

Transverse spin dynamics in a spin-polarized Fermi liquid

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The linear equations for transverse spin dynamics in a weakly polarized degenerate Fermi liquid with arbitrary relationship between temperature T and polarization γH are derived from Landau-Silin phenomenological kinetic equation with general form of two-particle collision integral. The temperature and polarization dependence of the spin current relaxation time is established. It is found in particular that at finite polarization transverse spin wave damping has a finite value at $T=0$. The analogy between temperature dependences of spin wave attenuation and ultrasound absorption in degenerate Fermi liquid at arbitrary temperature is presented. We also discuss spin-polarized Fermi-liquid in the general context of the Fermi-liquid theory and compare it with "Fermi liquid" with spontaneous magnetization.

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I. INTRODUCTION

The relaxation properties in a degenerate Fermi liquid are determined by the collisions of quasiparticles. Due to the Pauli exclusion principle only the quasiparticles near the Fermi surface in a layer with thickness of the order of temperature are effectively exchanged by energy and momentum. Consequently, the relaxation time is proportional to T^{-2} that leads to the temperature dependences of kinetic coefficients^{1,2} of viscosity $\eta \propto T^{-2}$ and thermal conductivity $\kappa \propto T^{-1}$. The longitudinal spin-diffusion coefficient D_{\parallel} determining the spin current in presence of a gradient of the absolute value of magnetization has been found³ proportional to T^{-2} . So, kinetic coefficients diverge when temperature tends to zero. A similar situation takes place in a spin-polarized Fermi liquid, where there are two Fermi distributions for spin-up and spin-down particles with different Fermi momenta p_F^{\uparrow} and p_F^{\downarrow} . So long as we deal with relaxation processes being determined by the collisions of quasiparticles from thermal vicinity of one of the Fermi surfaces [scattering of two quasiparticles with spins up (down)] or scattering of quasiparticles from thermal vicinities of two different Fermi surfaces (scattering of spin up and spin down quasiparticles), Refs. 4–6, the relaxation time and kinetic coefficients of viscosity and longitudinal spin diffusion are proportional to T^{-2} and the thermal conductivity to T^{-1} .

Another type of relaxation process characterizes the spin current due to gradient of direction of magnetization or so-called transverse spin diffusion.⁷ Indeed at $T=0$ all the states with momenta below the smaller of two Fermi surfaces (say for $p < p_F^{\downarrow}$) and with plus one-half or minus one-half projections of the spin to an arbitrary oriented quantization axis are completely occupied: $W_{\mathbf{p}}^{\uparrow} = W_{\mathbf{p}}^{\downarrow} = 1$. Hence the inhomogeneous rotation of magnetization does not change the equilibrium state of quasiparticles with momenta $p < p_F^{\downarrow}$. But for the Fermi particles with spin up and momenta in between two Fermi surfaces $p_F^{\downarrow} < p < p_F^{\uparrow}$ the probability to have spin-up projection to the rotated quantization axis, determined by equilibrium direction of magnetization, is deviated

from unity. Hence the relaxation process should involve all such particles even at $T=0$. In Fermi liquid this independent particle picture is changed due to interaction creating a dissipationless inhomogeneous magnetization rotation. However, in the presence of finite polarization a dissipative transverse diffusion motion is also present. Corresponding relaxation time does not diverge at zero temperature^{7–11} and transverse spin waves attenuate at $T=0$.

The calculations of transverse spin-diffusion coefficient D_{\perp} have been done in dilute degenerate Fermi gas with arbitrary polarization at $T=0$ in the papers by Jeon and Mullin,⁷ Meyerovich and Musaelian,⁸ and at $T \neq 0$ in Ref. 9. A derivation and an exact solution of the kinetic equation in the s -wave scattering approximation for dilute degenerate Fermi gas with arbitrary polarization at $T=0$ and for a small polarization $\mu H \ll \varepsilon_F$ at $T \neq 0$ have been obtained also in the papers by Golosov and Ruckenstein.¹⁰ For the treatment of this problem in a Fermi liquid the Matthiessen-type rule arguments (sum of temperature-driven and polarization-driven scattering rates) and simple relaxation-time approximation for the collision integral have been used.¹¹ Thus the zero-temperature attenuation of the transverse spin waves has been established.

This conclusion has been contested by Fomin,¹² who has proposed dissipationless spin-wave dispersion relation

$$\omega = \omega_L + \chi k^2, \quad (1)$$

where χ is a coefficient of proportionality between spin current and the chiral spin velocity arising at an inhomogeneous rotation of spin space. Fomin has not presented a calculation of this value. He just has established the spin-wave dispersion law in the assumptions that in a polarized Fermi liquid at $T=0$ all the spin current is the chiral current and it can be derived as response to the generalized gauge transformation or inhomogeneous spin space rotation like it has been done for superfluid ³He.^{13,14}

Certainly, if in the process of such a type of derivation we shall ignore the quasiparticles' finite scattering rate, one can obtain the dissipationless spin current originating from

Fermi-liquid interaction. However, besides the reactive part the total spin current calculated in presence of collisions includes dissipative or spin-diffusion part resulting in imaginary part of dispersion law for the transverse spin waves. At the microscopic level the collisions treatment is equivalent to derivation of kinetic equation that was done for the case of spin-polarized diluted Fermi gas.⁷⁻¹⁰

Being addressed to the same problem in polarized Fermi liquid, we need the kinetic equation. The derivation of it for a strongly interacting Fermi liquid is unreal problem. However, for a weakly polarized Fermi liquid it seems natural to work on the basis of semi-phenomenological Silin-type kinetic equation.¹⁵ Assuming its validity in the present paper we reexamine the derivation of transverse spin dynamics in weakly polarized Fermi liquid for arbitrary relationship between temperature T and polarization γH . At small space and time variations of transversal part of vectorial quasiparticle distribution function we shall obtain Leggett-type¹⁶ equations for spin and spin current densities. Then from the general form of two-particle collision integral similar to that was derived in Refs. 17,18 we deduce the spin current relaxation term. The latter is essentially simplified when both the temperature T and the polarization γH are much smaller than Fermi energy ε_F and the momenta of all excitations are confined to lie in the vicinity of both Fermi surfaces and therefore one may decouple the angular and energy variables in the collision integral in the manner first introduced by Abrikosov and Khalatnikov.²

