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Singular corrections to the Fermi-liquid theory

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We show that the singularities in the dynamical bosonic response functions of a two-dimensional Fermi liquid give rise to universal nonanalytic corrections to the results of the Fermi-liquid theory. In particular, we find a T^2 term in the specific heat, linear-in-T terms in the effective mass and in the uniform spin susceptibility $\chi_s(Q=0,T)$, and $|Q|$ term in $\chi_s(Q,T=0)$. The existence of these terms has been the subject of recent controversy, which is resolved in this paper. We present exact expressions for all nonanalytic terms to second order in a generic interaction $U(Q)$ and show that the nonanalytic terms originate exclusively from forward- and backward-scattering of particles with zero total momentum.

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The universal features of a Fermi liquid and their physical consequences continue to attract the attention of the condensed-matter community. In a three-dimensional Fermi liquid, the leading term in the real part of the on-shell self-energy, $\Sigma(\omega)$, behaves as ω for $\omega\rightarrow 0$ whereas the imaginary part vanishes as ω^2 or T^2. Such a regular behavior of the self-energy has a profound effect on, e.g., the specific heat and uniform spin and charge susceptibilities, which behave similarly to the free-fermion case, i.e., the specific heat is linear in T and the susceptibilities approach finite values at $T=0$. A regular behavior of the self-energy is also in line with a general argument that turning on the interaction in $D>1$ should not affect drastically the low-energy properties of a system,[1] unless special circumstances, e.g., a proximity to a quantum phase transition,[2] interfere.

The subject of this paper is the analysis of nonanalytic corrections (NAC) to the Fermi-liquid behavior. These corrections are universal in a sense that they are determined by fermions near the Fermi surface, and are of fundamental interest as they lead to anomalous temperature and momentum dependences of observable quantities. A well-known example is the $T^3\ln T$ term in the specific heat in three dimensions (3D).[3] Another example, discussed recently in the context of the metal-insulator transition in 2D,[4] is the linear-in-T correction to the conductivity of a weakly disordered 2D system.[5–7] NAC are also important for the theory of quantum critical phenomena in itinerant ferromagnets,[8] as a nonanalytic momentum dependence of the spin susceptibility may change the nature of the phase transition.[9] On the experimental side, both $T^3\ln T$ behavior of $C(T)$ and its analog in 2D (T^2) were observed in He^3.[10] There is also an evidence for the linear-in-T term in the spin susceptibility of a 2D compound Sr_2RuO_4.[11,12]

Nonanalyticities in observable quantities can be traced down to a behavior of $\Sigma(\omega)$, which does not have a regular expansion in integer powers of ω. The reason for this nonanalyticity can be understood by recalling that Landau's argument for the ω^2 (or T^2) behavior of Σ'' relies on the Fermi statistics of quasiparticles and on the assumption that the effective interaction is screened at large distances.[13] Long-range (current-current[14] or gauge[15]) interactions lead to the breakdown of the Fermi liquid. However, even if the bare interaction U is pointlike, the effective one contains a long-range part at finite frequencies. Indeed, already to the second order in U, the effective interaction $\tilde{U}=U^2\Pi(\Omega,q)$ is proportional to the *dynamical* polarization bubble of the electron gas, $\Pi(\Omega,q)$. In all dimensions, Π'' is universal and singular in q for $\Omega\ll v_F q\ll v_F k_F$,[13]

$$\Pi''(\Omega,q)=a_D\frac{\Omega}{v_F|q|}+\cdots,$$

where a_D is a coefficient, and v_F and k_F are the Fermi velocity and momentum, respectively. Due to this singular q behavior of Π'', $\tilde{U}(r)$ behaves as $1/r^{D-1}$ at distances $k_F^{-1}\ll r\ll v_F/|\Omega|$.

The induced long-range interaction affects the self-energy which, to the second order in U, is given by

$$\Sigma''(\omega)\simeq\frac{U^2}{v_F}\int_0^{|\omega|}d\Omega\int_{\sim|\Omega|/v_F}^{\sim k_F}dq\,q^{D-2}\Pi''(\Omega,q).$$

