Level statistics of *XXZ* spin chains with a random magnetic field

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The level-spacing distribution of a spin-$\frac{1}{2}$ XXZ chain is numerically studied under random magnetic field. We show explicitly how the level statistics depends on the lattice size L, the anisotropy parameter Δ, and the mean amplitude of the random magnetic field h. In the energy spectrum, quantum integrability competes with nonintegrability derived from the randomness, where the XXZ interaction is modified by the parameter Δ. When $\Delta \neq 0$, the level-spacing distribution mostly shows Wigner-like behavior, while when $\Delta = 0$, Poisson-like behavior appears although the system is nonintegrable due to randomness. Poisson-like behavior also appears for $\Delta \neq 0$ in the large h limit. Furthermore, the level-spacing distribution depends on the lattice size L, particularly when the random field is weak.

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I. INTRODUCTION

Random matrix theories have been successfully applied to analysis of the spectra of various physical systems such as quantum spin systems,[1–6] strongly correlated systems,[7] and disordered quantum systems.[8–11] In quantum spin chains, if a given Hamiltonian is integrable by the Bethe ansatz, the level-spacing distribution should be described by the Poisson distribution:

$$P_{\mathrm{Poi}}(s) = \exp(-s). \tag{1}$$

If it is not integrable, the level-spacing distribution should be given by the Wigner distribution:

$$P_{\mathrm{Wig}}(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right). \tag{2}$$

In the Anderson model of disordered systems, $P_{\mathrm{Poi}}(s)$ and $P_{\mathrm{Wig}}(s)$ characterize the localized phase and the metallic phase, respectively.[8]

The numerical observations should be important.[1–11] In fact, there has been no direct theoretical derivation for the suggested behavior of the level-spacing distribution. Furthermore, unexpected behavior has been recently found in the level statistics of some XXZ spin chains.[6] Robust non-Wigner behavior has been seen in the level-spacing distributions of next-nearest-neighbor coupled XXZ chains, although they are nonintegrable. The reason why it appears is not clear yet, although we have considered two possible reasons: extra symmetries or finite-size effects.

In this Brief Report, we discuss the level-spacing distribution of a disordered XXZ spin chain so that we may find possible clues to the unexpected behavior of the XXZ spin chains. We consider specifically the spin-$\frac{1}{2}$ XXZ spin chain with random magnetic field, where quantum integrability competes with nonintegrability due to the randomness. The Hamiltonian on L sites is given by

$$\mathcal{H} = J\sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) + \sum_{j=1}^{L} h_j S_j^z, \tag{3}$$

where $S^\alpha = (1/2)\sigma^\alpha$ and $(\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices; h_j is random magnetic field along the z axis at site j; the periodic boundary conditions are imposed. The random magnetic field h_j's are uncorrelated random numbers with a Gaussian distribution: $\langle h_j \rangle = 0$ and $\langle h_n h_m \rangle = h^2 \delta_{nm}$. Here we recall that the system (3) is integrable when $h=0$, while it is not when $h \neq 0$.

We show explicitly how the level-spacing distribution $P(s)$ depends on the lattice size L, the anisotropy parameter Δ, and the mean amplitude of the random magnetic field h. For some special cases of the Hamiltonian (3), the level statistics has been discussed. The present numerical results should make explicit connections among them and even extend them. When $\Delta \neq 0$, $P(s)$ mostly shows Wigner-like behavior. For the Heisenberg case ($\Delta = 1$), it has been shown that $P(s)$ coincides with $P_{\mathrm{Wig}}(s)$.[11] When $\Delta = 0$, Poisson-like behavior appears although the system is nonintegrable due to the randomness. However, the result is consistent with the Anderson localization of one-dimensional (1D) systems. The Wigner-like behavior for $\Delta \neq 0$ suggests that the Anderson localization in 1D systems should be broken by the interaction such as the XXZ coupling. Here we note that an analogous phenomenon has been observed in the 2D Anderson model with electron interaction.[9] In the large h and small h limits, Poisson-like behavior appears again for $\Delta \neq 0$, which is consistent with the numerical results for spin-glass clusters,[10] the open-boundary Heisenberg chain,[12] and the 3D Anderson model.[8] Furthermore, we find that $P(s)$ depends on the lattice size L, particularly when the random field is weak.

There is another motivation for the present study. The symmetry of the integrable quantum XXZ spin chain can be nontrivial. An extraordinary symmetry appears for special values of Δ: the XXZ Hamiltonian commutes with the sl_2 loop algebra when q is a root of unity, where q is defined by

TABLE I. Matrix size of the largest subspace and the number of samples calculated in this work for each lattice size.

