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# Weak localization of electrons in an external electric field

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The impact of an electric field on the electron localization problem is studied within the framework of a field-theoretic formulation. The investigation shows that the impact of the electric field on the localization corrections is governed by the interplay between two time scales, one set by the electric field, and the other by the phase relaxation rate. At very low temperatures the scaling of the conductivity is governed by the electric field. In this regime the conductivity depends logarithmically on the field, and an arbitrarily small electric field delocalizes the electron states. At higher temperatures the behavior of the conductivity is governed by the temperature scaling. In this regime the field has no impact on the observable leading localization corrections.

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## I. INTRODUCTION

The physics of weakly disordered electron systems has been the subject of considerable theoretical and experimental interest over the past years. According to semiclassical physics, the electrons in such systems move ballistically in between occasional scattering events due to impurities or phonons, which results in diffusive dynamics. The semiclassical picture has long been known to be incorrect in one dimension, where arbitrarily weak disorder leads to all electronic states being localized.[1] A completely new understanding of transport in two dimensions was obtained in the late 1970's by the scaling theory of Abrahams *et al.*[2] These authors showed that quantum interferences lead to the localization of all electronic states in two dimensions as well, no matter how weak the disorder. Accordingly, all two-dimensional systems are, strictly speaking, insulators.[3]

The same conclusion was reached by a field-theoretical approach that was pioneered by Wegner.[4] Using the replica trick to deal with the quenched disorder, he derived a generalized nonlinear $\sigma$ model whose coupling constant is the electrical conductivity. Wegner's original theory used bosonic fields, but later applications of his method derived the same model starting from a fermionic formulation.[5] A supersymmetric formulation, which avoids the replica trick, was also given.[6,7]

A third approach to the localization problem were mode-mode coupling theories. While the original formulation by Götze[8] missed the interference effects that lead to localization in two dimensions, a self-consistent diagrammatic approach by Vollhardt and Wölfle,[9] as well as an improvement on Götze's original method,[10] yielded results in agreement with both the scaling theory and the field-theoretic approaches. They also agree well with numerical simulations and with experiments.

All of the above approaches focus on the electronic diffusion coefficient, and make a connection to the electrical conductivity by means of an Einstein relation. By contrast, experimentally the conductivity is measured by applying an electric field. This raises the question how an externally applied electric field affects the localization phenomena.

In the literature, there is no clear answer to this question. Exact results are available for one-dimensional systems only.[11–13] These are characterized by a critical electric field $F_{\rm c}$ such that for field strengths $F<F_{\rm c}$ the states remain (power-law) localized, while for $F>F_{\rm c}$ the states are extended.

Unfortunately, these exact calculations for one-dimensional systems do not reveal how the electric field affects the quantum interference effects responsible for the weak localization in two dimensions. The impact of an electric field on the localization in two dimensional systems is therefore much less clear. It has been argued that a weak homogeneous electric field has no influence at all on the localization.[14–16] Other authors have concluded that an arbitrary small electric field already leads to delocalization.[17,18] In between these two extremes lies a theory that predicts a strong modification of the weak-localization corrections by an electric field,[19,20] and theories in which delocalization only occurs if the electric field exceeds a critical value.[21–23]

Experimentally, the situation is not clear either. While early experiments[24,25] seemed to find modifications of the weak-localization corrections by a field, no impact of the electric field on the localization corrections was observed in Ref. 26. Similarly, Ref. 27 concluded that an electric field has no impact on the localization corrections. On the other hand, the scaling argument of Ref. 18, which implies that an arbitrary small electric field leads to delocalization, is often used in the interpretation of experiments, as discussed in Ref. 28.

In the present paper we revisit this problem by means of field-theoretic techniques. We conclude that delocalization does indeed occur for arbitrarily small fields. However, this effect is not observable at realistic temperatures, which explains some of the apparent contradictions between theory and experiment. We also find that the leading contribution to the electric-field scaling, which was assumed in Ref. 18 to be the dominant effect, has a zero prefactor, and the actual ef-

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fect of the electric field is weaker. This is important for attempts to extract the dynamical critical exponents from experimental data.[28]

This paper is organized as follows. In Sec. II we introduce our model and formulate an effective field theory. This theory is investigated in a Gaussian approximation in Sec. III, which yields a generalized diffusion equation for density fluctuations. In Sec. IV we derive a generalized nonlinear $\sigma$ model, which is investigated in a one-loop approximation. In Sec. V we discuss our results and their connections to previous investigations. Some technical points are relegated to the appendix.

