$$(\varphi_2)_{\rm asymp}=-4\exp\left(\frac{x-\xi_2}{\sqrt{1-u^2}}\right). \tag{14}$$

After the substitution of the expressions (14) into Eq. (13) and calculation of the nonvanishing contribution to the first term from the right limit, $x=a$, and then getting back from the C frame to the laboratory reference frame (L frame), we arrive at an expression for the interaction potential for two moving fluxons in the infinitely long junction,

$$\delta H_{\rm int}=32\frac{\sqrt{1-u^2}}{\sqrt{1-V^2}}\exp\left(-\frac{|\xi_1-\xi_2|}{\sqrt{1-u^2}}\right), \tag{15}$$

where the contribution of the right half space is taken into account, and

$$V=\frac{1+\dot{\xi}_1\dot{\xi}_2-\sqrt{1-\dot{\xi}_1^2}\sqrt{1-\dot{\xi}_2^2}}{\dot{\xi}_1+\dot{\xi}_2} \tag{16}$$

is the velocity of the center of mass of the two-fluxon configuration in L frame. In the case of equal velocities, the potential (15) reduces to the well-known result of Karpman *et al.*[13]

Due to the ring geometry of the system, the potential (15) gives rise to *two* forces acting on each soliton, which should be added to the individual forces in Eq. (5). These interaction forces are

$$(F_{\rm int})_1=-(F_{\rm int})_2=-\frac{1}{8}\frac{d}{d\Delta x}\delta H_{\rm int}^{\rm ring} = \frac{4}{\sqrt{1-V^2}}\left[\exp\left(-\frac{\Delta X}{1-u^2}\right)-\exp\left(-\frac{L-\Delta X}{1-u^2}\right)\right], \tag{17}$$

where $\Delta X=|\xi_1-\xi_2|$, and 8 is for the effective mass of the fluxon in the present notation. Equations (5) and (17) describe the dynamics of the two-fluxon system in an annular LJJ.

III. NUMERICAL CALCULATIONS

In order to verify the theory presented above, we checked numerical solutions of the quasiparticle equations of motion against direct simulations of the full equation (1). The quasiparticle equations of motion were numerically solved by means of the fourth-order Runge-Kutta method. The time step was $\Delta t=0.05$. Equation (1) with boundary conditions (2) was numerically integrated using the stabilized "leap-frog" method.[17] The steps in time and space were taken as $\Delta t=\Delta x=0.05$. We verified the accuracy of the numerical routine by halving and doubling the discretization of the steps. Here we present numerical results for experimentally relevant values of $\alpha=0.02$ and $L=20$. The bias current γ was varied in steps of 0.002.

The general behavior of the system can be described as follows. While γ increases from zero, both fluxons originally stay pinned in the effective potential induced by the magnetic field, so that the voltage across the junction is zero. At a critical value of the current γ_c, the system switches into the R-O regime, in which one of the fluxons rotates, while the other one oscillates due to periodic collisions with the moving fluxon. This state is stable in a region $\gamma<\gamma_s$, up to another critical point γ_s. On the other hand, decreasing the bias current leads to a transition to the regime with both fluxons pinned (zero voltage) at a different value, $\gamma=\gamma_r$.

At $\gamma>\gamma_s$, the system operates in R-R regime with both fluxons rotating. Further increase of the bias current does not change the state of the system, up to a large value of the current, at which the junction switches to the "whirling" (resistive) state, with uniform rotation of the phase over all the system. When decreasing the bias current, the system

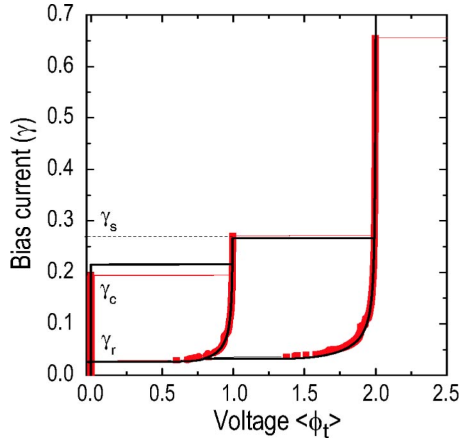

FIG. 2. The current-voltage characteristics of the long annular junction with two trapped fluxons found from direct numerical simulation of Eq. (1) (dots) and from the analytical model based on Eqs. (5) and (17) (solid line). In this case, the magnetic field is fixed to $h=0.3$.

switches first into the R-O regime and then into the zero-voltage one.

