

Ground-state cooling of mechanical resonatorsIvar Martin,¹ Alexander Shnirman,² Lin Tian,^{3,4} and Peter Zoller^{3,4}¹*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*²*Institut für Theoretische Festkörperphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany*³*Institute for Theoretical Physics, University of Innsbruck, A-6020, Austria*⁴*Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, 6020 Innsbruck, Austria*

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We propose an application of a single Cooper pair box (Josephson qubit) for active cooling of nanomechanical resonators. Latest experiments with Josephson qubits demonstrated that long coherence time of the order of microsecond can be achieved in special symmetry points. Here we show that this level of coherence is sufficient to perform an analog of the well known in quantum optics “laser” cooling of a nanomechanical resonator capacitively coupled to the qubit. By applying an ac driving to the qubit or the resonator, resonators with frequency of order 100 MHz and quality factors higher than 10^3 can be efficiently cooled down to their ground state, while lower-frequency resonators can be cooled down to micro-Kelvin temperatures. We also consider an alternative setup where dc-voltage-induced Josephson oscillations play the role of the ac driving and show that cooling is possible in this case as well.

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I. INTRODUCTION

Recently, fabrication of nanomechanical resonators with fundamental frequencies in the microwave range (100 MHz to 1GHz) has been achieved.¹ For such resonators, the quantum-mechanical level spacing is a few micro-eV, which is comparable to the lowest achievable cryogenic temperatures. Freezing out the mechanical degrees of freedom is favorable for ultrasensitive detection applications² due to reduced effects of thermal fluctuations. Even more spectacular applications can be envisioned if it is possible to cool the mechanical systems down to their motional ground states with high probability. Creation of exotic nonclassical states, entanglement with other quantum objects, e.g., spins or atoms, coherent quantum information transfer between quantum subsystems are just a few possibilities. However, reaching the motional ground state using conventional passive cooling techniques is practically unfeasible, and therefore other approaches need to be explored. Fortunately, the cooling problem is not unique to nanomechanics; a similar problem has been encountered and successfully solved in the field of ultracold atoms, where by using *active* cooling approaches it was possible to quench the vibrational motion of atoms and reach effective nano-Kelvin temperatures.³ The connection to quantum optics has been recently explored in several works. Hopkins *et al.*⁴ applied the quantum feedback control ideas to nanomechanical resonator cooling. Wilson-Rae *et al.*⁵ proposed an analog of resolved sideband laser cooling by coupling the resonator displacement to the level spacing of an attached semiconductor quantum dot, which is being irradiated by red-detuned laser. An advantage over the feedback-based techniques is that the sideband cooling does not require on-the-fly analysis of the output of a nearly ideal detector. On the other hand, direct implementation of the Ref. 5 approach appears rather challenging, from the fabrication stand point and due to stringent constraints on the quantum dot relaxation rate, which needs to be slower than

the resonator frequency for the vibrational sidebands to be resolved.

In this work we study an alternative realization of the laserlike cooling for nanomechanical resonators, where the role of the two-state system is played by a superconducting qubit (Cooper pair box, or CPB) capacitively coupled to the resonator. Interaction between the qubit and the resonator leads to splitting of the qubit states into equidistant vibrational sidebands. Latest experiments with Josephson qubits showed that long coherence time of the order of microsecond can be achieved in special symmetry points.⁶ When the qubit relaxation and dephasing rates are smaller than the oscillator frequency, one reaches the resolved sideband regime, favorable for cooling. By tuning microwave source frequency into the first red sideband (qubit level spacing minus the oscillator frequency), one can ensure that the microwave photon absorption processes are preferentially accompanied by simultaneous phonon emission from the resonator. The cooling cycle is completed when a photon is spontaneously emitted at the qubit natural frequency into an external bath. The emitted photon is blue-shifted relative to the source, as it carries away one resonator phonon energy. The exact reverse of this process is exponentially suppressed if the qubit level spacing is large relative to temperature, and hence the heating is determined by other—nonresonant or driving-induced—processes which are much slower than the dominant cooling mechanism. This makes cooling possible. A suggestion to use coupled nanoresonator-qubit system for laserlike cooling has been first made by Irish and Schwab.⁷ Here we consider two implementation of this scheme that use different ac driving sources: (1) microwaves directly applied to the CPB or the resonator, in the form of ac flux or voltage; (2) the ac Josephson effect on an auxiliary Josephson junction of the qubit. The advantage of the second approach is that it only requires dc bias for cooling; however, we find that it is not as effective as the ac biasing scheme. In the opposite limit of fast CPB relaxation when the vibrational sidebands are not resolved, we demonstrate that an-

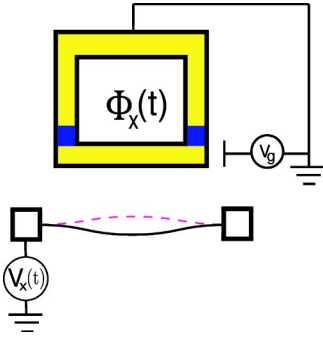


FIG. 1. (Color online) The system with ac driving.

other type of laser cooling, ‘‘Doppler’’ cooling,³ can be performed down to the temperature defined by the quantum dot level width. Although ground-state cooling in this regime is impossible, this technique could be practically attractive for noise reduction in local probe applications (e.g., atomic force microscopy, magnetic force microscopy, and magnetic resonance force microscopy), where typical resonator frequencies are below 1 MHz.

II. AC COOLING

System. The system under consideration is shown in Fig. 1. A mechanical resonator (horizontal beam) is coupled to a Cooper pair box (horizontal rectangle) through capacitance $C_x(x)$, which depends on the resonator displacement x . The charge, and hence the state, of CPB is separately controlled by the gate voltage V_g applied to capacitor C_g . CPB is coupled to a large superconductor through two Josephson junctions. The superconducting quantum interference device (SQUID) geometry is chosen to allow for application of an external ac flux to the system to provide the ac driving needed for cooling. Similar systems were considered in Refs. 8,7. The Hamiltonian of the system without dissipation reads

$$H = \frac{[Q - C_g V_g - C_x(x) V_x]^2}{2C_\Sigma} - E_J(\Phi_x) \cos \theta + H_x, \quad (1)$$

where Q is the charge on the island, $C_\Sigma \equiv C + C_g + C_x(x)$ is the total capacitance of the island, and $H_x \equiv (p^2/2m) + (m\omega_0^2 x^2/2)$. The total Josephson energy of the SQUID controlled by an external flux Φ_x is given by $E_J(\Phi_x) = 2E_J^0 \cos(\pi\Phi_x/\Phi_0)$, where E_J^0 is the Josephson energy of each of the junctions (we consider a symmetric setup) and $\Phi_0 = h/2e$ is the (superconducting) flux quantum. We assume the total gate charge $Q_g \equiv C_g V_g + C_x(x) V_x$ to be close to an odd number of electron charges, i.e., $Q_g = 2e(N + 1/2) + 2e\delta N$, where $|\delta N| \ll 1/2$. Then we can use a two-state approximation, i.e., $|\uparrow\rangle \equiv |Q = 2eN\rangle$, and $|\downarrow\rangle \equiv |Q = 2e(N + 1)\rangle$, and rewrite the Hamiltonian using the Pauli matrices as

$$H = 4E_C(x) \delta N(x) \sigma_z + E_C(x) [1 + 4\delta N^2(x)] - \frac{E_J}{2} \sigma_x + H_x, \quad (2)$$

where $E_C(x) \equiv e^2/2C_\Sigma$. The second term in the first line of Eq. (2) depends on x and, thus, added to the oscillator Hamiltonian H_x , renormalizes (slightly) the oscillator parameters. This term is also responsible for the direct coupling of the oscillator to the dissipation in the circuitry (see below). We, first, drop this term for clarity but later reintroduce it when discussing the direct coupling between the oscillator and the gate voltage fluctuations. Assuming that fluctuations of x are small relative to the resonator-CPB distance d , we obtain $C_x(x) \approx C_x - C_x x/d$. Thus $\delta N(x) \approx \delta N - N_x x/d$, where $N_x \equiv C_x V_x/2e$, and $E_C(x) \approx E_C + E_C(C_x/C_\Sigma)(x/d)$. Then the Hamiltonian simplifies to

$$H = 4E_C \delta N \sigma_z - \frac{E_J}{2} \sigma_x + \lambda(a^\dagger + a) \sigma_z + \hbar\omega_0 a^\dagger a, \quad (3)$$

where $\lambda \equiv -4E_C[N_x - \delta N(C_x/C_\Sigma)](\Delta x/d)$ and $\Delta x \equiv \sqrt{\hbar/(2m\omega_0)}$ is the amplitude of zero point motion, $x = \Delta x(a^\dagger + a)$. To increase the coupling one usually applies high gate voltage V_x so that $N_x \gg 1$. Thus, approximately