For $D>2$, the leading term in the momentum integral converges in the infrared, and $\Sigma''(\omega)\propto\omega^2$ in agreement with the Landau's argument. However, a subleading term is dominated by the lower limit, and behaves not as ω^4, as one might have expected, but as $|\omega|^D$ for $D<4$. For $D\leqslant 2$, already the leading term is infrared divergent and $\Sigma''(\omega)\propto|\omega|^D$, with an extra log for $D=2$. The breakdown of the expansion of $\Sigma''(\omega)$ in ω^2 does *not* mean a breakdown of the Fermi liquid: it is easy to see that $|\Sigma'(\omega)|\gg\Sigma''(\omega)$ for $D>1$, so that the quasiparticles are well defined. However, the nonanalyticity in Σ in all dimensions transfers into NAC to thermodynamic quantities.[16] Indeed, power counting indicates that the nonuniform charge and spin susceptibilities $\chi_{c,s}(Q,T)$ and the specific heat $C(T)$ may acquire nonanalytic corrections of the form $\max\{Q^{D-1},T^{D-1}\}$ (Refs. 12, 17 and 18) and T^D (Refs. 19 and 20), respectively (with extra logs for $D=1,3$).

Our motivation to study the NAC to the Fermi-liquid behavior is twofold. First, it is necessary to verify the power-counting arguments by carrying out explicit calculations of several observable quantities: $C(T)$, $\chi_{c,s}(Q,T)$, and the effective mass. That power counting may be misleading is seen, e.g., for the free-fermion susceptibility: according to power counting, it should also have a nonanalytic momentum dependence, whereas the exact result is a well-known Lindhard function which is analytic in Q for small Q. Second, it is very important to understand the origin of NAC in the Fermi-liquid theory. Without such an understanding, it is not clear why power counting fails in certain cases and also why explicit calculations show that only the spin but not charge susceptibility exhibits a nonanalytic behavior.

Results of prior explicit calculations are somewhat controversial. In $D=3$, the $T^3 \ln T$ term in $C(T)$ was found long ago.[3] More recently, Belitz, Kirkpatrick, and Vojta[17] (BKV) have shown that, to the second order in the interaction, the momentum dependence of the spin susceptibility is $Q^2 \ln Q$ in agreement with power counting. At the same time, the uniform $\chi_s(0,T)$ was found to scale as T^2 (Refs. 17 and 21) rather than as $T^2 \ln T$ predicted by power counting. The $T^2 \ln T$ term in $\chi_s(0,T)$ was, however, found in Ref. 22. In 2D, no explicit calculations of $\chi_s(Q,0)$ have been performed, although BKV conjectured that it should scale as $|Q|$. As far as the T dependences are concerned, Coffey and Bedell[20] obtained a T^2 term in $C(T)$ and Das Sarma *et al.*[23] found a linear in T term in the effective mass, $m^*(T)$. On the contrary, Chitov and Millis[12] (CM) argued that different contributions to $C(T)$ and $m^*(T)$ cancel each other, and only analytic corrections survive. Yet, CM found the T term in $\chi_s(0,T)$ for $D=2$, in agreement with power counting, but no such term in $\chi_c(0,T)$. Fratini and Guinea[24] extended the CM analysis to anisotropic Fermi surfaces.

In this paper, we present explicit results for the specific heat, effective mass, and spin and charge susceptibilities of an interacting, isotropic 2D Fermi system, to the second order in a finite-range interaction $U(q)$. We found that power-counting arguments are valid in 2D, i.e., $\chi_s(Q) \propto |Q|$, $\chi_s(T) \propto T$, $m^*(T) \propto T$, and $\delta C(T) \propto T^2$. These results agree with Refs. 18–20; the form of $\chi_s(T)$ agrees with that found by CM but the forms of $m^*(T)$ and $\delta C(T)$ disagree with those by CM. In agreement with CM, we found that different nonanalytic contributions to the charge susceptibility $\chi_c(Q,T)$ cancel each other. We also consider arbitrary D and explain why in $D=3$, $\chi_s(0,T) \propto T^2$ while $\chi_s(Q,0) \propto Q^2 \ln|Q|$.

A tractable model adopted in this paper allowed us to analyze in detail the origin of the NAC. We found that these corrections result from the singularities in the *dynamic* particle-hole response function, $\Pi(q,\Omega)$, near $q=0$ and $q=2k_F$, where $\Pi(q,\Omega)$ is nonanalytic. Physically, these two singularities give rise to a zero-sound mode and Friedel oscillation, respectively. The singularity near $q=0$ is entirely dynamic, while the one near $2k_F$ is also present in the static limit for $q>2k_F$. We found that the singularities in $\Pi(q,\Omega)$ are necessary ingredients which make power-counting arguments valid, i.e., they ensure that the prefactors in power-counting results do not vanish. As $q=0$ and $2p_F$ singularities are generic to a Fermi liquid,[13] the NAC that we found should retain their functional Q- and T forms for an arbitrary order in the interaction. These forms will change, however, for special anisotropic Fermi surfaces with, e.g., inflection points.[24]