Typical current-voltage (I-V) characteristics, which display all these states and transitions, are presented in Fig. 2. Points shown by dots correspond to the numerical solution of the full equation (1), while the lines depict solutions of the quasiparticle model based on Eqs. (5) and (17). As is seen, the analytical quasiparticle model making use of the expression (17) for the interaction forces is in good agreement with the direct simulations.

The comparison of the critical values γ_c of the bias current, obtained from the full simulations and from the analytical model, is shown in Fig. 3. A small difference between them is explained by the fact that, in the S-S regime, the actual distance between the pinned fluxons is small, hence the assumption of far separated ones is not accurate in this case. Indeed, the numerical computations show that the distance between the fluxons in the static case varies, depending on the magnetic field, in the range of 0.8–1.5.

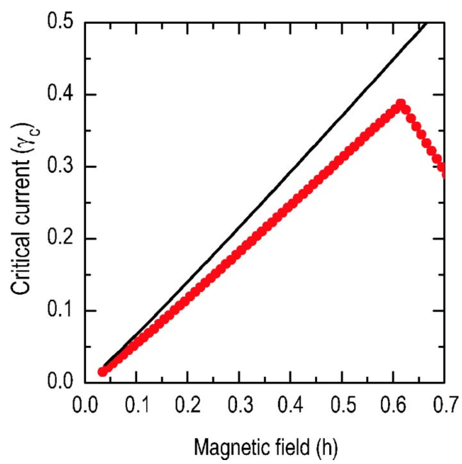

FIG. 3. The critical current γ_c vs the magnetic field h, obtained from direct numerical simulations of the full equation (1) (dots) and from the quasiparticle model (line).

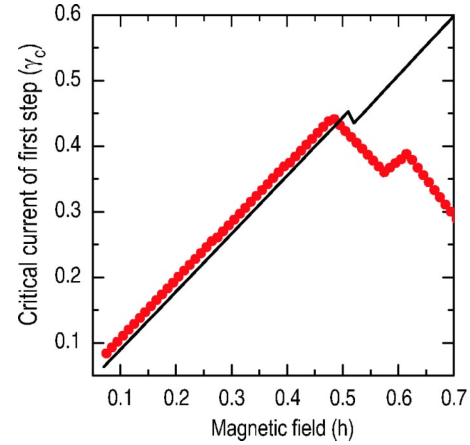

FIG. 4. The dependence of the maximum current of the first step in the I-V curve, γ_s, on the magnetic field h, as found from direct simulations of the full equation (1) (dots), and from the analytical model based on Eqs. (5) and (17) (line).

For small currents, the I-V curves for the R-O and R-R regimes, found from the direct simulations of Eq. (1), feature additional small steps. Additional analysis shows that they are due to resonant generation of radiation by the fluxons moving in the periodic potential; this phenomenon has been studied in detail before.[7]

The comparison of the other critical value of the bias current, γ_s (which corresponds to the first step of the I-V characteristics), again as found from the direct simulations and from the analytical model, is shown, vs the magnetic field, in Fig. 4. At small values of the field, these dependencies agree very well. However, for $h>0.48$ the curve generated by the direct simulations goes down with the increase of the field. Clearly, in this region the parameters γ and h are too large to apply the perturbation theory. The ansatz used in our approach breaks down, since large h generates extra fluxons in the junction. With the further increase of h, γ_s decreases until it becomes equal to γ_c. At a still stronger field $(h>0.58)$, the system jumps from the S-S regime directly into the whirling state.

On the other hand, the curve produced by the analytical approximation continues to go up with the field until $h=0.5$. For fields larger than 0.5, the system resonantly switches from the R-O regime into the R-R one, but at so large values of the field the quasiparticle model based on the perturbation theory becomes irrelevant.