$$\lambda \approx -4E_C N_x \frac{\Delta x}{d}. \quad (4)$$

Both gate voltages V_g and V_x as well as the external flux Φ_x fluctuate as they are provided by dissipative sources. This makes δn and E_J in Eq. (3) fluctuate. Moreover, in all real systems there are $1/f$ charge and flux noises. The charge $1/f$ noise can effectively be added to the noise of the gate charge δN . Experimentally, $1/f$ noise is the most severe factor limiting coherence. Long coherence times have been achieved in Ref. 6 by operating in a special point where the $1/f$ noise is less harmful. In the special point, the total-energy splitting of the qubit $\Delta E \equiv \sqrt{(8E_C)^2 \delta N^2 + E_J^2}$ is not sensitive to the fluctuations of δN and Φ_x in the linear order. This implies $\langle \delta N \rangle = 0$ and $\partial E_J / \partial \Phi_x = 0$. The fluctuations of δN are, nevertheless, still there and we obtain the following Hamiltonian characterizing the special point

$$H = -\frac{E_J}{2} \sigma_x + \hbar\omega_0 a^\dagger a + \lambda(a^\dagger + a) \sigma_z - \frac{X}{2} \sigma_z + g(a^\dagger + a) X, \quad (5)$$

where

$$X \equiv -\frac{4E_C}{e} (C_g \delta V_g + C_x \delta V_x).$$

By construction the coupling constant λ also contains a fluctuating part. We neglect this higher-order effect. The fluctuations of voltages δV_g and δV_x are caused by external impedances $Z(\omega)$ of the circuits that supply the voltages. The (unsymmetrized) noise power is given by $\langle \delta V_\omega^2 \rangle \equiv \langle \delta V(t) \delta V(0) \rangle_\omega = [\text{Re}Z(\omega)] \hbar \omega [\coth(\hbar\omega/2k_B T) + 1]$. Thus the term $X\sigma_z/2$ represents the coupling of the system to the harmonic (Gaussian) electromagnetic bath. The situation is similar to the quantum optics one where an atom is coupled to electromagnetic vacuum. The main differences are that in our case the coupling is longitudinal, i.e., it does not cause spin flips in the natural (charge) basis, and the bath temperature cannot always be neglected. The last term in Eq. (5) is the direct coupling between the bath and the oscillator.

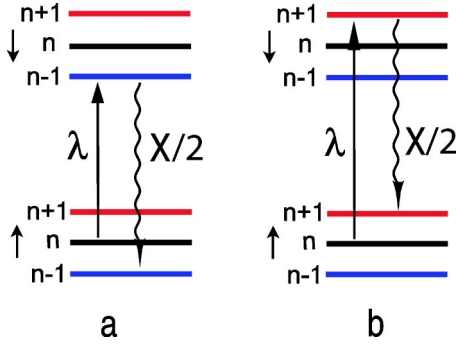


FIG. 2. (Color online) Dissipative processes due to the presence of the qubit: (a) $n \rightarrow n-1$; (b) $n \rightarrow n+1$. The spectra of the oscillator and the qubit are superimposed.

It originates from the second term in Eq. (2), and $g = N_x \Delta x / d$ at the symmetry point. In the weak-coupling case that we consider, the bath is fully characterized by its spectral function, and hence we do not need to include the self-Hamiltonian of the bath explicitly. The nonelectromagnetic bath acting on the resonator is later introduced through a phenomenological quality factor Q .

Equilibrium transition rates. To proceed, we first need to determine the resonator-qubit system relaxation rates caused by the electromagnetic environment. It is convenient to perform a $\pi/2$ rotation in the x - z plane, $\sigma_x \leftrightarrow \sigma_z$, to the eigenbasis of the qubit at the symmetry point. The Hamiltonian becomes

$$H = -\frac{E_J}{2} \sigma_z + \hbar \omega_0 a^\dagger a + \lambda (a^\dagger + a) \sigma_x - \frac{X}{2} \sigma_x + g (a^\dagger + a) X. \quad (6)$$

We choose $E_J \gg \omega_0, \lambda, T$. This ensures that without the driving the spin is all the time in the ground state $|\uparrow\rangle$. The qubit relaxation rate is

$$\Gamma_r \equiv \Gamma_{\downarrow \rightarrow \uparrow} = \frac{\langle X_{\omega=E_J}^2 \rangle}{4\hbar^2} \approx \pi \alpha_g E_J, \quad (7)$$

where

$$\begin{aligned} \langle X_\omega^2 \rangle &= 2\pi \hbar^2 \alpha_g \omega [\coth(\hbar \omega / 2k_B T) + 1], \quad \alpha_g \\ &\approx \frac{C_x^2 + C_g^2}{C_\Sigma^2} \frac{R}{R_Q}, \quad R_Q \equiv h/4e^2 \end{aligned}$$

(here we assumed independent δV_g and δV_x with similar external impedances $\text{Re } Z(\omega) \equiv R \sim 50\Omega$). The opposite excitation rate is exponentially suppressed.

The direct coupling between the bath and the oscillator gives the dissipative rates between the oscillator states $|n\rangle$: $\Gamma_{n \rightarrow n-1} \approx g^2 \langle X_{\omega=\omega_0}^2 \rangle n / \hbar^2$ and $\Gamma_{n \rightarrow n+1} \approx g^2 \langle X_{\omega=-\omega_0}^2 \rangle (n+1) / \hbar^2$. In addition, the oscillator can relax via the virtual excitations of the qubit. The corresponding processes are shown in Fig. 2. The rates for these processes are given by

$$\Gamma_{n \rightarrow n-1} \approx \frac{\lambda^2 \langle X_{\omega=\omega_0}^2 \rangle}{E_J^2 4\hbar^2} n, \quad (8)$$

$$\Gamma_{n \rightarrow n+1} \approx \frac{\lambda^2 \langle X_{\omega=-\omega_0}^2 \rangle}{E_J^2 4\hbar^2} (n+1). \quad (9)$$

These rates are bigger than those due to the direct coupling by a factor $(2E_C/E_J)^2$, and hence the direct coupling term in the Hamiltonian ($\propto g$) can be discarded. The above transition rates translate into the resonator quality factor caused by the coupling to the electromagnetic bath,

$$\frac{1}{Q_{\text{em}}} = \frac{\pi \lambda^2}{E_J^2} \alpha_g. \quad (10)$$

The charge noise at the (relatively low) frequency $\omega = \omega_0$ may be dominated by the $1/f$ contribution. The symmetrized correlator of this contribution has been studied, e.g., in echo experiments,⁹ and one can assume $S_X(\omega) = (\langle X_\omega^2 \rangle + \langle X_{-\omega}^2 \rangle) / 2 = E_{1/f}^2 / |\omega|$, where $E_{1/f} \equiv 4E_C \sqrt{\alpha_{1/f}}$ and typically $\alpha_{1/f} \approx 10^{-7}$. We are not aware of any study of the unsymmetrized correlators. Thus, we will introduce a phenomenological temperature $T_{1/f}$ for the particular frequency $\omega = \omega_0$ via $\exp[-\hbar \omega_0 / (k_B T_{1/f})] \equiv \langle X_{-\omega_0}^2 \rangle / \langle X_{\omega_0}^2 \rangle$. Then, for the corresponding quality factor we obtain

$$\frac{1}{Q_{1/f}} \approx \frac{\lambda^2}{E_J^2} \frac{8E_C^2}{(\hbar \omega_0)^2} \alpha_{1/f} \tanh \frac{\hbar \omega_0}{2k_B T_{1/f}}. \quad (11)$$

It is reasonable to assume that the effective $1/f$ noise temperature is not lower than the environment temperature. For further analysis we will assume $T_{1/f} \approx T$.