We confirm the results of the papers⁷⁻¹⁰ where the same problem were treated for the dilute Fermi gas. It is found in particular that at finite polarization spin-wave damping has a finite value at $T=0$. More precisely, at low temperatures it proves to be proportional to the number of collisions between quasiparticles

$$\frac{1}{\tau} \propto [(\gamma H)^2 + (2\pi T)^2]. \quad (2)$$

This corresponds to the law of zero sound attenuation¹⁹

$$\gamma \propto [\omega^2 + (2\pi T)^2], \quad (3)$$

which is also determined by the number of collisions between quasiparticles. One can find the results of recent measurements of low-temperature zero sound attenuation and surway of previous experimental works on this subject in Ref. 20.

In general, all regimes of the temperature behavior of the spin-wave absorption in the degenerate Fermi liquid can be juxtaposed with correspondent regimes in the absorption of ultrasound. We shall consider this analogy in the conclusion.

At the end we shall discuss spin-polarized Fermi liquid in the general context of the Fermi-liquid theory and compare it with an imaginary ferromagnetic Fermi liquid or liquid with spontaneous magnetization.

II. SPIN DYNAMICS EQUATIONS

The quasiparticle distribution function as well as quasiparticle energy are given by 2×2 matrix in spin space,

$$\hat{n}_{\mathbf{k}}(\mathbf{r}, t) = n_{\mathbf{k}}(\mathbf{r}, t) \hat{I} + \boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t) \hat{\boldsymbol{\sigma}}, \quad (4)$$

$$\hat{\varepsilon}_{\mathbf{k}}(\mathbf{r}, t) = \varepsilon_{\mathbf{k}}(\mathbf{r}, t) \hat{I} + \mathbf{h}_{\mathbf{k}}(\mathbf{r}, t) \hat{\boldsymbol{\sigma}}. \quad (5)$$

Here $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are Pauli matrices. The scalar and vector parts of matrix distribution function obey the coupled kinetic equations.¹⁵ As it was pointed out by Leggett¹⁶ in the case of small polarizations, the equation for the scalar part of the distribution function $n_{\mathbf{k}}(\mathbf{r}, t)$ decouples from the equation for the vector part of distribution function $\boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t)$ and we may put $n_{\mathbf{k}}$ equal to its equilibrium value, namely, usual Fermi function. At that the equation for $\boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t)$ still shall be nonlinear: as long as the polarization is small by its absolute value one can consider the arbitrary large variations of the direction of magnetization. On the other hand, the similar decoupling of the equation for the scalar part of the distribution function from the equation for the vector part of distribution function including the collision integrals (see below) takes place at arbitrary polarizations as long as we consider the small deviations of the magnetization direction from its equilibrium direction. We shall be interested in the latter case.

In general, the equation for the $\boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t)$ has the form

$$\frac{\partial \boldsymbol{\sigma}_{\mathbf{k}}}{\partial t} + \frac{\partial \varepsilon_{\mathbf{k}}}{\partial k_i} \frac{\partial \boldsymbol{\sigma}_{\mathbf{k}}}{\partial x_i} - \frac{\partial \varepsilon_{\mathbf{k}}}{\partial x_i} \frac{\partial \boldsymbol{\sigma}_{\mathbf{k}}}{\partial k_i} + \frac{\partial \mathbf{h}_{\mathbf{k}}}{\partial k_i} \frac{\partial n_{\mathbf{k}}}{\partial x_i} - \frac{\partial \mathbf{h}_{\mathbf{k}}}{\partial x_i} \frac{\partial n_{\mathbf{k}}}{\partial k_i} - 2(\mathbf{h}_{\mathbf{k}} \times \boldsymbol{\sigma}_{\mathbf{k}}) = \left(\frac{\partial \boldsymbol{\sigma}_{\mathbf{k}}}{\partial t} \right)_{coll}. \quad (6)$$

We divide all matrices in equilibrium and nonequilibrium parts,

$$\hat{n}_{\mathbf{k}} = \hat{n}_{\mathbf{k}}^0 + \delta \hat{n}_{\mathbf{k}}, \quad (7)$$

$$\hat{\varepsilon}_{\mathbf{k}} = \hat{\varepsilon}_{\mathbf{k}}^0 + \delta \hat{\varepsilon}_{\mathbf{k}}, \quad (8)$$

where

$$\hat{n}_{\mathbf{k}}^0 = \frac{1}{2}(n_0^+ + n_0^-) \hat{I} + \frac{1}{2}(n_0^+ - n_0^-) (\hat{\boldsymbol{\sigma}} \hat{\mathbf{m}}) \quad (9)$$

is the equilibrium distribution function of polarized Fermi liquid and

$$\hat{\varepsilon}_{\mathbf{k}}^0 = \varepsilon_{\mathbf{k}} \hat{I} - \frac{1}{2} \gamma (\mathbf{B} \hat{\boldsymbol{\sigma}}) \quad (10)$$

is the equilibrium quasiparticle energy. Here there are two Fermi distribution functions

$$n_0^{\pm}(\varepsilon_{\mathbf{k}}) = n_0 \left(\varepsilon_{\mathbf{k}} \mp \frac{\gamma H}{2} \right) = \frac{1}{\exp \left(\frac{\varepsilon_{\mathbf{k}} \mp \frac{\gamma H}{2} - \mu}{T} \right) + 1} \quad (11)$$

shifted on the value of polarization $\gamma H/2$, γ is the gyromagnetic ratio, Planck constant $\hbar=1$ throughout the paper, the polarization direction is determined by the unit vector $\mathbf{m}=\mathbf{H}/H$.

