Stability and quality factor of a one-dimensional subwavelength cavity resonator containing a left-handed material

Linfang Shen,[1] Sailing He,[1,2] and Sanshui Xiao[1]
[1]*Centre for Optical and Electromagnetic Research, State Key Laboratory for Modern Optical Instrumentation, Zhejiang University, Yu-Quan, Hangzhou 310027, People's Republic of China*
[2]*Division of Electromagnetic Theory, Alfven Laboratory, Royal Institute of Technology, S-100 44 Stockholm, Sweden*
(Received 9 October 2003; published 17 March 2004)

The characteristics of a one-dimensional subwavelength cavity resonator (SWCR) formed by a left-handed material layer and a right-handed material layer are studied. Some previously published conclusions for some fundamental issues are discussed and rectified. Our analysis for the stability and the quality factor of the SWCR shows that the two materials should be mismatched in the impedance in order to obtain a useful SWCR. It is shown that the resonant frequency of the cavity is closely related to the total thickness of the cavity. The total thickness can still be far less than the resonant wavelength if the thickness ratio of the two layers is chosen appropriately. The compactness of the SWCR is restricted by its stability and quality factor.

DOI: 10.1103/PhysRevB.69.115111 PACS number(s): 42.30.Wb, 78.20.Ci, 73.20.Mf

Materials that possess simultaneously negative permittivity and negative permeability have attracted considerable attention recently. These materials exhibit some unusual electromagnetic properties, such as the support of backward waves,[1] negative refraction,[2,3] amplification of evanescent waves,[4] and unusual photon tunneling,[5] etc. The electric field, the magnetic field, and the wave vector of an electromagnetic wave propagating in such a material obey the left-hand rule and thus it is called a left-handed material (LHM). Based on an intuitive thought of phase compensation between a LHM layer and a usual dielectric medium [i.e., right-handed material (RHM)] layer, Engheta[6] proposed an interesting idea for a one-dimensional (1D) subwavelength cavity resonator (SWCR), whose total thickness can be far less than the resonant wavelength.[7] A subwavelength cavity resonator is of great interest in various microwave or photonic applications, and a LHM seems to be able to provide a novel and effective way for its realization.

The SWCR Engheta considered is shown in Fig. 1, where a RHM layer of thickness d_1 and a LHM layer of thickness d_2 are sandwiched between two (electrically) perfectly conducting planes. Their permittivity and permeability are denoted by ϵ_i and μ_i $(i=1,2)$, respectively, where $\epsilon_1,\mu_1>0$ and $\epsilon_2,\mu_2<0$. This system has two independent polarized modes, which are degenerated. From Maxwell's equations together with the boundary conditions for the field components (including the continuity of the tangential fields at the medium interface), one can easily obtain the following characteristic equation (i.e., the resonating condition)[6]

$$\frac{tg(n_1k_0d_1)}{tg(n_2k_0d_2)}=-\frac{n_1\mu_2}{n_2\mu_1}, \tag{1}$$

where $n_1=\sqrt{\varepsilon_1\mu_1}$, $n_2=-\sqrt{\varepsilon_2\mu_2}$, and k_0 is the wave number in vacuum. For a SWCR, $n_ik_0d_i\ll 1$ $(i=1,2)$. Thus, Engheta obtained the following simple resonating condition as an approximation of Eq. (1),

$$\frac{d_1}{d_2}\simeq-\frac{\mu_2}{\mu_1}. \tag{2}$$

Engheta then concluded that the resonance of this cavity is determined by the thickness ratio (d_1/d_2) of the two layers and is independent of the total thickness $d(=d_1+d_2)$.[6] We show below that Engheta's conclusion is questionable and Eq. (2) can not be treated as a resonating condition.