A. Cooling by applying flux driving

For cooling we need to drive the system out of equilibrium. We propose to apply an external ac flux, $\Phi_x(t) = \Phi_{x,0} + D\Phi_0 \cos \omega_d t$, to the qubit. As we operate at the special point where $\partial E_J / \partial \Phi_x = 0$, linear driving is impossible. In the quadratic order, we obtain $E_J \rightarrow E_J + (\pi D/2)^2 E_J (\cos 2\omega_d t + 1)$. This gives

$$\begin{aligned} H &= -\frac{E_J + \Omega}{2} \sigma_z - \frac{X}{2} \sigma_x + \hbar \omega_0 a^\dagger a + \lambda (a^\dagger + a) \sigma_x \\ &\quad - \frac{\Omega}{2} \sigma_z \cos 2\omega_d t, \end{aligned} \quad (12)$$

where $\Omega = (\pi D/2)^2 E_J$. To describe the cooling process we use the Floquet picture. In other words, we count the number of energy quanta taken from the pumping source. This amounts to substituting the factors $e^{\pm 2\omega_d t}$ by raising/lowering operators $e^{\pm \chi}$ ($e^{i\chi}|m\rangle = |m+1\rangle$) and subtracting the energy taken from the source, $2\hbar \omega_d m$, from the Hamiltonian. Then the new Hilbert space of the problem is extended as $|\sigma\rangle|n\rangle|m\rangle$, where $|\sigma\rangle$ is the state of the qubit ($|\uparrow\rangle$ or $|\downarrow\rangle$), while n is the number of quanta in the resonator. Due to the new term in the Hamiltonian, $-2\hbar \omega_d m$, an un-

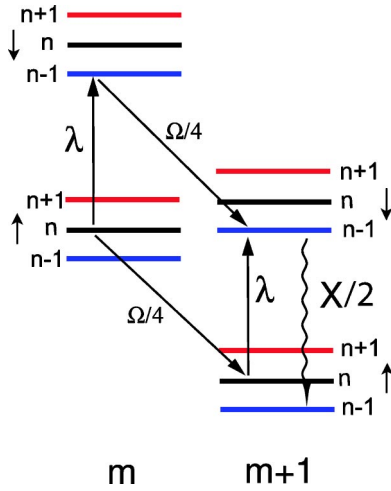


FIG. 3. (Color online) The cooling process.

bound staircase of states appears. Each step of the staircase (Floquet zone) is characterized by a number m and spans the Hilbert space $|\sigma\rangle|n\rangle$. The neighboring zones are shifted by energy $2\hbar\omega_d$ relative to each other. The states $|\sigma\rangle|n\rangle|m\rangle$ are the eigenstates of the Hamiltonian

$$H_0 \equiv -(E_J + \Omega)\sigma_z/2 + \hbar\omega_0 a^\dagger a - 2\hbar\omega_d m. \quad (13)$$

The other three terms form the perturbation

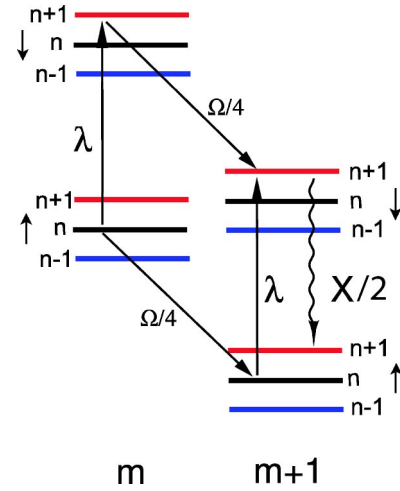
$$H' = -\frac{X}{2}\sigma_x + \lambda(a^\dagger + a)\sigma_x - \frac{\Omega}{2}\sigma_z \cos \chi. \quad (14)$$

This perturbation H' causes transitions inside of each zone and also down and up the staircase. At finite (noninfinite) temperature of the bath the down transitions prevail and the system propagates down the staircase. This corresponds to the flow of energy from the driving source to the bath. During this flow, the driving source quanta of energy can be up- or down-converted in frequency by amount ω_0 , that is, photons with frequency $2\omega_d \pm \omega_0$ are emitted into the bath. The former case corresponds to cooling while the latter to heating.

We first consider the up-conversion cooling process shown in Fig. 3. We choose the optimal detuning so that $2\omega_d = E_J + \Omega - \omega_0$ and the levels $|\uparrow, n, m\rangle$ and $|\downarrow, n-1, m+1\rangle$ are degenerate (in what follows we use the units $\hbar = 1$ for brevity). Thus we have to be careful when calculating the rates. First, we obtain the second-order matrix element between these two states equal to $\Delta\sqrt{n}$ where

$$\Delta \equiv \frac{\Omega\lambda}{2E_J}. \quad (15)$$

Note that the two paths shown in Fig. 3 interfere constructively. Then, the cooling rate depends on the relation between $\Delta\sqrt{n}$ and $\Gamma_r/2$. For the *weak driving* case $\Delta\sqrt{n} < \Gamma_r/2$ we have the cooling rate $\Gamma_{n \rightarrow n-1}^\Omega \approx (4\Delta^2/\Gamma_r)n$ (the superscript Ω is to emphasize that this rate is due to the driving). To justify this result we consider this process as tunneling from the level $|\uparrow, n, m\rangle$ to the broadened level $|\downarrow, n-1, m+1\rangle$. The retarded Green's function of the second


 FIG. 4. Level structure of the strong driving for ac cooling. Here $\psi_n^\pm \equiv (|n, \uparrow\rangle \pm |n-1, \downarrow\rangle)/\sqrt{2}$.

level is then given by $G^R(\omega) = 1/(\omega + i\Gamma_r/2)$. Using the Golden rule (the adiabatic elimination technique¹⁰), we obtain

$$\Gamma_{n \rightarrow n-1}^\Omega = 2\pi\Delta^2 n \left(-\frac{1}{\pi} \text{Im} G^R(\omega=0) \right) = \frac{4\Delta^2}{\Gamma_r} n. \quad (16)$$

In the opposite, *strong driving* case, $\Delta\sqrt{n} > \Gamma_r/2$, there are coherent oscillations between the two levels. The appropriate description is then to say that doublets of new eigenstates are formed which are split in energy by $2\Delta\sqrt{n}$, see Fig. 4. The doublets are defined for $n \geq 1$ as $\psi_n^\pm \equiv (|n, \uparrow\rangle \pm |n-1, \downarrow\rangle)/\sqrt{2}$ and for $n=0$ we have a single state $\psi_{n=0} \equiv |0, \uparrow\rangle$. There are four possible transitions between each state of the doublet and each state of another doublet shifted as $n \rightarrow n-1$ and $m \rightarrow m+1$. All four rates are equal $\Gamma_r/4$. In total this gives $\Gamma_{n \rightarrow n-1}^\Omega \approx \Gamma_r/2$. To summarize,

$$\Gamma_{n \rightarrow n-1}^\Omega = \begin{cases} (4\Delta^2/\Gamma_r)n & \text{if } \Delta\sqrt{n} < \Gamma_r/2 \\ \Gamma_r/2 & \text{if } \Delta\sqrt{n} > \Gamma_r/2. \end{cases} \quad (17)$$

Note that in the strong driving regime n denotes the doublet rather than the oscillator's level (see discussion below).

In addition to cooling, the ac driving induces competing heating processes. One, shown in Fig. 5, is a standard in

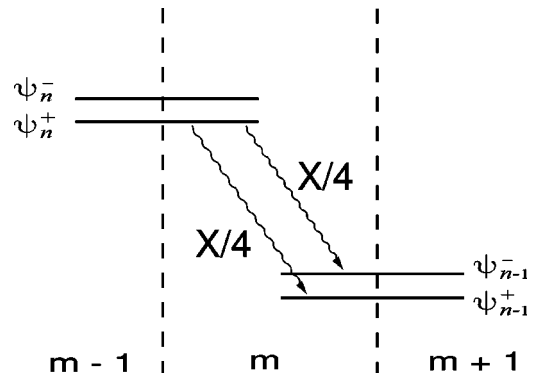


FIG. 5. (Color online) The heating process.

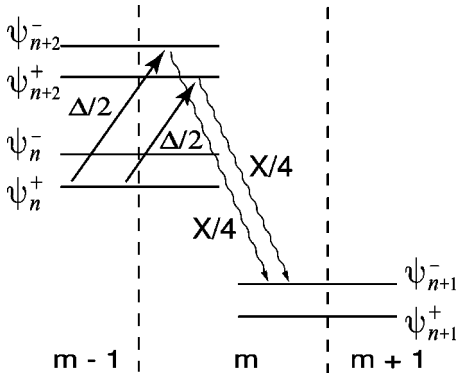


FIG. 6. Level structure of the strong driving for ac heating. The two shown processes interfere constructively.

quantum optics off-resonance process. Indeed, the states $|\uparrow, n, m\rangle$ and $|\downarrow, n+1, m+1\rangle$ are connected in the second order by the matrix element $\Delta\sqrt{n+1}$. These states are off-resonance (we restrict ourselves to the values of n such that level splitting $2\omega_0 \gg \Delta\sqrt{n+1}$). Thus, in the case $\Delta\sqrt{n} < \Gamma_r/2$ we obtain $\Gamma_{n \rightarrow n+1}^\Omega \approx (\Delta/2\omega_0)^2 \Gamma_r(n+1)$. In the strong driving case $\Delta\sqrt{n} > \Gamma_r/2$ we use again the basis of doublets, Fig. 6, and arrive at $\Gamma_{n \rightarrow n+1}^\Omega \approx (\Delta/4\omega_0)^2 \Gamma_r(n+1) \times (1 + \delta_{n,0})$. In this case the rates are between doublets as a whole. The factor $(1 + \delta_{n,0})$ is due to the fact that there is only one state in the ‘‘doublet’’ $n=0$. To summarize,

$$\Gamma_{n \rightarrow n+1}^\Omega = \frac{\Delta^2}{4\omega_0^2} \Gamma_r(n+1) \begin{cases} 1 & \text{if } \Delta\sqrt{n} < \Gamma_r/2 \\ \frac{1 + \delta_{n,0}}{2} & \text{if } \Delta\sqrt{n} > \Gamma_r/2. \end{cases} \quad (18)$$