The ‘‘effective’’ magnetic field \mathbf{H} is the field corresponding to the magnetization created by the external magnetic field \mathbf{H}_0 and by the pumping.²¹ The pumped part in view of very long time of longitudinal relaxation should be considered as equilibrium part of magnetization. The difference between \mathbf{B} and \mathbf{H} originates from the pumping that changes the quasiparticle distribution functions but does not directly affect on the energy of quasiparticles. Field \mathbf{B} consists of an external magnetic field \mathbf{H}_0 and the Fermi-liquid molecular field. To define \mathbf{B} we must consider the equilibrium distribution matrix (9) and equilibrium energy matrix (10) as deviations from the corresponding matrices for nonpolarized Fermi liquid,

$$\hat{n}_{\mathbf{k}}^0 = n_0(\varepsilon_{\mathbf{k}})\hat{I} + \delta\hat{n}_{\mathbf{k}}^0, \quad (12)$$

$$\begin{aligned} \hat{\varepsilon}_{\mathbf{k}}^0 &= \varepsilon_{\mathbf{k}}\hat{I} - \frac{1}{2}\gamma(\mathbf{B}\hat{\sigma}) = \varepsilon_{\mathbf{k}}\hat{I} - \frac{1}{2}\gamma(\mathbf{H}_0\hat{\sigma}) \\ &+ \frac{1}{2}Sp' \int d\tau' f_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'} \delta\hat{n}_{\mathbf{k}'}^0, \end{aligned} \quad (13)$$

where $d\tau=2d\mathbf{k}/(2\pi)^3$ and the Fermi-liquid matrix of interaction is

$$\begin{aligned} f_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'} &= f_{\mathbf{k}\mathbf{k}'}^s \hat{I}\hat{I}' + f_{\mathbf{k}\mathbf{k}'}^a \hat{\sigma}\hat{\sigma}' + f_{\mathbf{k}\mathbf{k}'}^b \hat{\mathbf{m}}(\hat{\sigma}\hat{I}' + \hat{I}\hat{\sigma}') \\ &+ f_{\mathbf{k}\mathbf{k}'}^c (\hat{\mathbf{m}}\hat{\sigma})(\hat{\mathbf{m}}\hat{\sigma}'). \end{aligned} \quad (14)$$

These three equation together give an equation for \mathbf{B} determination

$$\begin{aligned} \gamma\mathbf{B} &= \gamma\mathbf{H}_0 - \hat{\mathbf{m}} \int d\tau' [(f_{\mathbf{k}\mathbf{k}'}^a + f_{\mathbf{k}\mathbf{k}'}^c)(n_0^+ - n_0^-) \\ &+ f_{\mathbf{k}\mathbf{k}'}^b (n_0^+ + n_0^- - 2n_0)]. \end{aligned} \quad (15)$$

In the absence of a pumped magnetization the field, $\mathbf{B}=\mathbf{H}$ and Eq. (15) is just the self-consistency equation for the field \mathbf{H} determination as the function of an external field \mathbf{H}_0 . When the part of magnetization is created by pumping, \mathbf{H} presents an independent value and the total energy shift $\gamma(\mathbf{B}\hat{\sigma})/2$ is determined by means of two fields: external \mathbf{H}_0 and effective \mathbf{H} . We shall assume that they are parallel each other.

For the finite polarization the vector \mathbf{B} proves to be energy dependent. This reflects the impossibility to formulate a Fermi-liquid theory with finite polarization in terms of Landau Fermi-liquid parameters which are just numbers characterizing the intensity of interaction of quasiparticles near the Fermi surface. The problem is not resolved even by introduction of Fermi-liquid parameters separately for each Fermi sphere with Fermi momenta p_F^\downarrow and p_F^\uparrow . To avoid this complexity we shall limit ourselves by the case of small polarization assuming independence of the functions $f_{\mathbf{k}\mathbf{k}'}$ of energy. Then it is clear that \mathbf{B} is energy independent and

determined just by zeroth-order term in expansion of the functions $f_{\mathbf{k}\mathbf{k}'}^i$ on spherical harmonics.

We shall discuss the only perpendicular deviations from the initial equilibrium state,

$$\delta\hat{n}_{\mathbf{k}} = \delta\sigma_{\mathbf{k}}(\mathbf{r},t)\hat{\sigma}, \quad (\hat{\mathbf{m}}\delta\sigma_{\mathbf{k}}) = 0. \quad (16)$$

Then the energy deviation matrix has the form

$$\delta\hat{\varepsilon}_{\mathbf{k}} = \delta\mathbf{h}_{\mathbf{k}}\hat{\sigma}, \quad \delta\mathbf{h}_{\mathbf{k}} = \int d\tau' f_{\mathbf{k}\mathbf{k}'}^a \delta\sigma_{\mathbf{k}'} \quad (17)$$

and the kinetic equation (6) can be rewritten as

$$\begin{aligned} \frac{\partial\delta\sigma_{\mathbf{k}}}{\partial t} + \frac{\partial\varepsilon_{\mathbf{k}}^0}{\partial k_i} \frac{\partial\delta\sigma_{\mathbf{k}}}{\partial x_i} - \frac{1}{2} \frac{\partial(n_0^+ + n_0^-)}{\partial k_i} \frac{\partial\delta\mathbf{h}_{\mathbf{k}}}{\partial x_i} \\ - 2 \left[\left(-\frac{\gamma\mathbf{B}}{2} + \delta\mathbf{h}_{\mathbf{k}} \right) \left(\frac{1}{2}(n_0^+ - n_0^-)\hat{\mathbf{m}} + \delta\sigma_{\mathbf{k}} \right) \right] \\ = \left(\frac{\partial\sigma_{\mathbf{k}}}{\partial t} \right)_{coll}. \end{aligned} \quad (18)$$