In fact, when Engheta's condition $d_1/d_2=|\mu_2/\mu_1|$ is fulfilled, Eq. (1) leads to $tg(bk_0n_1|\mu_2|)/(bk_0n_1|\mu_2|)=tg(bk_0n_2\mu_1)/(bk_0n_2\mu_1)$, where $b=d_1/|\mu_2|=d_2/\mu_1$. Since $tg(x)/x$ is a monotonously increasing function for $x\in[0,1)$, this equation has only a trivial solution $k_0=0$ (i.e., the resonant frequency equals zero) except for the special case of $n_1\mu_2/n_2\mu_1=1$. This special case occurs when the

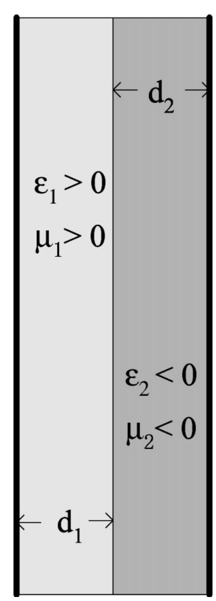

FIG. 1. Configuration of a 1D subwavelength cavity resonator formed a RHM layer and a LHM layer between two electrically perfectly conducting walls.

impedances of the two media are matched (i.e., $\epsilon_1\mu_2/\epsilon_2\mu_1=1$). In this case, arbitrary k_0 seems to satisfy Eq. (1) as long as $d_1/d_2=|\mu_2/\mu_1|$, however, the resonant frequency f_0 ($=ck_0/2\pi$, c is the light speed in vacuum) is then determined by the frequency dependence of the permittivity and permeability of the LHM (note that a LHM must be dispersive[3]). f_0 must satisfy simultaneously the following two equations: $\epsilon_2(f_0)/\mu_2(f_0)=\epsilon_1/\mu_1$ and $\mu_2(f_0)=-\mu_1 d_1/d_2$. Therefore, this special case may occur only at a certain particular frequency, and the resonant frequency f_0 is actually determined by the first equation rather than the second one, because only the ratio of d_1/d_2 has the possibility to be chosen arbitrarily. What's worse, as we will show later, for this special case with matched impedance the SWCR is very unstable and its quality factor is very low.

In a general case when the LHM and RHM layers are not matched in the impedance, nontrivial solution of Eq. (1) requires that $d_1/d_2\neq|\mu_2/\mu_1|$ and the solution should depend on the total thickness d of the cavity. In such a case, Engheta's condition (2) cannot be used to describe the SWCR since the simple approximation $tg(n_ik_0d_i)\simeq n_ik_0d_i$ ($i=1,2$) is not appropriate for use to simplify Eq. (1).

In this paper, by applying an improved approximation for the tangent functions in Eq. (1), we obtain an explicit algebraic resonating condition, which still contains all the important information of the SWCR. Through this explicit resonating condition, the resonant frequency of the SWCR is shown to be closely related to the total thickness of the cavity, and the stability and quality factor of the SWCR are analyzed.

First we introduce some normalized quantities for convenience: $\hat{\varepsilon}=-\varepsilon_2/\varepsilon_1$, $\hat{\mu}=-\mu_2/\mu_1$ (obviously $\hat{\varepsilon}>0$, $\hat{\mu}>0$), $\varsigma=d_1/d_2$, and $\phi=n_1k_0d=2\pi n_1 d/\lambda$. Equation (1) is then rewritten as

$$tg\left(\frac{\varsigma}{1+\varsigma}\phi\right)=\sqrt{\frac{\hat{\mu}}{\hat{\varepsilon}}}\,tg\left(\frac{\hat{n}}{1+\varsigma}\phi\right), \tag{3}$$

where $\hat{n}=\sqrt{\hat{\varepsilon}\hat{\mu}}=-n_2/n_1>0$. For a SWCR, $\phi\ll 1$. Thus, by applying Taylor's series expansion $tg(x)\approx x+x^3/3$ to Eq. (3), we obtain

$$\phi^2=\frac{3(\bar{\varsigma}-1)(1+\hat{\mu}\bar{\varsigma})^2}{\hat{\mu}^2(\alpha^3-\bar{\varsigma}^3)}, \tag{4}$$