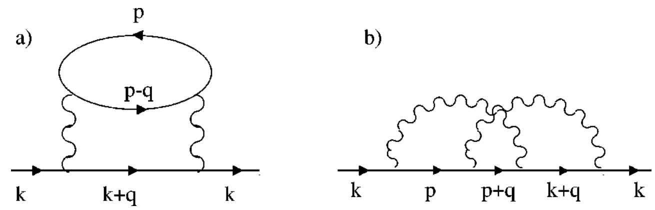

FIG. 1. Nontrivial second-order diagrams for the self-energy.

We also found that the nonanalytic corrections originate exclusively from a special type of essentially 1D collisions, i.e., when two incoming particles have opposite momenta *and* experience either forward or backward scattering. This implies that nonanalytic terms depend only on $U(0)$ and $U(2k_F)$ but not on the interaction averaged over the Fermi surface. Furthermore, as the two processes are parts of the same scattering amplitude $\Gamma_{\alpha,\beta;\gamma,\delta}(\mathbf{k},-\mathbf{k};\mathbf{k},-\mathbf{k}) = U(0)\,\delta_{\alpha\gamma}\delta_{\beta\delta} - U(2k_F)\,\delta_{\alpha\delta}\delta_{\beta\gamma}$, NAC to a Fermi-liquid behavior can be viewed *equivalently* as coming either from the singularity in the dynamical particle-hole bubble at $q=0$ *or* at $=2k_F$. As the $q=0$ singularity is entirely dynamical, the nonanalyticities are dynamical in nature as well.

The fact that the nonanalyticities originate from $q=0$ and $q=2k_F$ singularities in the particle-hole response functions also explains why they are present in the spin but not charge susceptibility. In the presence of a magnetic field, the interaction channels involving generation of electron-hole pairs of opposite spins acquire a finite energy gap (Zeeman splitting). As a result, the $q=0$ and $2p_F$ singularities in this channel are smeared by the magnetic field and a response to this field is singular. On the other hand, a variation of the chemical potential does not affect the singular parts of the particle-hole bubble, hence a charge response is regular.[25]

Effective mass and specific heat. To find the effective mass $m^*(T)$ one needs to know the real part of the fermionic self-energy, $\Sigma'(k,\omega)$, on the mass shell, i.e., at $\epsilon_k \equiv (k^2 - k_F^2)/2m = \omega$. The two nontrivial second-order diagrams for the Fermi energy are presented in Fig. 1. We evaluated $\Sigma''(k,\omega)$ first and then obtained $\Sigma'(\omega)$ on the mass shell via Kramers-Krönig transformation. The imaginary part of the self-energy reduces to the well-known forms[26] $\Sigma''(k,\omega) \propto \omega^2 \ln|\omega|$ and $\Sigma''(k,\omega) \propto T^2 \ln T$ for k near the Fermi surface and in the limits of $T \to 0$ and $\omega \to 0$, respectively. We obtained $\Sigma''(k,\omega,T)$ at arbitrary ω/T and ω/ϵ_k. Using this full $\Sigma''(k,\omega,T)$, we find that the real part of the self-energy on the mass shell has a regular, Fermi-liquid-type ω term, plus a nonanalytic piece

$$\Sigma'(\omega,T) = -\frac{m\overline{U}^2}{16\pi^2 v_F^2}\,\omega|\omega|\,g\!\left(\frac{\omega}{T}\right), \tag{1}$$

where $\overline{U}^2 = U^2(0) + U^2(2p_F) - U(0)U(2p_F)$ and

$$g(x) = 1 + \frac{4}{x^2}\left[\frac{\pi^2}{12} + \mathrm{Li}_2(-e^{-|x|})\right], \tag{2}$$

where $\mathrm{Li}_2(x)$ is a polylogarithmic function.