We deal with linear in $\delta\sigma_{\mathbf{k}}$ equation with coefficients independent of space and time variables. In the lowest order on polarization, these coefficients are expressed through Fermi-liquid parameters for nonpolarized Fermi liquid which are introduced as usual by

$$f_{\mathbf{k}\mathbf{k}'}^a = N_0^{-1} \sum_l F_l^a P_l(\hat{\mathbf{k}}, \hat{\mathbf{k}}'), \quad (19)$$

where $N_0 = m^*k_F/\pi^2$ is the density of states. As mentioned, for large polarizations the coefficients in the kinetic equation are not well determined, although the structure of the equation looks similar. So, for simplicity we limit ourselves by the treatment of the left hand side of the equation in the lowest order on polarization. One can neglect in this case F_l^b and F_l^c . Then following Leggett¹⁶ one may rewrite Eq. (18) as two equations for the first two harmonics (magnetization density and spin current density) of the distribution function

$$\mathbf{M}(\mathbf{r},t) = \frac{1}{2} \int d\tau \delta\sigma_{\mathbf{k}}, \quad (20)$$

$$\begin{aligned} \mathbf{J}_i(\mathbf{r},t) &= \frac{1}{2} \int d\tau \left[v_{Fi} \delta\sigma_{\mathbf{k}} - \frac{\partial n_0}{\partial k_i} \delta\mathbf{h}_{\mathbf{k}} \right] \\ &= \frac{1}{2} \left(1 + \frac{F_1^a}{3} \right) \int d\tau v_{Fi} \delta\sigma_{\mathbf{k}}. \end{aligned} \quad (21)$$

They are

$$\frac{\partial \mathbf{M}}{\partial t} + \frac{\partial \mathbf{J}_i}{\partial x_i} - \mathbf{M} \times \gamma \mathbf{H}_0 = 0, \quad (22)$$

$$\begin{aligned} \frac{\partial \mathbf{J}_i}{\partial t} + \frac{1}{3} v_F^2 (1 + F_0^a) \left(1 + \frac{F_1^a}{3} \right) \frac{\partial \mathbf{M}}{\partial x_i} - \mathbf{J}_i \times \gamma \mathbf{H}_0 \\ + \frac{4}{N_0} \left(F_0^a - \frac{F_1^a}{3} \right) (\mathbf{J}_i \times \mathbf{M}^{\parallel}) \\ = \frac{1}{2} \left(1 + \frac{F_1^a}{3} \right) \int d\tau v_{Fi} \left(\frac{\partial \boldsymbol{\sigma}_{\mathbf{k}}}{\partial t} \right)_{coll}. \end{aligned} \quad (23)$$

Here

$$\mathbf{M}^{\parallel} = \frac{\gamma N_0}{4} \mathbf{H}. \quad (24)$$

So far we developed the theory for the homogeneous external field. In this case one can excite transverse spin waves in an infinite medium. The situation has been described in Appendix A in Ref. 22. However, the theory is valid also when the magnetic field is coordinate dependent by its absolute value and includes a small but in general fast in time supplementary rf part directed in a perpendicular direction:

$$\begin{aligned} \mathbf{H}_0(\mathbf{r}, t) = \mathbf{H}_0(\mathbf{r}) + \mathbf{h}(\mathbf{r}, t), \quad \mathbf{H}_0(\mathbf{r}) = \hat{z} [H_0 + \delta H_0(\mathbf{r})], \\ (\hat{z} \mathbf{h}(\mathbf{r}, t)) = 0. \end{aligned} \quad (25)$$

This situation is typical for the spin-waves experiments (see, for instance Ref. 23). In this case we must introduce the additional longitudinal deviations in the distribution function and the energy of quasiparticles. So, Eqs. (16) and (17) are modified as follows:

$$\delta \hat{n}_{\mathbf{k}} = \delta \boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t) \hat{\boldsymbol{\sigma}} + \delta \sigma_{\mathbf{k}}^{\parallel}(\mathbf{r}, t) \hat{\sigma}_z, \quad \hat{z} \delta \boldsymbol{\sigma}_{\mathbf{k}} = 0, \quad (26)$$

$$\begin{aligned} \delta \hat{\varepsilon}_{\mathbf{k}} = -\frac{1}{2} \gamma [\hat{z} \delta H_0(\mathbf{r}) + \mathbf{h}(\mathbf{r}, t)] \hat{\boldsymbol{\sigma}} + \delta \mathbf{h}_{\mathbf{k}} \hat{\boldsymbol{\sigma}} \\ + \int d\tau' f_{\mathbf{k}\mathbf{k}'}^a \delta \sigma_{\mathbf{k}'}^{\parallel} \hat{\sigma}_z, \\ \delta \mathbf{h}_{\mathbf{k}} = \int d\tau' f_{\mathbf{k}\mathbf{k}'}^a \delta \boldsymbol{\sigma}_{\mathbf{k}'}. \end{aligned} \quad (27)$$

In linear approximation to the perpendicular to \hat{z} deviations the equations for the perpendicular and the parallel parts of the distribution function including the collision integral are independent. Thus for the transversal part we return back to the slightly modified system of equations (22) and (23):

$$\frac{\partial \mathbf{M}}{\partial t} + \frac{\partial \mathbf{J}_i}{\partial x_i} - \mathbf{M} \times \gamma \mathbf{H}_0(\mathbf{r}) - [\mathbf{M}^{\parallel} + \delta \mathbf{M}^{\parallel}(\mathbf{r})] \times \gamma \mathbf{h}(\mathbf{r}, t) = 0, \quad (28)$$

$$\begin{aligned} \frac{\partial \mathbf{J}_i}{\partial t} + \frac{1}{3} v_F^2 (1 + F_0^a) \left(1 + \frac{F_1^a}{3} \right) \frac{\partial \mathbf{M}}{\partial x_i} - \mathbf{J}_i \times \gamma \mathbf{H}_0(\mathbf{r}) \\ + \frac{4}{N_0} \left(F_0^a - \frac{F_1^a}{3} \right) (\mathbf{J}_i \times \mathbf{M}^{\parallel}) \\ = \frac{1}{2} \left(1 + \frac{F_1^a}{3} \right) \int d\tau v_{Fi} \left(\frac{\partial \boldsymbol{\sigma}_{\mathbf{k}}}{\partial t} \right)_{coll}, \end{aligned} \quad (29)$$

where $\delta \mathbf{M}^{\parallel}(\mathbf{r})$ is a change of spin density due to $\delta H(\mathbf{r})$.