Another driving-induced heating process, not characteristic for quantum optics, is due to the fact that in solid-state systems there is strong noise at low frequencies ($1/f$ noise). Thus, processes like the one shown in Fig. 7 become relevant. This process excites the qubit with the rate

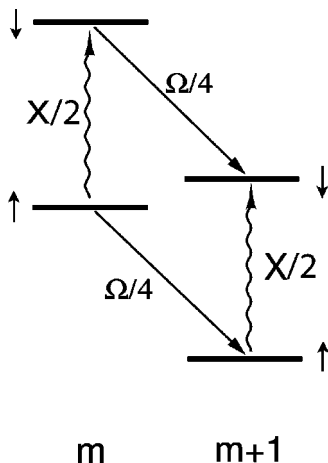


FIG. 7. Qubit heating induced by the applied drive.

$$\Gamma_e \equiv \Gamma_{\uparrow \rightarrow \downarrow} = \frac{\Omega^2}{4E_J^2} \frac{\langle X_{\omega=-\omega_0}^2 \rangle}{4\hbar^2}. \quad (19)$$

Since at $\omega = -\omega_0$ the noise is dominated by the $1/f$ contribution, rate (19) might dominate the heating. With this process, the heating is also resonant and, in analogy to Eq. (17), still assuming that $\Gamma_e \ll \Gamma_r$, we obtain

$$\Gamma_{n-1 \rightarrow n}^\Omega = \begin{cases} \Gamma_e (4\Delta^2/\Gamma_r^2)n & \text{if } \Delta\sqrt{n} < \Gamma_r/2 \\ \frac{\Gamma_e}{2} (1 + \delta_{n,0}) & \text{if } \Delta\sqrt{n} > \Gamma_r/2. \end{cases} \quad (20)$$

Master equation. Taking into account the internal dissipation of the resonator with quality factor Q , one can⁵ write down the master equation for the probability P_n to find the resonator in the state $|n\rangle$. First, we analyze the (strong driving) limit, $\Delta > \Gamma_r/2$. Then, it is convenient to write the rate equation in terms of the probabilities D_n to find the system in the doublet ψ_n^\pm ($n \geq 1$). The probability D_0 , then, is for the system to be in the (nondegenerate) ground state $\psi_0 = |\uparrow, 0\rangle$. The probabilities P_n are given by

$$P_n = \frac{1}{2} (D_n + D_{n+1}) + \frac{1}{2} D_n \delta_{n,0}. \quad (21)$$

The master equation reads

$$\begin{aligned} \dot{D}_n = & \frac{1}{2} \Gamma_r [D_{n+1} - D_n (1 - \delta_{n,0})] + \frac{\Delta^2}{8\omega_0^2} \Gamma_r [n D_{n-1} (1 + \delta_{n,1}) \\ & - (n+1) D_n (1 + \delta_{n,0})] + \frac{1}{2} \Gamma_e [D_{n-1} (1 - \delta_{n,0}) (1 + \delta_{n,1}) \\ & - D_n (1 + \delta_{n,0})] + \frac{\omega_0 [n_B(\omega_0) + 1]}{Q} \left[\left(n + \frac{1}{2} \right) D_{n+1} \right. \\ & \left. - \left(n - \frac{1}{2} \right) D_n (1 - \delta_{n,0}) \right] + \frac{\omega_0 n_B(\omega_0)}{Q} \left[\left(n - \frac{1}{2} \right) D_{n-1} \right. \\ & \left. \times (1 - \delta_{n,0}) (1 + \delta_{n,1}) - \left(n + \frac{1}{2} \right) D_n (1 + \delta_{n,0}) \right]. \end{aligned} \quad (22)$$

The unusual form of the second part of this master equation is due to the structure of the matrix elements between the different states of the doublets.

Multiplying Eqs. (22) by and summing over n we obtain

$$\begin{aligned} \frac{d}{dt} \langle \tilde{n} \rangle = & -\frac{1}{2} \Gamma_r (1 - D_0) + \frac{1}{2} \Gamma_e (1 + D_0) + \frac{\Delta^2}{8\omega_0^2} \Gamma_r (1 + D_0) \\ & - \left(\frac{\omega_0}{Q} - \frac{\Delta^2}{8\omega_0^2} \Gamma_r \right) \langle \tilde{n} \rangle + \frac{\omega_0}{2Q} (1 - D_0) + \frac{\omega_0 n_B(\omega_0)}{Q}, \end{aligned} \quad (23)$$

where $\langle \tilde{n} \rangle \equiv \sum_n n D_n$. We rewrite the right-hand side of Eq. (23) in terms of the phonon occupation expectation value

$$\langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} \left(n - \frac{1}{2} \right) D_n = \langle \tilde{n} \rangle - \frac{1-D_0}{2}. \quad (24)$$

We, then, obtain

$$\begin{aligned} \frac{d}{dt} \langle \tilde{n} \rangle = & -\frac{1}{2} \Gamma_r (1-D_0) + \frac{1}{2} \Gamma_e (1+D_0) + \frac{\Delta^2}{16\omega_0^2} \Gamma_r (3+D_0) \\ & - \left(\frac{\omega_0}{Q} - \frac{\Delta^2}{8\omega_0^2} \Gamma_r \right) \langle n \rangle + \frac{\omega_0}{Q} n_B(\omega_0). \end{aligned} \quad (25)$$

There are two main cooling regimes in Eq. (25). If the coefficient in front of $\langle n \rangle$ is positive, i.e., if $\omega_0/Q \gg (\Delta^2/8\omega_0^2)\Gamma_r$, we obtain the usual cooling with the cooling rate slowing down with decreasing $\langle n \rangle$. As this is the relevant regime for realistic parameters we will analyze only this case. Interestingly, however, in the opposite case of very high Q , when $\omega_0/Q \ll (\Delta^2/8\omega_0^2)\Gamma_r$, the cooling rate accelerates until $\langle n \rangle \approx 1$.

If $\langle \tilde{n} \rangle \gg 1$ (to be checked for self-consistency) the probability to be in the ground state is negligible, $D_0 \ll 1$. Then

$$\langle n \rangle \approx n_B(\omega_0) - \frac{(\Gamma_r - \Gamma_e)Q}{2\omega_0}. \quad (26)$$

This regime, thus, is realized when $n_B(\omega_0) > (\Gamma_r - \Gamma_e)Q/(2\omega_0)$. Clearly, for at least some cooling we need $(\Gamma_r - \Gamma_e) \gg (2\omega_0)/Q$. At lower temperatures, Eq. (26) gives negative $\langle n \rangle$ which means that the approximation breaks down and $\langle n \rangle \sim 1$ or less. Then, for an estimate we can use $1 - D_0 \sim \langle \tilde{n} \rangle$. In this regime $\langle \tilde{n} \rangle \approx 2\langle n \rangle$ [see Eq.(24)], therefore, $1 - D_0 \sim 2\langle n \rangle$. Thus, for $\Gamma_r \gg \Gamma_e$

$$\langle n \rangle \approx \frac{\Gamma_e + \frac{\omega_0 n_B(\omega_0)}{Q}}{\Gamma_r}. \quad (27)$$

For $n_B(\omega_0) > \Gamma_e Q/\omega_0$ we then obtain

$$\langle n \rangle \approx \frac{\omega_0 n_B(\omega_0)}{Q \Gamma_r}, \quad (28)$$

while in the opposite case the average occupation saturates at

$$\langle n \rangle \approx \frac{\Gamma_e}{\Gamma_r}. \quad (29)$$

From Eq. (23) it is clear that for $n_B(\omega_0) \gg 1$, the cooling is initially exponential in time, $\langle n \rangle_t \approx n_B(\omega_0) - \Gamma_r Q/(2\omega_0) [1 - \exp(-\gamma t)]$, with the decay rate determined by the oscillator bare damping, $\gamma = \omega_0/Q$. Only when the low occupancy regime ($\langle n \rangle_t < 1$) is reached, the rate of exponential decay increases to

$$\gamma = \frac{\Gamma_r - \Gamma_e}{2} - \frac{\omega_0}{2Q}. \quad (30)$$

If the driving is weaker, then for some low values of n the following master equation holds (see Refs. 3,5)

$$\begin{aligned} \dot{P}_n = & \left(A_- + \frac{\omega_0 [n_B(\omega_0) + 1]}{Q} \right) [(n+1)P_{n+1} - nP_n] \\ & + \left(A_+ + \frac{\omega_0 n_B(\omega_0)}{Q} \right) [nP_{n-1} - (n+1)P_n], \end{aligned} \quad (31)$$

where $A_- \equiv 4\Delta^2/\Gamma_r$ and $A_+ \equiv (\Delta/2\omega_0)^2 \Gamma_r + \Gamma_e (4\Delta^2/\Gamma_r^2)$. For the average occupation number $\langle n \rangle$ in the stationary state we obtain

$$\langle n \rangle = \frac{A_+ + \frac{\omega_0 n_B(\omega_0)}{Q}}{A_- - A_+ + \frac{\omega_0}{Q}}. \quad (32)$$