where $\bar{\varsigma}=\varsigma/\hat{\mu}$ and $\alpha=\sqrt[3]{\hat{\varepsilon}/\hat{\mu}}$. Note that $\bar{\varsigma}\approx 1$ corresponds to Engheta's "resonating condition" (2) and $\alpha=1$ corresponds to the case when the impedance is matched. For the general case of $\alpha\neq 1$, from Eq. (4) one has $\phi=0$ when $\bar{\varsigma}=1$, indicating that Engheta's resonating condition (2) gives zero resonating frequency. To obtain a nonzero resonating frequency, one should choose $\bar{\varsigma}\neq 1$. This agrees well with our previous analysis before taking the algebraic approximation. For given material parameters $\hat{\epsilon}$ and $\hat{\mu}$, the nonzero ϕ (related to the resonant frequency) is only determined by $\bar{\varsigma}$ (or $\varsigma=\hat{\mu}\bar{\varsigma}$). Thus, from the expression $\lambda=2\pi n_1 d/\phi$ one sees that the resonant wavelength is directly proportional to the total thickness d of the cavity (unlike the conclusion Engheta made in Ref. 6).

Since $\bar{\varsigma}\approx 1$ (however, not as the resonating condition) for a SWCR, one has $1+\hat{\mu}\bar{\varsigma}\approx 1+\hat{\mu}$, and $\alpha^3-\bar{\varsigma}^3\approx(\alpha-\bar{\varsigma})(\alpha^2+\alpha+1)$. Then Eq. (4) is further reduced to

$$N^2=\Gamma^2\frac{\alpha-\bar{\varsigma}}{\bar{\varsigma}-1}, \tag{5}$$

where $\Gamma=[2\pi\hat{\mu}/(1+\hat{\mu})]\sqrt{(\alpha^2+\alpha+1)/3}$ and $N\equiv 2\pi/\phi=\lambda/n_1 d$ represent the compactness of the *subwavelength* cavity resonator (we choose $N\geqslant 10$ in our numerical computation). Equation (5) can be rewritten in the following form,

$$\bar{\varsigma}=1+\frac{\Gamma^2}{\Gamma^2+N^2}(\alpha-1). \tag{6}$$

Unlike Engheta's resonating condition (2), our explicit resonating condition (5) or (6) still contains all the important information of the SWCR (as shown later). The above equation shows that $\Delta\equiv\bar{\varsigma}-1$ is very sensitive to $(\alpha-1)$, which reflects the degree of the impedance mismatch of the two materials. For example, when $\alpha(=\sqrt[3]{\hat{\varepsilon}/\hat{\mu}})$ decreases from 1.1 to 1.01, Δ drops by an order of magnitude. When α approaches 1 (i.e., the impedances of the two media are almost matched), $\Delta\to 0$. Otherwise, when $\alpha\neq 1$, Δ decreases as N increases. The dependence of Δ on N for the cases of $\hat{\varepsilon}=1.01$, 1.1, 2, and 4 is shown in Fig. 2(a) with $\hat{\mu}=1$. The results obtained from formula (6) and Eq. (1) are shown by the lines and marks, respectively, and they are in good agreement.

The quantity $\bar{\varsigma}$ is the relative ratio of d_1/d_2 and μ_2/μ_1 of the SWCR. From Eq. (5) one sees that as $\bar{\varsigma}$ is so close to 1 a small change in $\bar{\varsigma}$ may cause a significant change in N (and consequently a significant change in the resonant frequency). The stability of the resonant frequency of a SWCR to any parameter perturbation is very important for its design as well as its fabrication. Assume that a small perturbation of the quantity $\bar{\varsigma}$ (from its designed value $\bar{\varsigma}_0$) results in a shift of the resonant frequency from f_0 to f. Denote the relative shift of the resonant frequency as $\hat{f}=(f-f_0)/f_0$. From Eq. (5) one can easily obtain

$$\left.\frac{d\hat{f}}{d\bar{\varsigma}}\right|_{\bar{\varsigma}=\bar{\varsigma}_0}=\frac{1}{2\Delta}\left(1+\frac{\Gamma^2}{N^2}\right), \tag{7}$$

where $\Delta=\bar{\varsigma}_0-1$, $N=c/n_1 f_0 d$. Since $\Gamma^2\ll N^2$ in general, we have $d\hat{f}/d\bar{\varsigma}\approx 1/(2\Delta)$. Thus, the stability of the SWCR is mainly determined by the quantity Δ. When $\alpha\to 1$, we have $\Delta\to 0$ [cf. Eq. (6)] and then $d\hat{f}/d\bar{\varsigma}\to\infty$. In above derivation we neglect the dispersion of the LHM. However, the obtained result shows definitely the great difficulty for designing a SWCR with its impedances matched at an expected resonant frequency. Therefore, the stability of the SWCR re-