III. COLLISION INTEGRAL TREATMENT

The collision integral for the vectorial part of distribution function is determined through the general collision integral for the matrix distribution function as follows:

$$\left(\frac{\partial \boldsymbol{\sigma}_{\mathbf{k}}}{\partial t} \right)_{coll} = \frac{1}{2} S p \hat{\boldsymbol{\sigma}} \hat{I}_{coll}, \quad (30)$$

$$\begin{aligned} I_{coll}^{\alpha\beta} = -\frac{1}{4} \int d\tau_1' d\tau_2' d\mathbf{k}_2' \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') \\ \times \delta(\mathbf{k} + \mathbf{k}_1' - \mathbf{k}_2 - \mathbf{k}_2') F^{\alpha\beta}(\mathbf{k}, \mathbf{k}_1', \mathbf{k}_2, \mathbf{k}_2'). \end{aligned} \quad (31)$$

Here in absence of relativistic interactions $\varepsilon_1 = \varepsilon_{\mathbf{k}}$, $\varepsilon_2 = \varepsilon_{\mathbf{k}_2}$, etc. For only two-particle collisions the function $F^{\alpha\beta}$ must contain the products of the matrix distribution functions of the general form

$$(\hat{n}_{\mathbf{k}})^{\lambda_1 \lambda_2} (\hat{I} - \hat{n}_{\mathbf{k}_1'})^{\lambda_3 \lambda_4} (\hat{n}_{\mathbf{k}_2})^{\lambda_5 \lambda_6} (\hat{I} - \hat{n}_{\mathbf{k}_2'})^{\lambda_7 \lambda_8}$$

multiplied on some tensor function depending of all momenta and the spin indices. However, in absence of relativistic interactions in such the products the only matrix products of two types are possible:

$$\frac{1}{2} \{ [\hat{n}_{\mathbf{k}} (\hat{I} - \hat{n}_{\mathbf{k}_1'})]^{\alpha\beta} + [(\hat{I} - \hat{n}_{\mathbf{k}_1'}) \hat{n}_{\mathbf{k}}]^{\alpha\beta} \} S p [\hat{n}_{\mathbf{k}_2} (\hat{I} - \hat{n}_{\mathbf{k}_2'})]$$

and

$$\begin{aligned} \frac{1}{2} \{ [\hat{n}_{\mathbf{k}} (\hat{I} - \hat{n}_{\mathbf{k}_1'}) \hat{n}_{\mathbf{k}_2} (\hat{I} - \hat{n}_{\mathbf{k}_2'})]^{\alpha\beta} \\ + [(\hat{I} - \hat{n}_{\mathbf{k}_1'}) \hat{n}_{\mathbf{k}_2} (\hat{I} - \hat{n}_{\mathbf{k}_2'}) \hat{n}_{\mathbf{k}}]^{\alpha\beta} \}. \end{aligned}$$

As usual, in quantum mechanics one must take the symmetrized (Hermitian) products of operators and corresponding matrices. Adding to this expressions describing the scattering processes “going out” of initial state, the corresponding expressions for processes for “going in” initial state we can write the general form for the function \hat{F} determining the scattering integral for the binary collisions

$$\begin{aligned}
 F^{\alpha\beta}(\mathbf{k}, \mathbf{k}_1', \mathbf{k}_2, \mathbf{k}_2') &= \frac{1}{2} W_1(\mathbf{k}, \mathbf{k}_1', \mathbf{k}_2, \mathbf{k}_2') \left(\{[\hat{n}_{\mathbf{k}}(\hat{I} - \hat{n}_{\mathbf{k}_1'})]^{\alpha\beta} + [(\hat{I} - \hat{n}_{\mathbf{k}_1'})\hat{n}_{\mathbf{k}}]^{\alpha\beta}\} Sp[\hat{n}_{\mathbf{k}_2}(\hat{I} - \hat{n}_{\mathbf{k}_2'})] - \{[(\hat{I} - \hat{n}_{\mathbf{k}})\hat{n}_{\mathbf{k}_1'}]^{\alpha\beta} \right. \\
 &+ [\hat{n}_{\mathbf{k}_1'}(\hat{I} - \hat{n}_{\mathbf{k}})]^{\alpha\beta}\} Sp[(\hat{I} - \hat{n}_{\mathbf{k}_2})\hat{n}_{\mathbf{k}_2'}] + \frac{1}{2} W_2(\mathbf{k}, \mathbf{k}_1', \mathbf{k}_2, \mathbf{k}_2') \{[\hat{n}_{\mathbf{k}}(\hat{I} - \hat{n}_{\mathbf{k}_1'})\hat{n}_{\mathbf{k}_2}(\hat{I} - \hat{n}_{\mathbf{k}_2'})]^{\alpha\beta} \\
 &+ [(\hat{I} - \hat{n}_{\mathbf{k}_1'})\hat{n}_{\mathbf{k}_2}(\hat{I} - \hat{n}_{\mathbf{k}_2'})\hat{n}_{\mathbf{k}}]^{\alpha\beta} - [(\hat{I} - \hat{n}_{\mathbf{k}})\hat{n}_{\mathbf{k}_1'}(\hat{I} - \hat{n}_{\mathbf{k}_2})\hat{n}_{\mathbf{k}_2'}]^{\alpha\beta} - [\hat{n}_{\mathbf{k}_1'}(\hat{I} - \hat{n}_{\mathbf{k}_2})\hat{n}_{\mathbf{k}_2'}(\hat{I} - \hat{n}_{\mathbf{k}})]^{\alpha\beta} \}. \quad (32)
 \end{aligned}$$