Regimes similar to Eqs. (28) and (29) are clearly identified. However, a regime similar to Eq. (26) is not possible within master equation (31). The expression for the oscillator occupancy, Eq. (32), can be naturally interpreted in terms of two independent heat baths acting on the resonator, one being the equilibrium environment at the nominal external temperature T , coupled to the resonator by a coupling strength $\gamma_0 = \omega_0/Q$, while the other bath being introduced by the cooling process itself. The effective temperature of the latter, T^* , can be defined through $A_-/A_+ = \exp(\omega_0/T^*)$, and the effective coupling strength is $\gamma^* = \omega_0/Q^* = A_- - A_+$. Then, the final resonator occupancy can be reexpressed as

$$\langle n \rangle = \frac{\gamma_0 n_B(\omega_0) + \gamma^* n_B^*(\omega_0)}{\gamma_0 + \gamma^*}, \quad (33)$$

where n_B^* is the Bose distribution function at temperature T^* . Clearly, T^* is the lowest possible temperature for a given cooling process, which is achieved for $\gamma^* \gg \gamma_0$. The combined damping coefficient $\gamma = \gamma_0 + \gamma^*$ determines the rate of relaxation to the new stationary state.

Example. We consider a nanomechanical resonator with fundamental frequency 100 MHz ($\omega_0 = 2\pi \times 100$ MHz = 0.5 μ eV) and quality factor $Q = 10^5$. It is coupled to the qubit, which is characterized by the Josephson energy $E_J \approx 50$ μ eV and Coulomb charging energy $E_C \approx 160$ μ eV (corresponds to $C_\Sigma \approx 500$ aF). The coupling strength between the resonator and the qubit is determined by the mutual capacitance $C_x \approx 20$ aF and the gate voltage $V_x \approx 1$ V, such that $n_x \approx 60$ Cooper pairs (see Ref. 8). The gap between the resonator and the CPB is $d \approx 100$ nm. For these parameters, from Eq. (4), the resonator-CPB coupling strength is $\lambda \approx 0.1$ μ eV. Assuming that $C_g < C_x$, for the relaxation rate of CPB we find $\Gamma_r = 3 \times 10^{-3}$ μ eV.

The circuit-induced quality factor of the resonator is $Q_{em} \approx 4 \times 10^9$, which is significantly higher than the quality factors of typical resonators. For the $1/f$ contribution at $T_{1/f} > 10\omega_0$ we obtain $Q_{1/f} > 10^7$, which is still higher than a typical value. Hence this modification of the oscillator damping can be neglected compared to other environmental effects.

For cooling, we take $\Omega \approx 1$ μ eV, which corresponds to the modulation depth $D = 0.1$. We thus obtain $\Delta \approx 10^{-3}$ μ eV, and hence for all n we have $\Delta \sqrt{n} > \Gamma/2$ and

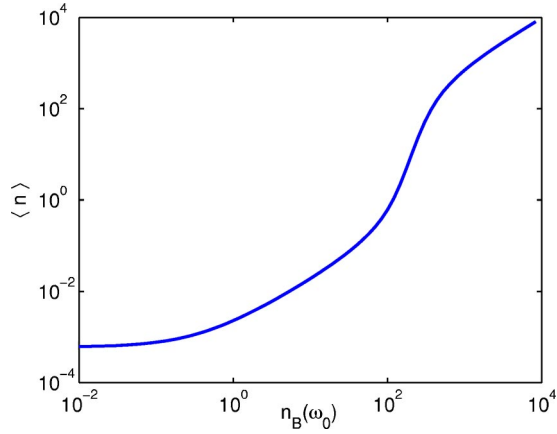


FIG. 8. (Color online) Cooling diagram for a 100 MHz resonator (detailed parameters in text).

$\Gamma_{n \rightarrow n-1}^\Omega \approx \Gamma_r/2$ (strong driving regime). The heating is indeed dominated by $1/f$ noise, with $\Gamma_e = (\Omega/2E_J)^2 (E_{1/f}^2/4\omega_0) \approx 1.8 \times 10^{-6} \mu\text{eV}$. In this regime we get the following results for cooling:

$$\langle n \rangle = \begin{cases} n_B(\omega_0) - 300 & \text{if } n_B(\omega_0) > 300 \\ 1.5 \times 10^{-3} n_B(\omega_0) & \text{if } 0.33 < n_B(\omega_0) < 300 \\ 0.5 \times 10^{-3} & \text{if } n_B(\omega_0) < 0.33. \end{cases}$$

The exact numerical solution of the rate equations, Eq. (22), is shown in Fig. 8.

B. Cooling by applying voltage driving

Another way to achieve ac cooling is by applying radio frequency voltage bias to the gates. In Fig. 1, apply a driving voltage $V_x = V_0 \cos \omega_d t$ on the resonator and another driving voltage $V_g = -(C_x/C_g)V_x$ on the CPB. The ac voltage V_x generates resonant coupling between the mechanical resonator and the CPB when $\omega_d = E_J - \omega_0$, which corresponds to the first red sideband coupling in quantum optics. The voltage V_x also generates an oscillating charge bias on the CPB with $\delta N_g^x = C_x V_x / 2e$; however, it is balanced by the bias V_g , which prevents harmful ac pumping of the CPB.

The scheme. The Hamiltonian of the voltage driving scheme is

$$H = -\frac{E_J}{2} \sigma_z + \hbar \omega_0 \hat{a}^\dagger \hat{a} + 4E_c \delta n_g \sigma_x + \lambda \cos \omega_d t (\hat{a}^\dagger + \hat{a}) \sigma_x - \frac{\hat{X}}{2} \sigma_x, \quad (34)$$

where the qubit works at the optimal point of $\delta n_g = 0$ to avoid the $1/f$ noise. The coupling is

$$\lambda = -4E_c \frac{C_x V_0}{2e} \frac{\Delta x}{d}. \quad (35)$$

Similar to the ac flux driving setup, we analyze the sideband cooling ($\omega_0 \gg \lambda, \Gamma_r$) in the regimes of weak driving and the strong driving; both can be reached in experiments.

The analysis is essentially the same as that in the flux driving case. Instead of the second-order matrix element Δ [see Eq. (15)], we have the direct coupling $\lambda/2$. Thus to obtain the cooling and the heating rates we substitute $\Delta \rightarrow \lambda/2$ into Eqs. (17) and (18). In this scheme there is no driving-induced contribution of the low-frequency ($1/f$) noise similar to Eq. (20).

Strong driving. The strong driving regime is achieved for $\lambda \sqrt{n}/2 \gg \Gamma_r/2$. As the $1/f$ noise does not contribute to the heating, i.e., the rates (20) do not appear, the leading heating mechanism is the off-resonance coupling [see Eq. (18)]. Thus, the dynamics is described by the rate equation (22) with $\Gamma_e = 0$. We obtain the cooling results as follows: For $n_B > Q(\Gamma_r/2\omega_0)$ we obtain $\langle n \rangle \approx n_B - Q(\Gamma_r/2\omega_0)$. In the intermediate regime $Q(\Gamma_r/2\omega_0) > n_B > Q(\Gamma_r/\omega_0)(\lambda/4\omega_0)^2$ the result is $\langle n \rangle \approx \omega_0 n_B / Q\Gamma_r$. Finally, for $n_B < Q(\Gamma_r/\omega_0)(\lambda/4\omega_0)^2$, the occupation number saturates at $\langle n \rangle \approx (\lambda/4\omega_0)^2$.

For example, let $\Gamma_r = 5 \times 10^{-3} \mu\text{eV}$, $\lambda = 25 \times 10^{-3} \mu\text{eV}$ (with the voltage bias $V_0 \sim 50$ mV) and $Q = 10^5$. We have $(\lambda/4\omega_0)^2 \Gamma_r \approx 8 \times 10^{-7} \mu\text{eV}$. The stationary occupation number is

$$\langle n \rangle \approx \begin{cases} n_B(\omega_0) - 500 & \text{if } n_B(\omega_0) > 500 \\ 10^{-3} n_B(\omega_0) & \text{if } 0.15 < n_B(\omega_0) < 500 \\ 1.5 \times 10^{-4} & \text{if } n_B(\omega_0) < 0.15, \end{cases}$$

which gives $\langle n \rangle = 2.5 \times 10^{-3}$ at the temperature of 20 mK.