A. The resonance condition

A noteworthy feature of the two-fluxon dynamics in the R-O regime is a possibility of a resonance between the natural frequency of oscillations of the trapped fluxon and periodic excitation due to its collisions with the rotating one. The resonance condition can be predicted by equating the small-oscillation frequency ω_0, which is given by Eq. (6), and the rotation frequency $\omega_r=q\langle\dot{\xi}_r\rangle$, where $\dot{\xi}_r$ is the velocity of the rotating fluxon, that can be obtained from Eq. (8). This yields

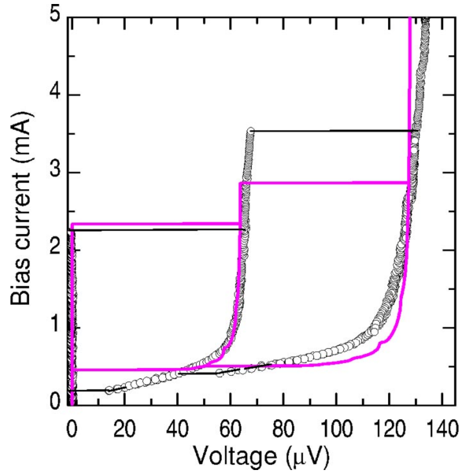

FIG. 5. Current-voltage characteristics of the annular Josephson junction with two trapped fluxons. The open circles correspond to the experimental data and the solid line to the numerical solutions of Eq. (1) with $\alpha = 0.025$, $L = 28.5$, and $h = 0.3$.

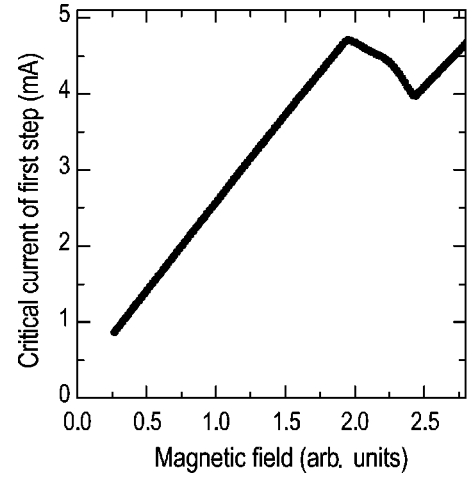

FIG. 6. The experimentally found critical current for the jump from the one-fluxon step of the *I*-*V* curve to the two-fluxon one, as a function of the magnetic field.

$$h = \gamma_{\rm res} \cosh\left(\frac{\pi^2}{L}\right) \sqrt{1 + \frac{64\pi^4 \gamma_{\rm res}^2}{L^2(\pi^2 \gamma_{\rm res}^2 + 16\alpha^2)^2}}, \qquad (18)$$

where $\gamma_{\rm res}$ is the value of the bias current corresponding to the resonance.

It may be expected that this resonance would result in a resonant switching from the R-O regime into the R-R one, and there would appear a drop in the dependence of the first-step critical current γ_s vs the magnetic field h. This drop is predicted by the numerical solution of the analytical model (the solid line in Fig. 4). However, at low magnetic fields the system actually switches into the R-O branch at a higher current than the current value corresponding to the resonance. At larger magnetic fields, it switches directly into the R-R branch, because the total perturbation (field and bias current) becomes too strong.

In fact, the perturbation approach is not applicable in this situation, as a more detailed consideration of the analytical model shows that it formally predicts strong periodic overlapping between the two fluxons, which cannot take place in reality. With the increase of the junction's length L, the intersection point of the curves $\gamma_c(h)$ and $\gamma_{\rm res}(h)$ moves upward in bias current, while its dependence on magnetic field is very weak.