Due to the total quasiparticle density $Sp f d\mathbf{k} \hat{I}_{coll}$ and the total quasiparticle spin density $Sp \hat{\boldsymbol{\sigma}} f d\mathbf{k} \hat{I}_{coll}/2$ conservation, the functions W_1 and W_2 obey the following conditions: $W_1(\mathbf{k}, \mathbf{k}_1', \mathbf{k}_2, \mathbf{k}_2') = W_1(\mathbf{k}_1', \mathbf{k}, \mathbf{k}_2', \mathbf{k}_2)$ and $W_2(\mathbf{k}, \mathbf{k}_1', \mathbf{k}_2, \mathbf{k}_2') = W_2(\mathbf{k}_1', \mathbf{k}_2, \mathbf{k}_2', \mathbf{k})$. The two-particle collision integral for the matrix distribution function determined by Eqs. (31) and (32) corresponds to collision integral derived in Born approximation by Silin¹⁷ (see also Ref. 18). When all the distribution function matrices are diagonal, the collision integral (26), Eq. (29) reduces to the diagonal form of two-particle collision integral²⁴ with $W_1 = 2W_{\uparrow\downarrow}$ and $W_1 + W_2 = W_{\uparrow\uparrow}$.

For the case of low temperature $T \ll \varepsilon_F$ and small polarization $\gamma H \ll \varepsilon_F$ when all the quasiparticle momenta of scattering particles lie near the Fermi surface of nonpolarized Fermi liquid one can suppose as in Ref. 2 that functions W_1 and W_2 depend only on the angle θ between \mathbf{k} and \mathbf{k}_2 and on the angle ϕ between the planes $(\mathbf{k}, \mathbf{k}_2)$ and $(\mathbf{k}_1', \mathbf{k}_2')$. Now we must take Eq. (32) in the linear approximation on devia-

tions $\delta \hat{n}$. We deal only with the terms containing deviations $\delta \hat{n}_{\mathbf{k}} = \delta \boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t) \hat{\boldsymbol{\sigma}}$ and do not consider the terms containing deviations $\delta \hat{n}_{\mathbf{k}_1'}$ etc. because after all integrations in the collision integral (31) they are independent of the \mathbf{k} direction (equilibrium distribution matrix $\hat{n}_{\mathbf{k}}^0$ is isotropic in the \mathbf{k} space). Hence they disappear at final stage after integration in the right hand side of Eq. (29). We choose the local direction of the quantization axis $\hat{\mathbf{m}}$ along \hat{z} direction, such that $(\hat{z} \delta \boldsymbol{\sigma}_{\mathbf{k}}) = 0$. We can perform the integration over \mathbf{k}_2' in Eq. (31) eliminating δ function of momenta and also, following the procedure of Ref. 2 reproduced in Ref. 25 in somewhat different manner, reexpress the integration over momentum space as

$$d\mathbf{k}_2 d\mathbf{k}_1' = \frac{(m^*)^3}{2 \cos(\theta/2)} d\varepsilon_2 d\varepsilon_1' d\varepsilon_2' \sin \theta d\theta d\phi d\phi_2.$$

So, the linear part of the collision integral is

$$\begin{aligned}
 \delta \hat{I}_{coll} &= -\delta \hat{n}_{\mathbf{k}} \frac{m^{*3}}{2(2\pi)^5} \int d\varepsilon_2 d\varepsilon_1' d\varepsilon_2' \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') \{ \bar{W}_1 \{ [1 - n_0^+(\varepsilon_1') + 1 - n_0^-(\varepsilon_1')] [n_0^+(\varepsilon_2)(1 - n_0^+(\varepsilon_2')) \\
 &+ n_0^-(\varepsilon_2)(1 - n_0^-(\varepsilon_2'))] + [n_0^+(\varepsilon_1') + n_0^-(\varepsilon_1')] [(1 - n_0^+(\varepsilon_2))n_0^+(\varepsilon_2') + (1 - n_0^-(\varepsilon_2))n_0^-(\varepsilon_2')] \} \\
 &+ \bar{W}_2 \{ (1 - n_0^+(\varepsilon_1'))n_0^+(\varepsilon_2)(1 - n_0^+(\varepsilon_2')) + (1 - n_0^-(\varepsilon_1'))n_0^-(\varepsilon_2)(1 - n_0^-(\varepsilon_2')) \\
 &+ n_0^+(\varepsilon_1')(1 - n_0^+(\varepsilon_2))n_0^+(\varepsilon_2') + n_0^-(\varepsilon_1')(1 - n_0^-(\varepsilon_2))n_0^-(\varepsilon_2') \} \}, \quad (33)
 \end{aligned}$$

where

$$\bar{W}_i = \int W_1(\theta, \phi) \sin \frac{\theta}{2} d\theta d\phi, \quad i=1,2. \quad (34)$$

Integration over energies is easily performed. Let us do it for one particular term in this expression.

$$\begin{aligned}
 &\int \int d\varepsilon_1' d\varepsilon_2' [1 - n_0^+(\varepsilon_1')] n_0^+(\varepsilon_1' + \varepsilon_2' - \varepsilon_1) \\
 &\times [1 - n_0^+(\varepsilon_2')] \\
 &= T^2 \int \int dx dy [1 - f(x)] f(x + y - t + h) [1 - f(y)]
 \end{aligned}$$

$$\begin{aligned}
 &= T^2 \int dx f(-x) \frac{x + h - t}{e^{x+h-t} - 1} \\
 &= T^2 \frac{\pi^2 + (h-t)^2}{2} f(h-t). \quad (35)
 \end{aligned}$$