## II. MODEL AND FIELD-THEORETIC FORMULATION

We consider the Hamilton operator

$$H=H_x+V(\boldsymbol{x}). \tag{2.1}$$

Here

$$H_x=-\frac{\Delta}{2m}+\boldsymbol{F}\cdot\boldsymbol{x} \tag{2.2}$$

is the Hamilton operator for free particles in the presence of an electric field $\boldsymbol{F}$, with $\Delta$ the Laplace operator and $m$ the electron mass, and $V(\boldsymbol{x})$ is a random potential. We use units such that $\hbar$ and the electron charge are equal to unity. $V$ is characterized by a Gaussian distribution with zero mean and second moment

$$\langle V(\boldsymbol{x})V(\boldsymbol{x}')\rangle_{\mathrm{dis}}=\frac{1}{2\pi\nu\tau}\,\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{2.3}$$

Here $\langle\cdots\rangle_{\mathrm{dis}}$ denotes the disorder average, $\nu$ is the density of states per spin at the Fermi level, and $\tau$ is the single-particle scattering time. Unless otherwise noted, we consider a two-dimensional system $d=2$ in the weak-disorder regime, $\mu\tau=k_{\mathrm{F}}\ell/2\gg 1$, where $\ell$ is the mean-free path, and $\mu$ and $k_{\mathrm{F}}$ are the chemical potential and the Fermi wave number, respectively, in equilibrium.

In our investigation we focus on configuration averages of retarded and advanced Green functions and their products. These functions are solutions of the differential equations

$$(\pm i\omega+E-H)\,G^{R,A}(\boldsymbol{x},\boldsymbol{x}'|E;\omega)=\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{2.4}$$

In equilibrium, $\boldsymbol{F}=0, E=\mu$ is the chemical potential. For $\boldsymbol{F}\neq 0$, the quantity $E-\boldsymbol{F}\cdot\boldsymbol{x}\equiv\mu_x$ is the chemical potential in a local equilibrium approximation, i.e., the kinetic energy of an electron located at point $\boldsymbol{x}$. As a boundary condition, we require that the Green functions vanish at infinity.[17,23] The solutions of Eq. (2.4) then are symmetric with respect to an interchange of $\boldsymbol{x}$ and $\boldsymbol{x}'$,

$$G^{R,A}(\boldsymbol{x},\boldsymbol{x}'|E;\omega)=G^{R,A}(\boldsymbol{x}',\boldsymbol{x}|E;\omega). \tag{2.5a}$$

This symmetry is a direct consequence of the time-reversal invariance, which is not broken by the electric field.[14]

The electric field does, however, break the translational invariance in real space. In the absence of the random potential, or after disorder averaging, a translation by a vector $\boldsymbol{a}$ leads to an energy change $\boldsymbol{F}\cdot\boldsymbol{a}$. The configuration averages $\overline{G}^{R,A}$ of the Green functions therefore satisfy the relationship

$$\overline{G}^{R,A}(\boldsymbol{x}+\boldsymbol{a},\boldsymbol{x}'+\boldsymbol{a}|E;\omega)=\overline{G}^{R,A}(\boldsymbol{x},\boldsymbol{x}'|E-\boldsymbol{F}\cdot\boldsymbol{a};\omega). \tag{2.5b}$$

The symmetry properties expressed by Eqs. (2.5) will be important later.

We now consider a generating functional for our Green functions. Following Ref. 29, we define

$$Z^{R,A}[j_{R,A}]=\int D[\Phi]\exp\left(\pm iS^{R,A}/2+\int d\boldsymbol{x}\,j_{R,A}(\boldsymbol{x})\Phi(\boldsymbol{x})\right), \tag{2.6}$$

where

$$S^{R,A}=\int d\boldsymbol{x}\,\Phi(\boldsymbol{x})(\pm i\omega+E-H)\Phi(\boldsymbol{x}), \tag{2.7}$$

with $\Phi(\boldsymbol{x})$ a real scalar field. The retarded and advanced Green functions are obtained according to the rule

$$G^{R,A}(\boldsymbol{x},\boldsymbol{x}'|E;\omega)=\left.\frac{\mp i\,\delta^2}{\delta j_{R,A}(\boldsymbol{x})\,\delta j_{R,A}(\boldsymbol{x}')}\right|_{j=0}\ln Z^{R,A}[j_{R,A}]. \tag{2.8}$$