Lattice size	Matrix size	Number of samples
8	70×70	10000
10	252×252	3000
12	924×924	1000
14	3432×3432	900

$\Delta=(q+1/q)/2$.[13] The loop algebra is an infinite-dimensional Lie algebra, and the dimensions of degenerate eigenspaces are given by some exponential functions of the system size, which can be extremely large.[14,15] It should, therefore, be nontrivial how the large degeneracies are resolved by nonintegrability.

II. NUMERICAL PROCEDURE

In the Hamiltonian (3), total S^z is conserved. The eigenstates with different S^z are uncorrelated. Therefore, we consider only the largest subspace $S^z=0$. The largest sectors for the lattice size L=8,10,12,14 have 70,252,924,3432 eigenvalues, respectively (see Table I).

To find universal statistical properties of the Hamiltonians, one has to deal with unfolded eigenvalues instead of raw eigenvalues. The unfolded eigenvalues are renormalized values, whose local density of states is equal to unity everywhere in the spectrum. In this Brief Report, the unfolded eigenvalues x_i are obtained from the raw eigenvalues E_i in the following method. Let us define the integrated density of states as

$$n(E)=\sum_{i=1}^{N} \theta(E-E_i). \tag{4}$$

Here $\theta(E)$ is the step function and N is the number of the eigenvalues. We choose some points of coordinates: $(E_i,n(E_i))$ for $i=1,21,41,\ldots,N$. The average of integrated density of states $\langle n(E)\rangle$ is approximated by the spline interpolation through the chosen points. The unfolded eigenvalues are defined as

$$x_i=\langle n(E_i)\rangle. \tag{5}$$

The level-spacing distributions are given by the probability function $P(s)$, where $s=x_{i+1}-x_i$.

We have calculated 10 000,3000,1000,900 samples of $P(s)$ for L=8,10,12,14, respectively (see Table I) and averaged the samples for each L. To calculate the eigenvalues, we have used standard numerical methods, which are contained in the LAPACK library.

III. LEVEL-SPACING DISTRIBUTION

Depending on the anisotropic parameter Δ $(0\leqslant\Delta\leqslant1)$, the level-spacing distribution $P(s)$ changes between the Wigner distribution $P_{\mathrm{Wig}}(s)$ and the Poisson distribution $P_{\mathrm{Poi}}(s)$ as shown in Fig. 1, where $L=14$ and $h/J=0.5$. When $\Delta=0$, $P(s)$ almost coincides with $P_{\mathrm{Poi}}(s)$ although the system is nonintegrable due to the random magnetic field. As Δ increases, $P(s)$ rapidly changes to $P_{\mathrm{Wig}}(s)$.

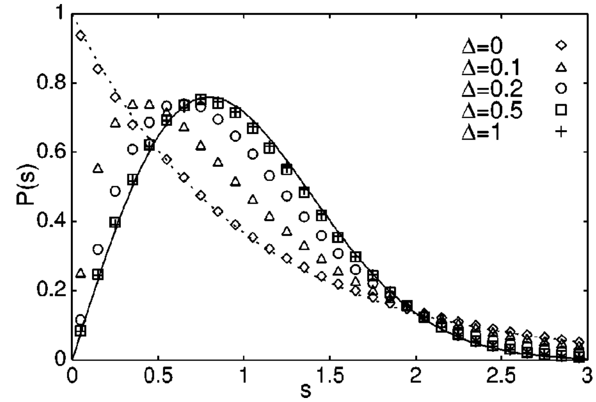

FIG. 1. Level-spacing distributions for $L=14$, $h/J=0.5$, $\Delta=0,0.1,0.2,0.5,1$. Broken line, the Poisson distribution; solid line, the Wigner distribution.

Let us explain the Poisson-like behavior of $\Delta=0$ in terms of the Anderson localization. The Hamiltonian (3) can be mapped into a model of interacting 1D free fermions under random potential:

$$\mathcal{H}=\frac{J}{4}2\left[\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1}+c_{j+1}^\dagger c_j)-(-1)^M(c_L^\dagger c_1+c_1^\dagger c_L)\right]+\Delta\frac{J}{4}\sum_{j=1}^{L}(4c_j^\dagger c_j c_{j+1}^\dagger c_{j+1}-4c_j^\dagger c_j+1)+\sum_{j=1}^{L}h_j\left(\frac{1}{2}-c_j^\dagger c_j\right). \tag{6}$$