In the two limits, $g(\infty)=1$ and $g(x\ll 1)\approx 4\ln 2/x$. The first limit corresponds to $T=0$ in which case Eq. (1) gives $\Sigma'(\omega)\propto\omega|\omega|$. This nonanalytic form agrees with power counting. For small ω/T, i.e., for $x\ll 1$, the $1/x$ form of $g(x)$ leads to the ωT term in $\Sigma'(\omega,T)$ for $\omega\ll T$. This in turn implies that the quasiparticle mass $m^*(T)$ acquires a linear-in-T correction

$$m^*(T)=m\left(1-m^2\overline{U}^2\frac{\ln 2}{8\pi^2}\frac{T}{E_F}\right). \tag{3}$$

We next consider the specific heat. The general expression is given in Ref. 13 [see also Eq. (4) below]. We verified that for our $\Sigma(k,\omega)$ it reduces to the frequency integral of the on-shell $\Sigma'(\omega,T)$. Using Eqs. (1) and (2), we obtained

$$\begin{aligned}\delta C(T)&=\frac{2}{\pi}T\frac{\partial}{\partial T}\left[\frac{1}{T}\int_{-\infty}^{\infty}\frac{d^2k}{(2\pi)^2}\int_{-\infty}^{\infty}d\omega\,\omega\frac{\partial n}{\partial\omega}\right.\\&\left.\qquad\times\mathrm{Im}[\Sigma(k,\omega)G(k,\omega)]\right]\\&=-\frac{2m}{\pi}T\frac{\partial}{\partial T}\left[\frac{1}{T}\int_{-\infty}^{\infty}d\omega\,\omega\frac{\partial n}{\partial\omega}\Sigma'(\omega,T)\right]\\&=-AC_{FL}(m\overline{U})^2\left(\frac{T}{E_F}\right),\end{aligned}\tag{4}$$

where n is a Fermi function, $C_{FL}=\pi Tm/3$ is the Fermi-gas result, and $A=9\zeta(3)/(4\pi^4)=0.03$. We see that a nonanalyticity in the fermionic self-energy gives rise to the T^2 term in the specific heat. This term comes only from fermions in a near vicinity of the Fermi surface and from this perspective is model independent.

Spin susceptibility. The relevant diagrams for the spin susceptibility are presented in Fig. 2. Evaluation of the diagrams is rather tedious but straightforward. We calculated all diagrams in two ways: (i) explicitly, by exploring the nonanalyticities in the particle-hole bubble near $Q=0$ and $2k_F$, and (ii) by retaining only vertices in which both total and transferred momenta are small. We obtained identical results in both methods, which proves that only a single scattering amplitude is relevant.

The nonanalytic contributions to the spin susceptibility from individual diagrams are as follows:

$$\chi_1(Q,T)=\chi_0K(Q,T)[U^2(0)+U^2(2k_F)],$$

$$\chi_2(Q,T)=-\chi_0K(Q,T)U(0)U(2k_F),$$

$$\chi_3(Q,T)=\chi_0K(Q,T)[U^2(2k_F)-U^2(0)],$$

$$\chi_4(Q,T)=\chi_0K(Q,T)U(0)U(2k_F),$$

$$\chi_5=\chi_6=\chi_7=0, \tag{5}$$

where $\chi_0=m/\pi$, and $K(Q,0)$ and $K(0,T)$ are given by

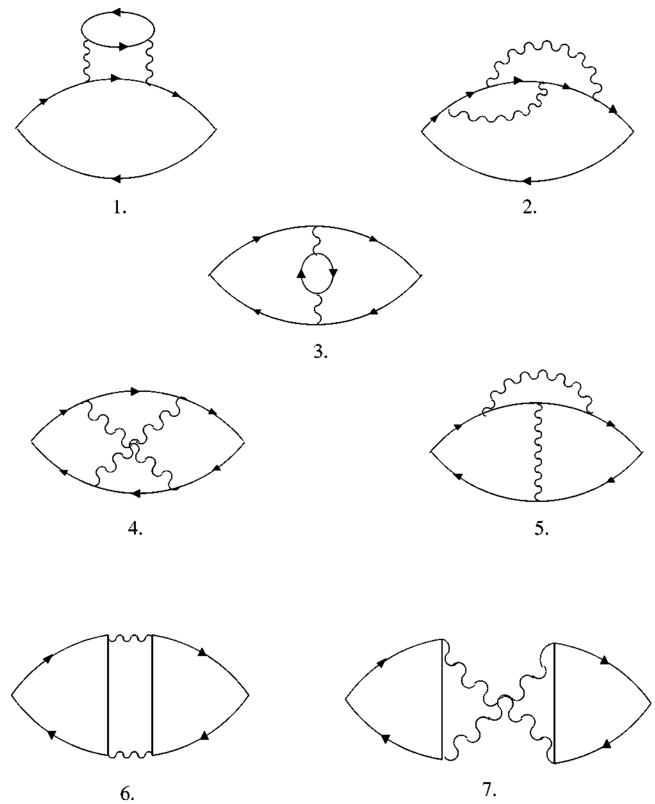

FIG. 2. Relevant second-order diagrams for spin and charge susceptibilities. The last two diagrams are nonzero only for the charge susceptibility.