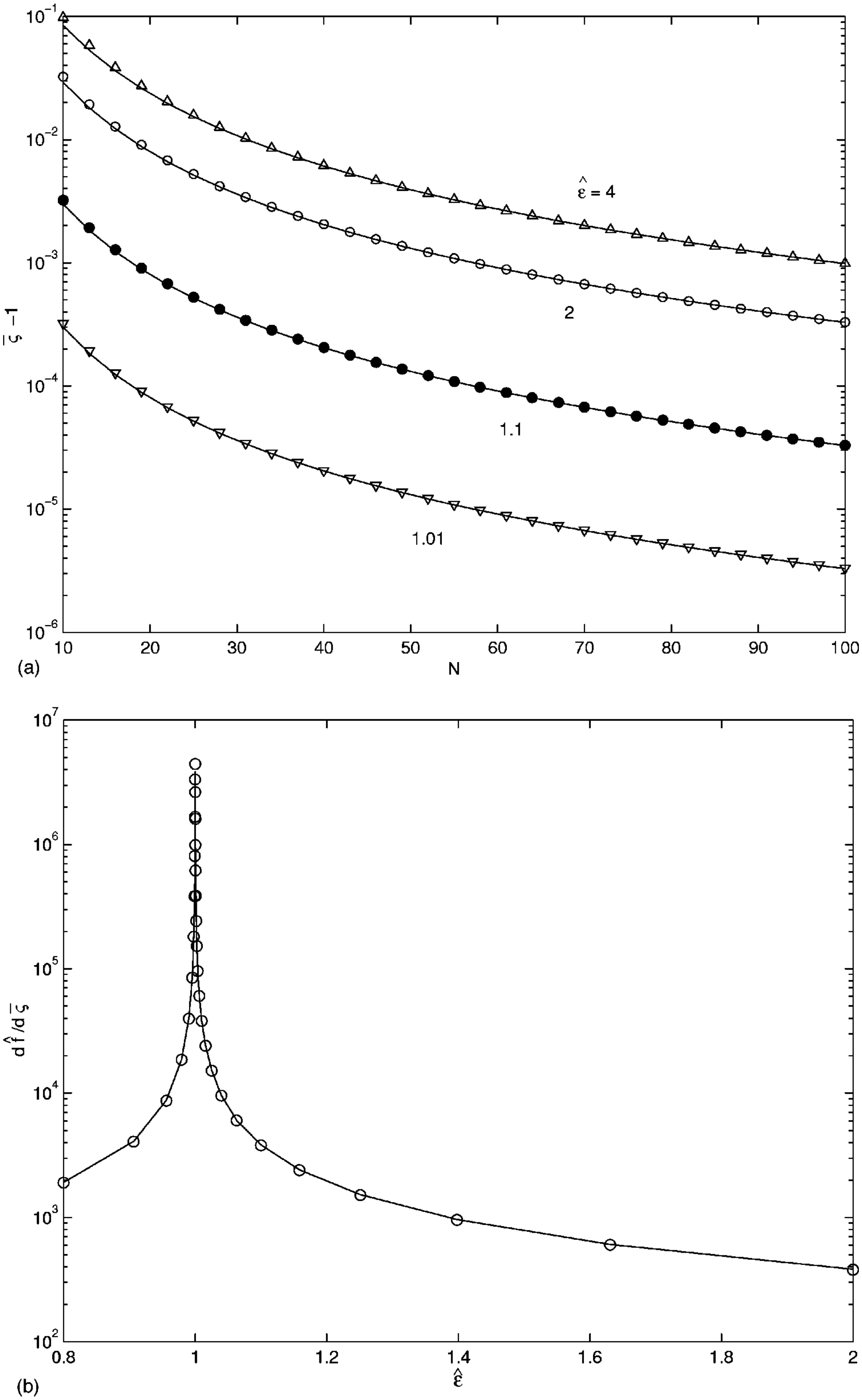

FIG. 2. (a) Deviation of $\overline{\varsigma}$ from 1 as $N(=\lambda/d)$ increases for $\hat{\varepsilon}=1.01$, 1.1, 2, and 4. The solid lines and the marks correspond to the results obtained from formula (6) and Eq. (1), respectively. (b) Derivative $d\hat{f}/d\overline{\varsigma}$ as $\hat{\varepsilon}$ increases for $N=50$. The solid lines and the circles correspond to the results obtained from formulas (7) and (8), respectively. $\hat{\mu}=1$ is used in both (a) and (b).

quires an impedance mismatch between the LHM layer and the RHM layer. Furthermore, since Δ decreases (and then the stability decreases) as N increases, the compactness of the SWCR is restricted by the stability requirement. One can also derive the following exact expression for the derivative $d\hat{f}/d\overline{\varsigma}$ from Eq. (1):

$$\left.\frac{d\hat{f}}{d\overline{\varsigma}}\right|_{\overline{\varsigma}=\overline{\varsigma}_0}=\frac{(1+\hat{\mu})+g(1+\hat{\varepsilon})}{(1+\hat{\mu}\overline{\varsigma})[(1-\overline{\varsigma})+g(\hat{\varepsilon}/\hat{\mu}-\overline{\varsigma})]}, \qquad (8)$$

where $g=tg^2(2\pi\hat{\mu}\overline{\varsigma}/(1+\hat{\mu}\overline{\varsigma})N)$. Unlike Eq. (7), the above