Here, L is the number of sites; M is the number of fermions; $c_j^\dagger$ and c_j are the creation and annihilation operators of fermions on the jth site, respectively. And the Anderson model of noninteracting disordered fermions is given by

$$\mathcal{H}=\sum_j \varepsilon_j c_j^\dagger c_j+\sum_{\langle i,j\rangle}V(c_i^\dagger c_j+c_j^\dagger c_i), \tag{7}$$

where ε_j is the random potential at the jth site; V is a constant hopping integral; $\langle i,j\rangle$ denotes summation over nearest-neighbor sites. One can find that Eq. (6) for $\Delta=0$ corresponds to Eq. (7) of the 1D case. It is known that localization always occurs in the 1D case, while the 3D Anderson model has the metallic phase and the localized phase. Here we recall that the metallic phase corresponds to $P_{\mathrm{Wig}}(s)$ and the localized phase to $P_{\mathrm{Poi}}(s)$. Thus, the observed Poisson-like behavior for $\Delta=0$ is consistent with the Anderson localization.

Let us discuss a consequence of the Wigner-like behavior for $\Delta\neq0$. The Hamiltonian (3), namely, Eq. (6), for $\Delta\neq0$ corresponds to interacting 1D fermions under random potential. Thus, the Wigner-like behavior of $P(s)$ for $\Delta\neq0$ might suggest that the interaction among fermions should break the Anderson localization in 1D chains. The suggestion could be consistent with the observation on the 2D interacting lattice

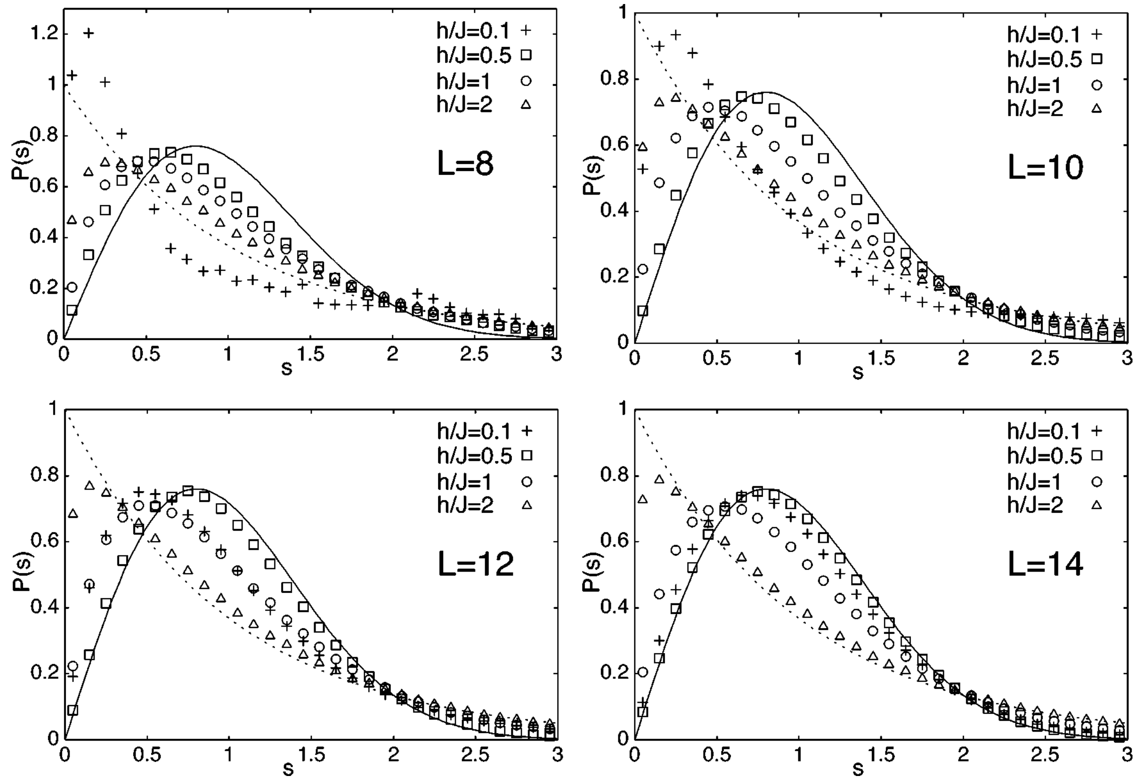

FIG. 2. Level-spacing distributions for $L=8,10,12,14$, $h/J=0.1,0.5,1,2$, $\Delta=0.5$. Broken lines, the Poisson distribution; solid lines, the Wigner distribution.

fermions where the level-spacing distribution changes from $P_{\rm Poi}(s)$ to $P_{\rm Wig}(s)$ as the electron-electron interaction increases from zero.[9]