$$K(Q,0)=\frac{2}{3\pi}\left(\frac{m}{4\pi}\right)^2\frac{|Q|}{k_F},\quad K(0,T)=\left(\frac{m}{4\pi}\right)^2\frac{T}{E_F}. \tag{6}$$

Collecting all contributions, we find

$$\chi_s(Q,T)=\sum_{i=1}^{7}\chi_i(Q,T)=\chi_0[1+2K(Q,T)U^2(2k_F)]. \tag{7}$$

We see that all nonanalytic contributions with $U(0)$ cancel out, and the final result depends only on $U(2k_F)$.

We now take a deeper look into the origin of the nonanalytic contributions to the susceptibility. The power-counting argument does not rely on the singularity in the particle-hole bubble, i.e., a singular piece in $\Pi(\Omega_m,q)$ near $q=0$, $\Pi_{\rm sing}=\nu_2\Omega_m/\sqrt{(v_Fq)^2+\Omega_m^2}$, has a scaling dimension zero and hence is treated as a constant in power counting. However, we found that for each diagram, a replacement of $\Pi(\Omega_m,q)$ by a constant does not give rise to a linear-in-$|Q|$ term in $\chi_s(Q,0)$ because in the prefactors to this term all poles are located in the same half plane of q, and the integral over q vanishes. This vanishing could not be detected by power counting. The substitution of $\Pi_{\rm sing}$ instead of a constant changes the situation as this term contains a branch-cut singularity which is present in both half planes of q. The q integral then does not vanish, and the $|Q|$ term emerges in $\chi_s(Q,0)$, in agreement with power counting.

A nonanalytic T dependence of $\chi(0,T)$ is also associated with the singularity in $\Pi(Q,\Omega)$ but the interplay between the two is somewhat different from that at $T=0$ as a singular piece in $\chi_s(0,T)$ comes only from a zero bosonic Matsubara frequency and not from a set of $\omega_m \sim T$. We found that in any $2 \leqslant D \leqslant 3$, the result for $\chi_s(0,T)$ can be expressed as

$$\chi_s(0,T) = AU^2(2p_F)J, \tag{8}$$

where A is a positive constant, and

$$J = \frac{(D-2)(4-D)}{2}\left(\frac{T}{E_F}\right)^{D-1}\int_0^\infty \frac{dz\, z^{D-2}}{e^z-1}. \tag{9}$$

For $D \to 2$, $J \to (T/E_F)$ which yields Eqs. (6) and (7). For arbitrary $2<D<3$, $\chi_s \propto J \propto T^{D-1}$ which parallels $\chi_s(Q,0) \propto |Q|^{D-1}$. However, in $D=3$, Eq. (9) yields $J \propto T^2$ without logarithmic corrections, i.e., $\chi_s(0,T) \propto T^2$. This explains why the $Q^2 \ln|Q|$ term is *not* accompanied by $T^2 \ln T$ in 3D.

Charge susceptibility. The charge susceptibility $\chi_c(Q,T)$. Here we have two additional contributions given by diagrams 6 and 7 in Fig. 2 (for the spin susceptibility these diagrams vanish after spin summation). We find

$$\chi_6(Q,T) = \chi_0 \; K(Q,T)[U^2(0) - U^2(2k_F)],$$

$$\chi_7(Q,T) = -\chi_0 \; K(Q,T)[U^2(0) + U^2(2k_F)], \tag{10}$$

where $K(Q,T)$ is given by Eq. (6). Combining this last result with Eq. (5), we find that all nonanalytic terms from individual diagrams cancel out, i.e., $\chi_c(Q,T)$ is regular. This agrees with the result by CM.[12]

To conclude, in this paper we demonstrated that the universal singularities in the bosonic response functions of a Fermi liquid give rise to universal nonanalytic corrections to the Fermi-liquid forms of the self-energy, the specific heat and the spin, but not charge, susceptibility. We obtained explicit results in 2D for $\delta C(T) \propto T^2$, $\chi_s(Q,T=0) \propto |Q|$, $\chi_s(Q=0,T) \propto T$. We demonstrated that these nonanalytic terms come from the processes with both transferred *and* total momentum close to zero. We also demonstrated that thermal ($\propto T$) and quantum ($\propto |Q|$) corrections to the spin susceptibility are of different origin. This explains why in 3D, $\chi_s(Q,0) \propto Q^2 \ln Q$, while $\chi_s(0,T) \propto T^2$.

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