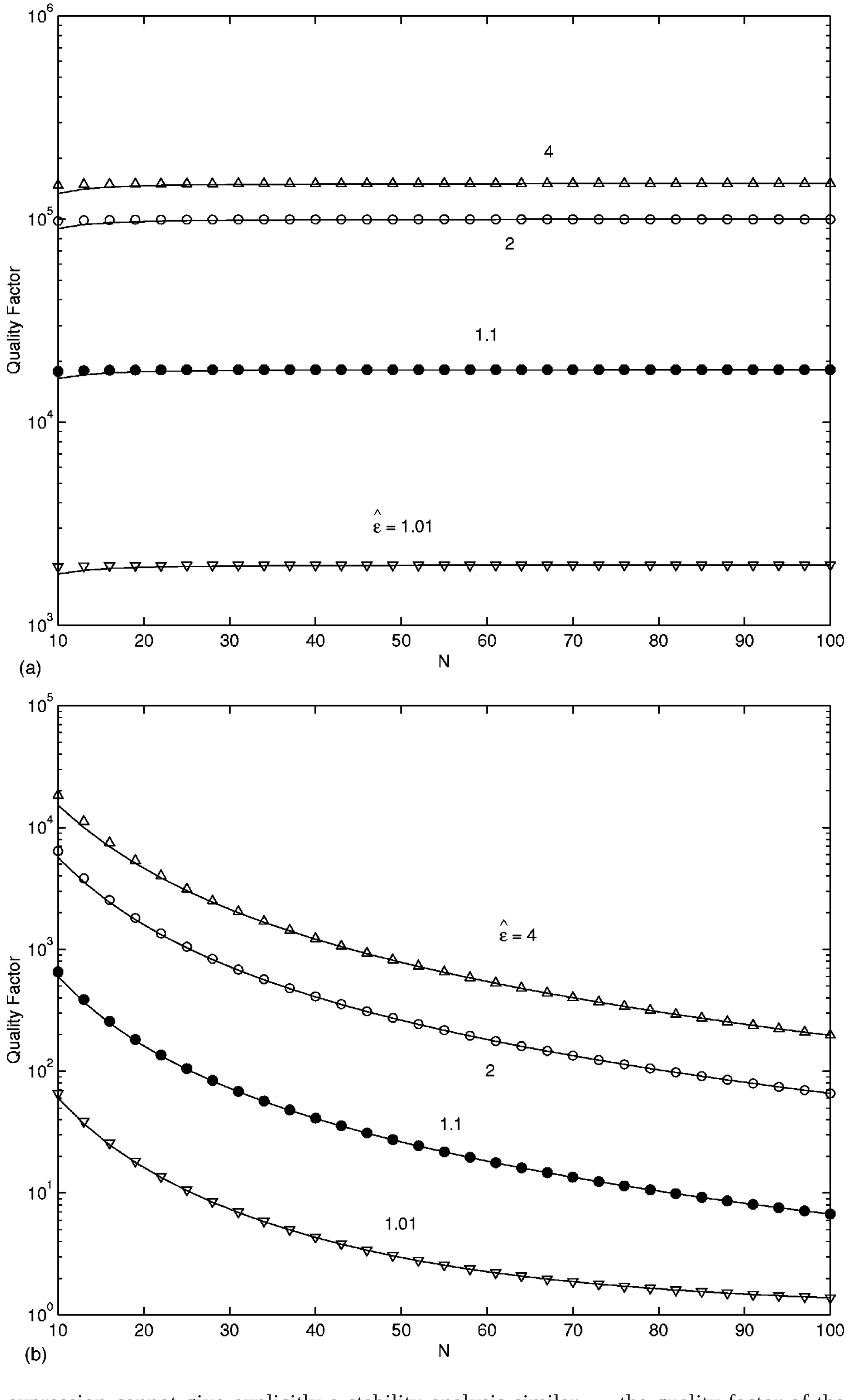

FIG. 3. (a) The quality factor (degraded by the permittivity loss) of the SWCR as N increases for $\mathrm{Re}\{\hat{\varepsilon}\}=1.01$, 1.1, 2, and 4. Here $\hat{\mu}=1$ and $\delta_e=10^{-5}$. The marks and the solid lines correspond to the results obtained from Eq. (1) and formula (10), respectively. (b) The quality factor (degraded by the permeability loss) of the SWCR as N increases for $\hat{\varepsilon}=1.01$, 1.1, 2, and 4. Here $\mathrm{Re}\{\hat{\mu}\}=1$ and $\delta_m=10^{-5}$. The marks and the solid lines correspond to the results obtained from Eq. (1) and formula (11), respectively.

expression cannot give explicitly a stability analysis similar to the one given between Eqs. (7) and (8). The values of $d\hat{f}/d\overline{\varsigma}$ calculated from Eqs. (7) and (8) as $\hat{\varepsilon}$ varies (with $\hat{\mu}=1$ and $N=50$) are shown in Fig. 2(b) by the solid line and the circles, respectively. Again, our explicit resonant condition gives a reliable result. Below we show how our explicit resonating condition (5) can be used conveniently to study the quality factor of the SWCR.

In the above analysis, the LHM is assumed to be lossless. However, it is well known that an actual LHM is always lossy (i.e., ϵ_2 or μ_2 has an imaginary part).[8] Thus, we need to study the quality factor Q of the SWCR, which measures the lifetime of its eigenstate. For the LHM, its loss can come from the imaginary part of its electric permittivity (ϵ_2) or magnetic permeability (μ_2). Due to the requirement that the tangential electric field should be zero on the two walls of the cavity, one can expect that the influence of the loss from the permeability would have a different impact on the quality factor as compared to the influence of the loss from the permittivity. Thus, we analyze the quality factor of the SWCR separately in two cases: (i) the case when only ϵ_2 has an imaginary part, (ii) the case when only μ_2 has an imaginary part. In both cases we assume that the imaginary part of ϵ_2 or μ_2 is far less than the corresponding real part.