Weak driving. For $\lambda \sqrt{n}/2 \ll \Gamma_r/2$, the dynamics of the resonator is described by the rate equation (31) with $A_- = \lambda^2/\Gamma_r$ and $A_+ = (\lambda/4\omega_0)^2 \Gamma_r$. Note that the low-frequency noise does not appear in this scheme ($\Gamma_e = 0$), which improves the cooling efficiency. For $n_B(\omega_0) < A_+ Q/\omega_0$ the average occupation is given by $\langle n \rangle = A_+/A_-$; for $n_B(\omega_0) > A_+ Q/\omega_0$ we obtain $\langle n \rangle = n_B(\omega_0) \omega_0 / A_- Q$. As an example, let $\Gamma_r = 50 \times 10^{-3} \mu\text{eV}$, $\lambda = 5 \times 10^{-3} \mu\text{eV}$ (with the bias voltage of $V_0 \sim 10$ mV), and $Q = 10^5$. We have $A_- = 0.5 \times 10^{-3} \mu\text{eV}$ and $A_+ = 3 \times 10^{-7} \mu\text{eV}$. Then, the cooling results are as follows:

$$\langle n \rangle = \begin{cases} 10^{-2} n_B(\omega_0), & n_B(\omega_0) > 0.06 \\ 6 \times 10^{-4}, & n_B(\omega_0) < 0.06, \end{cases} \quad (36)$$

where at the temperature of 20 mK with $n_B(\omega_0) = 5$, we have $\langle n \rangle = 0.05$. Thus better cooling can be achieved in the strong driving regime than that in the weak driving regime.

Discussion. The relaxation rate Γ_r can be adjusted by varying the external circuit of the CPB. For example, by varying the gate capacitance C_g , the relaxation rate changes as $\Gamma_r \propto C_g^2$. The coupling constant λ can be adjusted by varying the bias V_0 . In this scheme, we choose V_0 to be in the range of 10–100 mV and Γ_r in the range of $(1-100) \times 10^{-3} \mu\text{eV}$, which includes both the weak driving regime ($\lambda \sqrt{n}/2 \ll \Gamma_r/2$) and the strong driving regime ($\lambda \sqrt{n}/2 \gg \Gamma_r/2$). These parameter regimes have been realized in charge qubit experiments.

One practical issue of this scheme concerns the accuracy of the gate compensation. Both gate voltages generate an

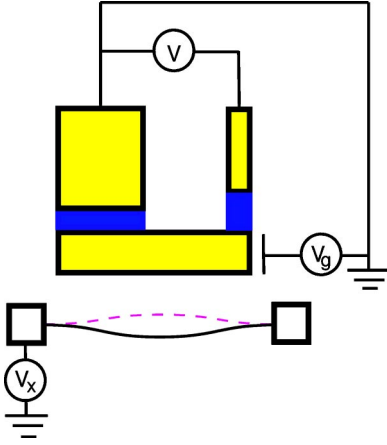


FIG. 9. (Color online) The system with dc driving.

extra part of the CPB's charging energy $\delta H = 4E_c \delta n_g \sigma_x$, with $\delta n_g = (C_x V_x + C_g V_g)/2e$. By controlling the voltage with an accuracy of microvolts, which can be achieved with standard technology, the oscillating bias on the CPB can be neglected.

Compared with the ac flux driving cooling scheme in this paper, the resonator can now be cooled to a lower temperature because the heating process only involves the electromagnetic noise of the circuit at frequency E_J , while the low-frequency noise, which is the dominant heating factor in the previous scheme, does not affect the system.

III. DC COOLING

Effectively, ac driving can be achieved applying a dc transport voltage to an auxiliary Josephson junction. We modify the system as shown in Fig. 9 so that it becomes effectively an SET transistor. Dissipative Cooper pair and quasiparticle transport is similar systems was considered in Refs. 11,12. The Hamiltonian in the charge basis reads

$$H = \frac{[Q - C_R V - C_g V_g - C_x(x) V_x]^2}{2C_\Sigma} - E_{J,L} \cos \theta - E_{J,R} \cos(\theta + 2eVt/\hbar) + H_x, \quad (37)$$

where θ is the phase on the island (and simultaneously on the left junction as the left lead is grounded), $C_\Sigma \equiv C_L + C_R + C_g + C_x(x)$. Due to the transport voltage V , the Hamiltonian is time dependent and, thus, represents a driven system. We repeat the steps described above and arrive (in the charge basis) at

$$H = -\frac{E_{J,L}}{2} \sigma_x - \frac{X}{2} \sigma_z + \hbar \omega_0 a^\dagger a + \lambda (a^\dagger + a) \sigma_z - \frac{E_{J,R}}{2} (\sigma_+ e^{i\omega_J t + i\delta\phi} + \text{H.c.}), \quad (38)$$

where $X \equiv -(4E_c/e)(C_R \delta V + C_g \delta V_g + C_x \delta V_x)$, $\delta\phi \equiv (2e/\hbar) \int \delta V dt$, and the Josephson frequency $\omega_J \equiv 2eV/\hbar$. We see that the right junction's Josephson energy assumes the role of pumping amplitude Ω , while ω_J is the pumping

frequency. Unfortunately V is noisy and, thus, the pumping source has a substantial linewidth. The transport voltage should not be sensitive to the background charges, therefore V is assumed to have Ohmic noise spectrum (no $1/f$ component).

After a $\pi/2$ rotation in the x - z plane, $\sigma_x \leftrightarrow \sigma_z$, we obtain

$$H = -\frac{E_{J,L}}{2} \sigma_z - \frac{X}{2} \sigma_x + \hbar \omega_0 a^\dagger a + \lambda (a^\dagger + a) \sigma_x - \frac{E_{J,R}}{4} [(\sigma_z + i\sigma_y) e^{i\omega_J t + i\delta\phi} + \text{H.c.}]. \quad (39)$$

While the Hamiltonians (12) and (39) look similar, there are two important differences. One, already discussed, is the fact that the pumping frequency ω_J in Eq. (39) is fundamentally noisy, while ω_d in Eq. (12) can be made coherent. The second (very important) difference is that in Eq. (12) the pumping is applied to σ_z only, while in Eq. (39) it couples to σ_z and σ_y . Both these facts hinder the cooling. Indeed, the coupling to σ_y gives a direct matrix element $E_{J,R}/4$ between the states $|\uparrow, n, m\rangle$ and $|\downarrow, n, m+1\rangle$. This interaction repels the levels and we must choose $E_{J,R} \ll 4\omega_0$ so that the resonant detuning as in Fig. 3 is possible. In addition, the noise of the transport voltage translates into the linewidth for the transition $|\uparrow, n, m\rangle \rightarrow |\downarrow, n-1, m+1\rangle$ equal to $\Gamma_\varphi = 2\pi\alpha_{tr} k_B T/\hbar$, where $\alpha_{tr} \equiv R/R_Q$. The fluctuations of the transport voltage are not screened by the ratio of capacitances as it happens for the gate charge. Therefore $\alpha_{tr} \approx 10^{-2}$. Because of these additional constraints the applicability of the dc cooling scheme is limited to higher-frequency/quality factor resonators. For an estimate, consider an oscillator with $\omega_0 = 2\pi \times 1 \text{ GHz} \approx 5 \mu\text{eV} \approx 50 \text{ mK}$ at temperature $T = 50 \text{ mK}$. We then obtain $\Gamma_\varphi \approx 0.3 \mu\text{eV}$, which significantly exceeds Γ_r . Hence, we have to substitute Γ_r by Γ_φ in all formulas. For the Josephson coupling in the right junction we take $E_{J,R} = 2 \mu\text{eV}$. Then, instead of Eq. (15), we find $\Delta \approx E_{J,R} \lambda / (2E_{J,L}) \approx 2 \times 10^{-3} \mu\text{eV}$ (we assume $E_{J,L} \approx 50 \mu\text{eV}$). The cooling rate can again be represented as $A_- n$, where $A_- \approx 2\Delta^2/\Gamma_\varphi \approx 2 \times 10^{-5} \mu\text{eV}$. Thus, cooling becomes possible only if $Q > \omega_0/A_- \approx 2.5 \times 10^5$.