Here $x = (\varepsilon_1' - \mu - \gamma H/2)/T$, $y = (\varepsilon_2' - \mu - \gamma H/2)/T$, $h = \gamma H/2T$, and $t = (\varepsilon - \mu)/T$, $f(x) = (e^x + 1)^{-1}$. It should be stressed that the definitions of variables of integration x and y depend on particular products like $n^+(1 - n^-)n^-$ under the integral. On the contrary, the variable t and the parameter h have an invariant definition for all the terms. Thus we obtain

$$\begin{aligned}
\delta\hat{l}_{coll} &= -\delta\hat{n}_{\mathbf{k}} \frac{m^{*3}}{2(2\pi)^5} (2\bar{W}_1 + \bar{W}_2) \frac{T^2}{2} [(\pi^2 + (h-t)^2) \\
&\quad \times f(h-t) + (\pi^2 + (h+t)^2)f(-h-t) \\
&\quad + (\pi^2 + (t-h)^2)f(t-h) + (\pi^2 + (t+h)^2)f(t+h)] \\
&= -\delta\hat{n}_{\mathbf{k}} \frac{m^{*3}}{2(2\pi)^5} (2\bar{W}_1 + \bar{W}_2) \\
&\quad \times \left[(\pi T)^2 + \left(\frac{\gamma H}{2} \right)^2 + (\varepsilon - \mu)^2 \right]. \quad (36)
\end{aligned}$$

Now we must substitute this expression in the right-hand side of Eq. (29),

$$\begin{aligned}
&\frac{1}{2} \left(1 + \frac{F_1^a}{3} \right) \frac{1}{2} S p \int d\tau v_{Fi} (\hat{\boldsymbol{\sigma}} \delta\hat{l}_{coll}) \\
&= -\frac{1}{2} \left(1 + \frac{F_1^a}{3} \right) \frac{m^{*3}}{2(2\pi)^5} (2\bar{W}_1 + \bar{W}_2) \\
&\quad \times \int d\tau v_{Fi} \left[(\pi T)^2 + \left(\frac{\gamma B}{2} \right)^2 + (\varepsilon - \mu)^2 \right] \delta\boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t). \quad (37)
\end{aligned}$$

Taking into account the definition (21) one can directly express part of this integral containing $(\pi T)^2 + (\gamma H/2)^2$ through the spin current density. The integral as a whole is not in general expressed in terms of the current. That prevents to consider Eqs. (28) and (29) as the closed system of equations for the spin density \mathbf{M} and the spin current density \mathbf{J}_i . However, one can make an assumption which is plausible for weakly polarized Fermi liquid that the energy dependence of $\delta\boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t)$ is factorized from the space and direction of $\hat{\mathbf{k}}$ dependences:

$$\delta\boldsymbol{\sigma}_{\mathbf{k}}(\mathbf{r}, t) \propto [n_0^+(\varepsilon) - n_0^-(\varepsilon)] [\mathbf{A}(\mathbf{r}, t) + \mathbf{B}_i(\mathbf{r}, t) \hat{k}_i]. \quad (38)$$

In this case in the lowest order of the ratio $\gamma H/\mu$ one can rewrite the expression (37) as

$$-\frac{m^{*3}}{6(2\pi)^5} (2\bar{W}_1 + \bar{W}_2) [(2\pi T)^2 + (\gamma H)^2] \mathbf{J}_i(\mathbf{r}, t). \quad (39)$$

IV. RESULTS AND DISCUSSION

So, finally we have come to the closed system of equations for the spin density \mathbf{M} and the spin current density \mathbf{J}_i ,

$$\begin{aligned}
\frac{\partial \mathbf{M}}{\partial t} + \frac{\partial \mathbf{J}_i}{\partial x_i} - \mathbf{M} \times \boldsymbol{\gamma} \mathbf{H}_0(\mathbf{r}) - [\mathbf{M}^{\parallel} + \delta\mathbf{M}^{\parallel}(\mathbf{r})] \times \boldsymbol{\gamma} \mathbf{h}(\mathbf{r}, t) = 0, \\
\quad (40)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{J}_i}{\partial t} + \frac{1}{3} v_F^2 (1 + F_0^a) \left(1 + \frac{F_1^a}{3} \right) \frac{\partial \mathbf{M}}{\partial x_i} - \mathbf{J}_i \times \boldsymbol{\gamma} \mathbf{H}_0(\mathbf{r}) \\
+ \frac{4}{N_0} \left(F_0^a - \frac{F_1^a}{3} \right) (\mathbf{J}_i \times \mathbf{M}^{\parallel}) = -\frac{\mathbf{J}_i}{\tau}, \quad (41)
\end{aligned}$$

where the current relaxation time is

$$\frac{1}{\tau} = \frac{m^{*3}}{6(2\pi)^5} (2\bar{W}_1 + \bar{W}_2) [(2\pi T)^2 + (\gamma H)^2]. \quad (42)$$

This system has the same structure as the Leggett spin dynamic equations.¹⁶ However, unlike the Leggett equations these are linear equations for the \mathbf{M} and \mathbf{J}_i in presence of small but finite polarization. They contain all the information about spin rotation or Leggett-Rice effect. The current relaxation time is proved to be finite at zero temperature thereby the result that has been obtained earlier for the dilute Fermi gas^{9,10} is confirmed.

The damping response of the transverse magnetization on the transverse rf field $\mathbf{h}(\mathbf{r}, t)$ has been found in Ref. 23 and in more rigorous manner in Ref. 22. The corresponding transverse magnetization fluctuations can be established by means of standard fluctuation-dissipation relations.