Consider the first case, i.e., when μ_2 is real and $\varepsilon_2 = \varepsilon_{2r}(1 - i\delta_e)$ [all the field components are assumed to have the time dependence of form $\exp(i2\pi ft)$]. Here ε_{2r} and δ_e are real, and $0 < \delta_e \ll 1$. The solution of Eq. (5) for the frequency becomes complex, i.e., $f = f_r + i f_i$ (f_r and f_i are real). Then the quality factor of the SWCR is determined by $Q = f_r / f_i$. Let f_0 denote the resonant frequency corresponding to the ideal case of $\varepsilon_2 = \varepsilon_{2r}$. From Eq. (5), we have

$$\left(\frac{f}{f_0}\right)^2 = \frac{\Gamma_r^2(\alpha_r - \overline{\varsigma})}{\Gamma^2(\alpha - \overline{\varsigma})}, \tag{9}$$

where $\alpha_r = \sqrt[3]{\hat{\varepsilon}_r / \hat{\mu}}$ ($\hat{\varepsilon}_r = -\varepsilon_{2r}/\varepsilon_1$), $\Gamma_r = [2\pi\hat{\mu}/(1 + \hat{\mu})]\sqrt{(\alpha_r^2 + \alpha_r + 1)/3}$. Note that the parameter $\overline{\varsigma}$ $(= |d_1\mu_2/d_2\mu_1|)$ is determined by the desired resonant frequency f_0, i.e., $\overline{\varsigma} = \Delta + 1$, where $\Delta = (\alpha_r - 1)\Gamma_r^2/(\Gamma_r^2 + N^2)$ and $N = c/n_1 f_0 d$. When the impedances of the two media are so closely matched that $|\alpha_r - 1| \ll \delta_e$, we can obtain (after some derivation) $f/f_0 \approx \sqrt{3}N(1+i)/2\Gamma_r$. Then we obtain $Q = 1$ and $f_r = \sqrt{3}Nf_0/2\Gamma_r \gg f_0$. In such a case, a tiny loss of the LHM leads to a large change in the resonant frequency and furthermore the resonator fails to select frequency with such a low Q. Therefore, the impedances of the two materials in the SWCR should be mismatched enough in order to obtain a high Q. When the impedance mismatch is large, a wave propagating in the RHM layer is strongly reflected at the RHM-LHM interface and then a considerable amount of the energy flow comes back to the lossless RHM layer. This effectively reduces the dissipative rate of the total energy in the cavity since only the LHM layer is lossy.

When the impedance mismatch is large enough so that $|\alpha_r - 1| \gg \delta_e$, one can find from Eq. (9) that the frequency shift (from f_0 to f_r) is only of order $(\delta/\Delta_r)^2$ and the quality factor has the following explicit expression,

$$Q = 2\delta_e^{-1}\frac{|\alpha_r^3 - 1|}{\alpha_r^3}\left|1 - \left(\frac{\Gamma^2}{2N^2}\right)\left(1 + \frac{1}{\alpha_r} + \frac{1}{\alpha_r^2}\right)\right|. \tag{10}$$

Obviously, Q depends very much on the mismatch degree of the two materials and is almost independent of N (since $\Gamma^2 \ll N^2$ in general). For example, when $\delta_e = 10^{-5}$, we have $Q \approx 2\times10^3$, 2×10^4, and 2×10^5 for $\hat{\varepsilon}_r/\hat{\mu} = 1.01$, 1.1, and 2, respectively. The quality factor calculated from formula (10) for $\delta_e = 10^{-5}$ is shown by the solid lines in Fig. 3(a) as N increases for $\hat{\varepsilon}_r = 1.01$, 1.1, 2, and 4 (with fixed $\hat{\mu} = 1$). The results are verified by calculating the quality factor directly from Eq. (1) [however, in a quite complicated way; see the marks in Fig. 3(a)].