IV. DISCUSSION AND COMPARISON WITH QUANTUM OPTICS

For comparison with quantum optics cooling schemes, we present here an analysis of some of the processes described above using the quantum optics language. In quantum optics literature one usually employs the transformation to the rotating frame and/or other canonical transformations together with the rotating wave approximation (RWA) in order to single out the near-resonant terms responsible for the studied transitions. We start with Hamiltonian (12) and transform it into the interaction representation with respect to $H_0(t) \equiv -[E_J + \Omega + \Omega \cos 2\omega_d t] \sigma_z / 2 + \omega_0 a^\dagger a$. This amounts to

$$\sigma_+ \rightarrow \sigma_+ e^{-i(E_J t + \Omega t + (\Omega/2\omega_d) \sin 2\omega_d t)} \quad \text{and} \quad a \rightarrow a e^{-i\omega_0 t}. \quad (40)$$

As $\Omega \ll 2\omega_d \sim E_J$ we can expand the factor $\exp[-i(\Omega/2\omega_d)\sin 2\omega_d t]$. The near-resonant (RWA) parts of the coupling term $H_\lambda \equiv \lambda(a^\dagger + a)\sigma_x$ read, then, after the transformation as

$$H_\lambda^{\text{RWA}} = -\frac{\Omega\lambda}{4\omega_d}(a\sigma_- + a^\dagger\sigma_-e^{2i\omega_d t} + \text{H.c.}), \quad (41)$$

where the resonance condition $2\omega_d = E_J + \Omega - \omega_0$ was assumed. The first term of Eq. (41) clearly corresponds to the resonant transition provided by the matrix element $\Delta\sqrt{n}$ [see Eq. (15)]. The second term corresponds to the off-resonant transition. In analogy with atom optics,³ we can introduce Lamb-Dicke parameter $\eta \equiv \lambda/2\omega_d \approx \lambda/E_J$. Note that, unlike the case of trapped atoms/ions we do not have near-resonant terms flipping just the spin, of the type $\Omega\sigma_\pm \exp(\pm i\omega_0 t)$. Such an interaction would repel the levels, and, for a resonant detuning, one would need $\Omega \ll \omega_0$. In our case, a much stronger driving is allowed. We do, however, encounter this limitation in the dc cooling scheme due to the presence of the near-resonant term in the Hamiltonian Eq. (39) which is proportional to σ_y .

Also, in the dissipative term $H_X \equiv -(X/2)\sigma_x$, we find a slow contribution:

$$H_X^{\text{slow}} = \frac{X}{2} \frac{\Omega}{4\omega_d} (\sigma_- e^{i\omega_0 t} + \text{H.c.}), \quad (42)$$

which is responsible for the heating process (18) dominated by the low-frequency noise. In quantum optics this contribution is typically neglected as there are no strong low-frequency sources.

In the case of ac voltage cooling, in the rotating wave approximation the Hamiltonian is

$$\begin{aligned} H^{\text{RWA}} = & -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega_0\hat{a}^\dagger\hat{a} \\ & + \frac{\lambda}{2}(\hat{a}^\dagger\sigma_+ + \hat{a}\sigma_- + \hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \\ & - \frac{\hat{X}}{2}(\sigma_+ e^{-i\omega_d t} + \sigma_- e^{i\omega_d t}), \end{aligned} \quad (43)$$

where the interaction includes the resonant coupling $\hat{a}^\dagger\sigma_+ + \hat{a}\sigma_-$ between the states $|\uparrow, n\rangle$ and $|\downarrow, n-1\rangle$, and the off-resonant coupling $\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+$ between the states $|\uparrow, n\rangle$ and $|\downarrow, n+1\rangle$ with an energy difference of $2\hbar\omega_0$. The time dependence in the last term shows that only high-frequency fluctuations on the order of E_J induces relaxation.

The voltage driving scheme for ac cooling presents a direct analog to the laser cooling of quantum optics systems as is obvious from Fig. 10; however, it is a simpler scheme as the coupling is a direct bilinear coupling instead of the polaron coupling in quantum optics.

V. “DOPPLER” COOLING

For smaller oscillator frequencies ω_0 and/or larger qubit relaxation rates Γ_r , it may happen that the vibrational side-

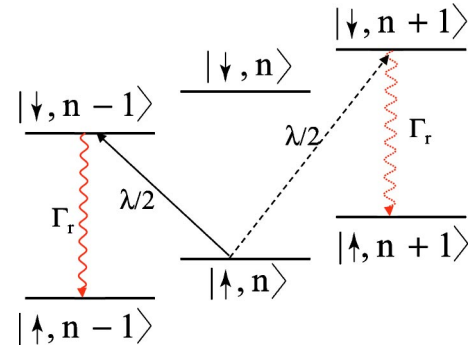


FIG. 10. (Color online) Level structure of the voltage driving scheme for ac cooling. The solid lines indicate the cooling process; and the dotted lines indicate the heating process.

bands are no longer resolved, i.e., $\Gamma_r > \omega_0$. Then, cooling to the ground state is impossible but the less effective “Doppler” cooling is still feasible. In the standard Doppler cooling, cold atoms moving in a laser field experience different light pressure depending on their velocity. For instance, if the laser is red-detuned relative to the atomic transition, then atom moving towards the light source will be absorbing photons more than an atom moving away from the source, which will result in a velocity-dependent force on the atom. By properly arranging multiple lasers it is then possible to achieve significant reduction of the atom’s effective temperature. The same reasoning applies to an atom in a trap, when the trap frequency is much smaller than the atomic transition linewidth. Equivalently, in this case, the Doppler cooling can be reinterpreted in terms of the cooling and heating transition rates for the oscillating atom. For red-detuned laser, photon absorption processes with simultaneous phonon emission dominate the ones where an additional phonon is created. One can show that, similar to the free atom case, this leads to cooling down to temperatures proportional to the atomic transition linewidth. In this section we will demonstrate that an analog of Doppler cooling can be performed in both ac and dc cooling setups described in the preceding sections in the regime of nonresolved sidebands.

ac cooling scheme. We start with the ac-driving scheme described by Hamiltonian (12). In the Doppler case it is enough to study, first, the spin’s dynamics without the oscillator. We perform the following transformation

$$\sigma_+ \rightarrow \sigma_+ e^{-i(2\omega_d t + (\Omega/2\omega_d)\sin 2\omega_d t)}, \quad (44)$$

and keep only the RWA terms in the coherent part of the Hamiltonian but all the terms in the part describing the interaction with the bath. The transformed Hamiltonian reads

$$H_{\text{spin}}^{\text{RWA}} = \frac{\delta\omega}{2}\sigma_z - \frac{X}{2}(\sigma_+ e^{-2i\omega_d t} + \text{H.c.}) + \frac{\Omega}{4\omega_d} \frac{X}{2}\sigma_x, \quad (45)$$

where $\delta\omega$ is the detuning $\delta\omega \equiv 2\omega_d - E_J - \Omega$. The last term in the Hamiltonian, as before [see Eqs. (19) and (18)], generates transitions between the ground and excited states of the qubit. The corresponding rate, however, is typically much smaller than the relaxation rate generated by the second term

of the Hamiltonian, and hence can be neglected. This is in contrast to the ac ground-states cooling scheme discussed above, where the qubit heating rate determined the lowest possible occupation number of the resonator. The slow (RWA) part of the interaction Hamiltonian reads

$$H_{\lambda}^{\text{slow}} = -\frac{\Omega}{4\omega_d}\lambda(a^{\dagger}+a)\sigma_x. \quad (46)$$

With Hamiltonian (45) we write down the Bloch-Redfield equations^{13,14} for the spin's density matrix $\dot{\rho}=R\rho$, where R is the Redfield tensor. In this equation, the density matrix is treated as a four-vector. We choose the representation $\rho=(1/2)s\sigma_0+\gamma\sigma_z+\alpha\sigma_++\alpha^*\sigma_-$, which gives $\rho=(s,\gamma,\alpha,\alpha^*)$ and $\text{Tr}\rho=s$. For proper density matrices $s=1$. In this representation

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\Gamma_r}{2} & -\Gamma_r & 0 & 0 \\ 0 & 0 & -i\delta\omega - \frac{\Gamma_r}{2} & 0 \\ 0 & 0 & 0 & i\delta\omega - \frac{\Gamma_r}{2} \end{pmatrix}. \quad (47)$$

The rate Γ_r originates from the second term of Eq. (45),

$$\Gamma_r = \frac{\langle X_{\omega=2\omega_d}^2 \rangle}{4\hbar^2}, \quad (48)$$

[cf. Eq. (7)]. For simplicity we have assumed that at high frequency $2\omega_d$ the temperature is effectively zero, i.e., $\langle X_{\omega=-2\omega_d}^2 \rangle \ll \langle X_{\omega=2\omega_d}^2 \rangle$.