The known dispersion law of the transversal spin waves following from Eqs. (40) and (41) is

$$\omega = \omega_L + (D'' - iD')k^2, \quad (43)$$

where $\omega_L = \gamma H_0$ is the Larmor frequency,

$$D' = \frac{v_F^2 (1 + F_0^a) (1 + F_1^a/3) \tau}{3[1 + (\kappa \tau \gamma H)^2]}, \quad D'' = \kappa \tau \gamma H D' \quad (44)$$

are correspondingly the diffusion coefficient and its reactive part, $\kappa = F_0^a - F_1^a/3$. The spin wave damping is determined by diffusion coefficient D' . In hydrodynamic region $|\kappa| \tau \gamma H \ll 1$ its temperature dependence is determined by the time of scattering $\tau \propto T^{-2}$. Then, passing through the maximum at $|\kappa| \tau \gamma H \sim 1$ (Leggett-Rice effect, see Ref. 26), at lower temperatures $|\kappa| \tau \gamma H > 1$ the diffusion coefficient starts to be proportional to the number of collisions between the quasiparticles $\propto \tau^{-1}$. Finally, at very low temperatures $T < \gamma H/2\pi$ and finite polarization it comes to the finite constant value. Thus the transverse spin waves in a Fermi liquid with finite polarization have a finite attenuation at $T=0$.

The behavior of transverse spin wave damping is similar to, and in fact has the same origin as, the attenuation of ultrasound with frequency ω in a degenerate Fermi liquid.^{19,2} The latter decreases as $\tau \propto T^{-2}$ in hydrodynamic region $\omega \tau \ll 1$, then it passes through the maximum at $\omega \tau \sim 1$ and behaves as the number of collisions $\propto 1/\tau \propto [\omega^2 + (2\pi T)^2]$ in collisionless region $\omega \tau \gg 1$ (see Ref. 27). It keeps the finite value $\propto \omega^2$ at $T=0$ (see Ref. 20).

It will be appropriate to repeat here the conditions under which our derivation is valid. The most important is that we were working in the linear on the space and time variations of the transverse part of vectorial quasiparticle distribution

function. In polarized Fermi liquid the large deviations of the magnetization direction are always accompanied by the changes of its longitudinal part, therefore we cannot uncouple the kinetic equations for the scalar and vectorial distribution functions in a Fermi liquid with finite polarization. We also lose the possibility to transform the matrix products in the collision integral as we did.

Thus the important point in our treatment of transversal spin motion was its independence of longitudinal degrees of freedom. So long we consider only linear dynamics there is no coupling between transversal and longitudinal parts unless we do not take into account spin nonconserving collisions (see below). In that sense we do not see a necessity in limitations of the developed here theory like $|\omega - \omega_L| \tau \ll 1$ [see Eq. (43)] with both τ^\perp given by Eq. (42) and $\tau^\parallel \propto T^{-2}$ as discussed in Ref. 10. Nevertheless, it is worth noting that till now the measurements have been performed in frame of these hydrodynamic conditions.²⁸

Another important assumption is the condition of the weak polarization $\gamma H \ll \varepsilon_F$ of degenerate Fermi liquid with $T \ll \varepsilon_F$, however, with arbitrary relation between polarization γH and temperature T . This confines all the quasiparticle momenta to lie in the vicinity of the Fermi surface of nonpolarized Fermi liquid, and therefore one may decouple the energy and the angular variables in the collision integral like it was done in Ref. 2 and finally to obtain the closed system of the equations.

Experimentally, the transverse spin current relaxation time is determined by measurements of the spin echo attenuation or damping of the standing spin waves. The results on whether or not the transverse relaxation time saturates at low temperatures in spin-polarized Fermi liquids so far have been contradictory. The recent spin echo experiments most probably suggest that spin current relaxation remains finite as temperature tends to zero.²⁹⁻³¹ At first sight the large angle deviations, which are the principal feature of the spin echo method, prevent of application of our theory where the transverse spin attenuation is calculated in linear on the transverse perturbation approximation. Nevertheless, the linear theory does work because the fastly varying in time “transverse” magnetization is always kept small during the whole course of the spin echo experiment if one chooses as the natural axis of quantization slowly varying in time the direction of local magnetization.³²

On the contrary, the direct measurements of spin waves³³ demonstrates much smaller damping than expected on the basis of the spin echo experiments. Here, however, the absence of zero-temperature attenuation is probably masked by not enough precise temperature determination.³⁴

V. CONCLUDING REMARKS: SPIN-POLARIZED FERMION LIQUID VERSUS FERROMAGNETIC “FERMI LIQUID”

The established zero-temperature attenuation of transversal spin waves in polarized Fermi liquid is transferred to longitudinal modes like paramagnons and sound waves via spin nonconserving magnetic dipole-dipole interaction.³⁵ In its turn the finite damping of longitudinal collective motions causes the finite damping of the Fermi-liquid quasiparticles. The latter means that, strictly speaking, a polarized Fermi liquid at $T=0$ is not the Fermi liquid. This effect, however, being proportional to the square of the amplitude of dipole-dipole interaction will manifest itself at extremely low temperatures.

A Fermi-liquid theory for the spin waves in a ferromagnetic metal has been developed by Abrikosov and Dzialoshinskii.³⁶ It was done in neglecting of quasiparticle collisions, hence the dampingless spectrum has been obtained. However, the following was noted by Herring:³⁷ “For a ferromagnetic metal . . . if the spin of quasiparticle at the Fermi surface is reversed, the corresponding quasiparticle state will no longer be closed to the Fermi surface, and it will have a finite, rather than an infinitesimal, decay rate.” The finite decay rate of quasiparticle states produces the zero-temperature spin wave attenuation. The latter certainly contradicts to the Goldstone theorem for a isotropic ferromagnetic ground state. So, starting from a Fermi-liquid approach to the itinerant ferromagnet we come to the contradiction. The resolution of this paradox is that in an itinerant ferromagnet the formation of off-diagonal deviations of the momentum-dependent distribution function $\delta \sigma_{\mathbf{k}}(\mathbf{r}, t)$ and its time-space variations according to the kinetic equation will be blocked up by the alteration of orbital part of the electron wave function and corresponding increase of an interaction energy. So, the equation of motion of magnetic degrees of freedom is not formulated in (\mathbf{k}, \mathbf{r}) or phase space but only in the space of coordinates \mathbf{r} as has been done by Landau-Lifshitz³⁸ for the magnetization density $\mathbf{M}(\mathbf{r}, t)$.

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