IV. EXPERIMENT

Measurements of the *I*-*V* characteristics of the two-fluxon state were performed in long annular Nb-Al-AlO_x-Nb junctions. Due to the magnetic flux quantization in the superconducting ring, the number of initially trapped fluxons (solitons) is conserved. Trapping of a magnetic flux in the junction was achieved while cooling the sample below the critical temperature of niobium, $T_c^{\rm Nb} = 9.2$ K, in the presence of a small bias current passing through the junction. The number of trapped fluxons was determined from the highest voltage of the tallest resonant branch on the current-voltage characteristics (see Fig. 5). Experiments were performed by applying the bias current I from the top electrode of the junction to the bottom one and measuring the dc voltage generated due to the motion (*rotation*, in terms of the theoretical consideration) of the trapped fluxons. The results presented below were obtained for a junction with the mean diameter 100 μm and ring's width 3 μm. The circumference (length of the annular junction) in the normalized units was $L \simeq 28.5$. The measurements were performed at 4.2 K.

At zero magnetic field, depinning of a fluxon was observed as a switching from the zero voltage state into the single-fluxon step of the *I*-*V* curve (the state with one moving fluxon), at the current I_c that was smaller by a factor of about 65 than the critical current for the same junction, measured without trapped fluxons. This fact indicates a high degree of homogeneity of the junction (a strong local inhomogeneity would give rise to a much larger value of the fluxon-depinning critical current).

We measured the *I*-*V* curves of the state with two trapped fluxons for different strengths of the applied magnetic field. One of these curves is shown in Fig. 5 (open circles). The solid line corresponds to the numerical solution of Eq. (1) with parameters found from experimental data: $\alpha = 0.025$, $L = 28.5$, and $h = 0.3$ (solid line in Fig. 5). The experimental curves agree quite well with those predicted by the above analysis. Two branches of the *I*-*V* characteristic are observed. The first branch, at the voltage of about 65 μV, corresponds to the R-O regime, and the second one, observed at about 130 μV, clearly pertains to R-R regime. The actual losses in the experiment were apparently stronger than that taken for numerical simulations. This explains why small steps (generated by the resonant emission of plasma waves by the fluxon) present in numerical data are suppressed in the experiment.

Figure 6 shows the measured value of the switching current for the transition from the R-O regime to the R-R one vs the external magnetic field. The curve is similar to the dependence $\gamma_s(h)$ (dots in Fig. 4), which was obtained above from the direct integration of the full sine-Gordon model (1).

V. SUMMARY

In this paper, we have reported results of theoretical and experimental studies of two-fluxon dynamics in a long annular Josephson junction in the presence of the external magnetic field. The analytical expression for the interaction force between two fluxons moving at different velocities has been derived, without assuming the motion nonrelativistic (which is an essentially unique element of the analysis). Solutions of the system of the two resultant coupled quasiparticle equations of motion for the fluxons demonstrate good agreement with direct numerical simulations of the two-fluxon state in the full sine-Gordon model including all the perturbations.

Three distinct dynamical regimes of the two-fluxon state have been thus identified. First, both fluxons may be pinned in the potential induced by the magnetic field (the S-S regime). There is some discrepancy in the prediction of the critical current, which destroys this static regime, between the quasiparticle model and direct simulations, due to the fact that the separation between the two trapped fluxons is rather small, while the analytical model assumes them to be well separated.

In the second regime, one of the fluxons rotates around the junction, while the other one oscillates in the potential well. For this R-O regime, the maximum current is very accurately predicted by the quasiparticle model, if compared to the direct simulations. For this case, we have also investigated the possibility of the resonant excitation of the trapped fluxon by periodic collisions with the rotating one, and eventually concluded that this case, although seeming superficially natural, is irrelevant, due to limitations on the applicability of the perturbation theory.

In the third regime (the R-R one), both fluxons rotate. The corresponding I-V curves found in the direct simulations demonstrate several additional small steps, which are explained by resonant generation of small-amplitude plasma waves by fluxons moving in the periodic potential; the latter effect was considered earlier for the single-fluxon case.[7]

The method developed in this work to calculate, in the general case, the effective interaction force acting between two moving solitons may find application in other soliton-bearing systems.

ACKNOWLEDGMENTS

A.A.A. is grateful to F. Kh. Abdullaev and E. N. Tsoy for useful discussions. We also wish to thank A. Kemp for his help in the experiment. B. A. M. appreciated hospitality of Physikalisches Institut III at Universität Erlangen-Nürnberg.

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