Next we employ the ‘‘quantum regression theorem’’ (approximation) to obtain (for $t>0$) the correlator

$$C_x(t) \equiv \langle \sigma_x(t)\sigma_x(0) \rangle = \text{Tr}(\sigma_x e^{Rt}\sigma_x\rho_{\infty}), \quad (49)$$

where ρ_{∞} is the stationary density matrix, $R\rho_{\infty}=0$. For $t<0$ we can use $C_x(-t)=C_x^*(t)$. In the four-vector representation the operator σ_x multiplying from the left is given by

$$\sigma_x = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1/2 & 1/2 \\ 1/2 & -1 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \end{pmatrix}. \quad (50)$$

Finally, after the Fourier transform we obtain

$$C_x(\omega) = -2 \text{Re} \text{Tr}(\sigma_x [R+i\omega]^{-1}\sigma_x\rho_{\infty}), \quad (51)$$

which is easy to calculate (using MATHEMATICA). Expanding near $\omega=0$ we obtain $C_x(\omega)=S_x+\eta_x\omega$, where

$$S_x = \frac{4\Gamma_r}{4\delta\omega^2 + \Gamma_r^2}, \quad (52)$$

and

$$\eta_x = -\frac{32\delta\omega\Gamma_r}{(4\delta\omega^2 + \Gamma_r^2)^2}. \quad (53)$$

From the spin-correlation functions, we can now determine the transition rates for the resonator, $A_{\pm}=(\Omega\lambda/4\omega)^2 C_x(\mp\omega_0)$. Note, that the same expressions for A_{\pm} can be obtained from Eq. (16) generalized to arbitrary detuning. Then, the secondary ‘‘heat bath’’ introduced by cooling is characterized by the effective temperature

$$T^* \equiv \frac{S_x}{2\eta_x} = -\frac{4\delta\omega^2 + \Gamma_r^2}{16\delta\omega}. \quad (54)$$

Optimizing with respect to the detuning $\delta\omega$, we find that the minimum (positive) temperature $T_{\min}\approx\Gamma_r/4$ is reached for the optimal red detuning $\delta\omega_{\text{opt}}\approx-\Gamma_r/2$. For the effective quality factor due to the spin we obtain

$$\frac{1}{Q^*} = \frac{A_- - A_+}{\omega_0} = \frac{\Omega^2\lambda^2\eta_x}{8\omega_d^2}. \quad (55)$$

The final resonator occupancy can now be obtained from Eq. (33). Clearly, Q^* must be smaller than the oscillator's own quality factor in order for the cooling to be effective. For the same parameters as used in the ac cooling section, we find that at the optimal detuning $Q^*\sim 1$, which corresponds to a nearly overdamped regime, similar to optical molasses in atom optics. Hence, a resonator with a frequency $\omega_0\leq\Gamma_r/2=300$ kHz can be cooled down to temperature $T_{\min}\approx\Gamma_r/4\sim 10$ μ K. Note, that the effect of the dominant qubit heating mechanism due to the $1/f$ noise, which we neglected here, if necessary, can be managed by reducing the driving strength Ω (at the expense of reduced cooling power, i.e., larger Q^*).

Similar analysis applies to the voltage driving scheme for ac cooling in the Doppler regime. For $\lambda\sqrt{n}/2\ll\Gamma_r/2$, from Eq. (16)

$$A_- = \frac{\lambda^2\Gamma_r}{\Gamma_r^2 + 4(\delta\omega + \omega_0)^2}, \quad A_+ = \frac{\lambda^2\Gamma_r}{\Gamma_r^2 + 4(\delta\omega - \omega_0)^2}, \quad (56)$$

where $\delta\omega=\omega_d-E_J$ is the detuning of the driving frequency from the qubit's ground-to-excited-state transition. Cooling is achieved when $\delta\omega<0$. Neglecting the effect of the finite intrinsic Q -factor, the final phonon number from Eq. (32) is

$$\langle n \rangle_f = \frac{\Gamma_r^2 + 4(\delta\omega + \omega_0)^2}{16|\delta\omega|\omega_0}. \quad (57)$$

The scheme provides optimal cooling at $\delta\omega_{\text{opt}}=-\Gamma_r/2$ with the temperature $T_{\min}=\Gamma_r/4$, same as the flux driving case.

dc cooling scheme. Analysis of the dc cooling scheme proceeds analogously. We start with Hamiltonian (39) and perform transformation

$$\sigma_+ \rightarrow \sigma_+ e^{-i(\omega_J t + \frac{\Omega}{\omega_J} \sin \omega_J t)}, \quad (58)$$

where, for now, $\Omega \equiv E_{J,R}$, and we remind that $\omega_J \equiv (2e/\hbar)V$. For simplicity we consider V noiseless now but later will introduce the low-frequency (classical) noise of V . Then, we obtain

$$H_{\text{spin}}^{\text{RWA}} = \frac{\delta\omega}{2} \sigma_z - \frac{\Omega}{4} \sigma_x - \frac{X}{2} (\sigma_+ e^{-Vt} + \text{H.c.}) + \frac{\Omega}{2V} \frac{X}{2} \sigma_x, \quad (59)$$

where $\delta\omega \equiv \omega_J - E_{J,L}$. The main differences with Eq. (45) are: (i) the second term of Eq. (59) is absent in Eq. (45) (recall the extra matrix element in the dc scheme); (ii) the detuning $\delta\omega$ is noisy due to the noise of V . This will give an extra ‘‘pure’’ dephasing rate $\Gamma_\varphi = 2\pi\alpha_{\text{tr}}k_B T/\hbar$, where $\alpha_{\text{tr}} \equiv R/R_Q$. As in the ac Doppler case, neglecting the excitation rate coming from the last term in the Hamiltonian, for the Redfield tensor we obtain

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\Gamma_r}{2} & -\Gamma_r & -\frac{i\Omega}{4} & \frac{i\Omega}{4} \\ 0 & -\frac{i\Omega}{2} & -i\delta\omega - \tilde{\Gamma} & 0 \\ 0 & \frac{i\Omega}{2} & 0 & i\delta\omega - \tilde{\Gamma} \end{pmatrix}, \quad (60)$$

where $\tilde{\Gamma} = \Gamma_\varphi + \Gamma_r/2$. The general expressions for S_x and η_x are quite complicated. An analysis shows that the simplest cooling regime is achieved when $\Omega \ll \Gamma_r \approx \Gamma_\varphi$. That is, it does not make sense to keep small Γ_r when the linewidth is anyway given by large Γ_φ . The relaxation rate Γ_r can easily be increased by choosing bigger gate capacitances C_x and/or C_g . Assuming $\Gamma_r = \Gamma_\varphi$ and $\Omega \ll \Gamma_\varphi$, we obtain

$$S_x = \frac{12\Gamma_\varphi}{4\delta\omega^2 + 9\Gamma_\varphi^2}, \quad (61)$$

and

$$\eta_x = -\frac{96\delta\omega\Gamma_\varphi}{(4\delta\omega^2 + 9\Gamma_\varphi^2)^2}. \quad (62)$$

Optimizing with respect to $\delta\omega$, we find that the minimum temperature $T_{\text{min}} \approx (3/4)\Gamma_\varphi$ is achieved for the optimal detuning $\delta\omega \approx -(3/2)\Gamma_\varphi$. The effective quality factor due to the cooling environment, similarly to Eq. (55), is

$$\frac{1}{Q^*} = \frac{\Omega^2 \lambda^2 \eta_x}{2\omega_J^2}. \quad (63)$$

The regime $\Gamma_r \ll \Gamma_\varphi$ is more subtle and requires further analysis.

Example. Choosing $C_x \approx 200$ aF instead of previously assumed 20 aF, we obtain $\Gamma_r \approx 0.3 \mu\text{eV} \approx \Gamma_\varphi$. We choose $\Omega = E_{J,R} \approx 0.1 \mu\text{eV}$ (note that this is quite a small value for usual Josephson junctions). Then we obtain $T_{\text{min}} \approx 0.25 \mu\text{eV} \approx 2.5$ mK for $\delta\omega \approx 0.5 \mu\text{eV} \approx 2\pi \times 100$ MHz. Near optimal detuning point, we obtain $\eta_x \approx 6(\mu\text{eV})^{-2}$. For the coupling constant λ we can take $\lambda \approx 1 \mu\text{eV}$ instead of $0.1 \mu\text{eV}$ as we have allowed ten times bigger capacitance C_x . Then we obtain $Q^* \approx 0.8 \times 10^5$. Thus, for cooling to be effective, resonator Q -factor should exceed 10^5 and the final resonator temperature is determined according to Eq. (33) with $T^* = 2.5 \mu\text{K}$.

VI. CONCLUSIONS

We considered several approaches to active cooling of mechanical resonators using a coupling to a superconducting Josephson qubit. In the resolved vibrational sideband regime, when the qubit level width is smaller than the resonator frequency, we proposed two schemes for ground-state cooling of the resonator. In the first scheme, the ac driving required for cooling is provided by an external microwave source. We find that for a 100 MHz oscillator coupled to a practically realizable Josephson qubit, at the external temperatures below 1 K, it is possible to reduce the thermal occupancy of the oscillator mode by three order of magnitude. In the second scheme, the ac driving is generated by the ac Josephson oscillations on an auxiliary junction of the qubit. This scheme is attractive since there is no need for an external ac driving source; however, in the present realization, we find that it is not as effective as the one with an explicitly applied ac driving. We also demonstrate that even in the regime when the vibrational sidebands are not resolved, it is possible to perform an analog of Doppler cooling with the final resonator temperature limited by the qubit linewidth.

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