Now we consider the second case when ε_2 is real and $\mu_2 = \mu_{2r}(1 - i\delta_m)$. Here μ_{2r} and δ_m are real, and $0 < \delta_m \ll 1$. Similar to the first case, when $|\alpha_r - 1| \ll \delta_m$ (here $\alpha_r = \sqrt[3]{\hat{\varepsilon}/\hat{\mu}_r}$ and $\hat{\mu}_r = -\mu_{2r}/\mu_1$), a tiny loss of the LHM leads to a large shift in the resonant frequency and $Q \approx 1$. Thus, the impedances of the two media are required to be mismatched enough in order to obtain a high Q. For $|\alpha_r - 1| \gg \delta_e$, we can obtain the following expression for the quality factor

$$Q \approx \frac{2|\Delta|}{\delta_m}. \tag{11}$$

This expression shows that Q depends almost only on the quantity Δ (when the loss δ_m is fixed). Since $\Delta = (\alpha_r - 1)\Gamma_r^2/(\Gamma_r^2 + N^2) \sim (\alpha_r - 1)\Gamma_r^2/N^2$, Q decreases as N increases in this case. Figure 3(b) shows the dependence of Q on N when $\delta_m = 10^{-5}$ [the other parameters are the same as used in Fig. 2(a)]. The solid lines and the marks correspond to the results obtained from formula (11) and Eq. (1), respectively, and they are in good agreement. Apparently, this strong dependence of Q on N leads to another limitation on the compactness of the SWCR.

Comparing formulas (10) and (11), one sees that the quality factors in the two cases differ greatly when $\delta_e \sim \delta_m$ (the quality factor for a nonzero δ_m is much smaller than that for a nonzero δ_e, particularly when N is very large). The permeability loss has much more influence on the quality factor than the permittivity loss. This "asymmetry" of the influence is due to the requirement that the tangential electric field should be zero on the two walls of the cavity. We have also analyzed the case when both ε_2 and μ_2 are complex [i.e., $\varepsilon_2 = \varepsilon_{2r}(1 - i\delta_e)$ and $\mu_2 = \mu_{2r}(1 - i\delta_m)$] and found that the quality factor is still determined mainly by δ_m (even when δ_e is significantly larger than δ_m) for a subwavelength cavity resonator (i.e., when N is large enough).

In conclusion, we have studied the characteristics of a 1D SWCR, which contains a RHM layer and a LHM layer. A simple explicit resonating condition [i.e., Eq. (6)], which still contains all the important information of the SWCR, has been presented. The second term in the right-hand side of Eq. (6), which Engheta neglected in Ref. 6, is very important in a characteristic analysis of the SWCR. The rectified explicit resonating condition has been used to analyze (with an explicit expression and a physical insight) what influences the stability of the SWCR and how the loss of the LHM degrades the quality factor of the SWCR. Our analysis for the stability and the quality factor has shown that the impedances of the LHM and RHM should be mismatched in order to obtain a useful SWCR. We have also shown that the resonant frequency for such a resonator is closely related to the total thickness d of the cavity and d can still be far less than the resonant wavelength λ. Although the thickness ratio of the two layers in the SWCR has a rough approximation $d_1/d_2 \approx |\mu_2/\mu_1|$, the deviation $\Delta = |d_1\mu_1/d_2\mu_2| - 1$ is a key quantity (besides the total thickness d) for designing a SWCR with a desired resonant frequency. It has been shown that the permeability loss has a much stronger influence on the quality factor of a SWCR than the permittivity loss. Both the stability and the quality factor decrease as λ/d increases, and thus the compactness of the SWCR is restricted by its stability and quality factor.

The partial support of Natural Science Foundation of China under a project (Grant No. 60277018) and a key project (Grant No. 90101024) is gratefully acknowledged.

[1] V.G. Veselago, Sov. Phys. Usp. **10**, 509 (1968).
[2] R.A. Shelby, D.R. Smith, and S. Schultz, Science **292**, 77 (2001).
[3] D.R. Smith and N. Kroll, Phys. Rev. Lett. **85**, 2933 (2000).
[4] J.B. Pendry, Phys. Rev. Lett. **85**, 3966 (2001).
[5] Z.M. Zhang and C.J. Fu, Appl. Phys. Lett. **80**, 1097 (2002).
[6] N. Engheta, IEEE Antennas Wireless Propagat. Lett. **1**, 10 (2002).
[7] D.R. Smith, W.J. Padilla, D.C. Vier, S.C. Nemat-Nasser, and S. Schultz, Phys. Rev. Lett. **84**, 4184 (2000).
[8] N. Garcia and M. Nieto-Vesperinas, Phys. Rev. Lett. **88**, 207403 (2002).