We now discuss how the level-spacing distribution $P(s)$ depends on the random magnetic field h. We consider only the case of $\Delta\neq 0$. In Fig. 2 the graphs of $P(s)$ are shown for some values of h/J and L, where $\Delta=0.5$. We first consider the case of large h. As h/J increases from the value of 0.5, we observe that the form of $P(s)$ changes from $P_{\rm Wig}(s)$ to $P_{\rm Poi}(s)$. The observation suggests that the effect of random magnetic field on each site should become larger than that of the correlation between adjacent spins, as the random field h/J increases. The spins should become more independent of each other as h/J increases, since the effect of correlation decreases effectively. Thus, the Poisson-like behavior of $P(s)$ should appear in the limit of large h/J. Similar shifts from $P_{\rm Wig}(s)$ to $P_{\rm Poi}(s)$ as randomness increases have been discussed for the 3D Anderson model,[8] the spin-glass clusters,[10] and the open-boundary Heisenberg chain.[12]

For the case of small h, the level-spacing distribution $P(s)$ strongly depends on the system size L, and the behavior of $P(s)$ is dominated by finite-size effects. In Fig. 2, we observe that the form of $P(s)$ for $h/J=0.1$ is different from that of the standard Wigner distribution, particularly when L is small. When L is small, random magnetic field is irrelevant to energy levels if it is smaller than the order of $1/L$. In fact, energy differences should be at least in the order of $1/L$, and random magnetic field can be neglected if it is much smaller than some multiple of $1/L$. Thus, for the case of small h, the level statistics should show such a behavior as that of $h=0$. In fact, the Hamiltonian for $h=0$ is the integrable XXZ spin chain, which should have Poisson-like behavior. Furthermore, the integrable XXZ Hamiltonian at $\Delta=0.5$ has the sl_2 loop algebra symmetry,[13] and the level-spacing distribution should show a peak at $s=0$.[6] In Fig. 2, the graph of $P(s)$ for $L=8$ and $h/J=0.1$ suggests such behavior.

Let us discuss finite-size effects on the level-spacing distributions. In order to observe the size dependence of $P(s)$ clearly, we employ the following parameter: $\eta=\int_0^{s_0}[P(s)-P_{\rm Wig}(s)]ds/\int_0^{s_0}[P_{\rm Poi}(s)-P_{\rm Wig}(s)]ds$, where $s_0=0.4729\ldots$ is the intersection point of $P_{\rm Poi}(s)$ and $P_{\rm Wig}(s)$.[10,12] Thus, we have $\eta=0$ when $P(s)$ coincides with $P_{\rm Wig}(s)$, and $\eta=1$ when $P(s)$ coincides with $P_{\rm Poi}(s)$. In

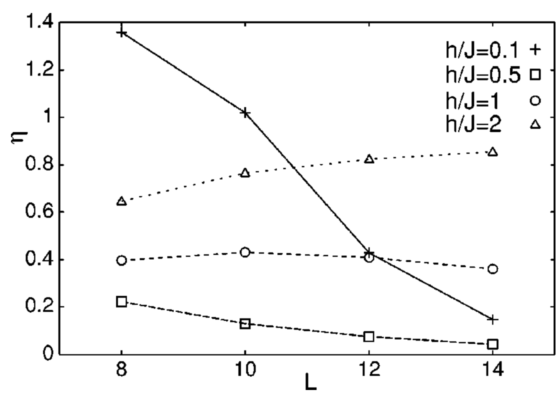

FIG. 3. Dependence of the parameter η on the lattice size L for $h/J=0.1,0.5,1,2$ and $\Delta=0.5$. $\eta=0$ corresponds to the Wigner distribution and $\eta=1$ to the Poisson distribution.

Fig. 3, the value of η for $h/J=0.1$ strongly depends on the lattice size L. Moreover, we observe that as L increases, η decreases for $h/J=0.5$, while η increases for $h/J=2$. The observation suggests that η approaches either the value 0 or 1 as L increases. In other words, it should become more definite whether $P(s)$ has Wigner-like behavior or not, as the system size becomes large.

IV. CONCLUSIONS

In conclusion, we have calculated the level-spacing distributions of finite spin-$\frac{1}{2}$ XXZ chains under random magnetic field, and shown how the level-spacing distributions change between the Poisson distribution and the Wigner distribution depending on the lattice size L, the anisotropy parameter Δ, and the mean amplitude of the random magnetic field h. For $\Delta=0$, the level-spacing distribution $P(s)$ almost coincides with the Poisson distribution although the system has the randomness. As Δ increases from zero, $P(s)$ rapidly shifts to the Wigner distribution. The behaviors of $P(s)$ have been explained in terms of Anderson localization. For $\Delta\neq 0$, $P(s)$ strongly depends on L when h is small. When L is finite, $P(s)$ should show Poisson-like behavior in the small h limit. In the large h limit, however, $P(s)$ should become close to the Poisson distribution independent of L.

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