To calculate the configuration average we use the replica trick. To this end we consider $2n$ copies of the original generating functional ($n$ for the retarded degrees of freedom, and $n$ for the advanced ones), and take the limit $n\to 0$ at the end of the calculation. The replicated generating functional takes the form

$$\begin{aligned}(Z^RZ^A)^n=&\int\prod_{\alpha}D[\Phi_\alpha]\exp\left(\sum_{\alpha}\int d\boldsymbol{x}\,j_\alpha(\boldsymbol{x})\Phi_\alpha(\boldsymbol{x})\right)\\&\times\exp\left[\frac{i}{2}\sum_{\alpha}\int d\boldsymbol{x}\,\Phi_\alpha(\boldsymbol{x})(i\omega_\alpha\right.\\&\left.+E-H)\Lambda_\alpha\Phi_\alpha(\boldsymbol{x})\right].\end{aligned} \tag{2.9}$$

Here $\alpha=-(n-1),\ldots,n$, $\Lambda_\alpha=-1$ for $\alpha\leqslant 0$ and $\Lambda_\alpha=1$ for $\alpha>0$, $\omega_\alpha=\omega\Lambda_\alpha$, and $j_\alpha=j_A$ for $\alpha\leqslant 0$ and $j_\alpha=j_R$ for $\alpha>0$. Now we calculate the configuration average $\overline{Z}=\langle(Z^RZ^A)^n\rangle_{\mathrm{dis}}$, which generates the averaged Green functions $\overline{G}^{R,A}$. If one performs a Hubbard-Stratonovich transformation to decouple the resulting term quartic in $\Phi$, and then integrates out $\Phi$, one finds

$$\overline{Z}[j=0]=\int D[Q]e^{A[Q]}, \tag{2.10}$$

where

$$A[Q]=\frac{\pi\nu}{8\tau}\int d\boldsymbol{x}\,\mathrm{tr}[Q(\boldsymbol{x})]^2-\frac{1}{2}\mathrm{tr}\ln G_Q^{-1}, \tag{2.11}$$

with

$$(G_Q^{-1})_{\alpha\alpha'}(\boldsymbol{x},\boldsymbol{x}'|E;\omega) = \left[(i\omega_\alpha + E - H_x)\delta_{\alpha\alpha'} + \frac{i}{2\tau}Q_{\alpha\alpha'}(\boldsymbol{x})\right]\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{2.12}$$

It is easy to show that $G_Q$, averaged with respect to the action $A$, and taken in the limit $n\to 0$, is diagonal in the replica indices, and equal to $\overline{G}^R$ ($\overline{G}^A$) for $\alpha>0$ ($\alpha<0$). The $Q$-matrix fields in the Eqs. (2.11) and (2.12) are real-valued matrix fields which satisfy the relationship

$$\Lambda_\alpha Q_{\alpha\alpha'}(\boldsymbol{x}) = \Lambda_{\alpha'} Q_{\alpha'\alpha}(\boldsymbol{x}). \tag{2.13}$$

Equations (2.11)–(2.13) define the effective action which forms the basis for our further investigations.

## III. TRANSPORT AND HEATING IN GAUSSIAN APPROXIMATION

### A. Saddle-point Green functions

We now look for a saddle-point solution $Q^{\rm SP}$ of the effective action $A[Q]$, defined by

$$\left.\frac{\delta A[Q]}{\delta Q}\right|_{Q=Q^{\rm SP}} = 0. \tag{3.1}$$

The saddle-point values of $Q$ and $G_Q$ are diagonal in the replica indices, $Q^{\rm SP}_{\alpha\alpha'} = \delta_{\alpha\alpha'}Q_\alpha$, $(G_{Q^{\rm SP}})_{\alpha\alpha'} = \delta_{\alpha\alpha'}G_\alpha$, as are the exact expectation values. From Eqs. (2.11) and (2.12) we find

$$Q_\alpha(\boldsymbol{x}) = \frac{i}{\pi\nu}G_\alpha(\boldsymbol{x},\boldsymbol{x}|E;\omega) = \frac{i}{\pi\nu}G_\alpha(0,0|E-\boldsymbol{F}\cdot\boldsymbol{x};\omega), \equiv \tilde{Q}_\alpha(\mu_x). \tag{3.2}$$

In the second equation we have used the generalized translational invariance property of the Green function, Eq. (2.5b), and the notation in the third equation emphasizes that the saddle-point field depends on $\boldsymbol{x}$ and the direction of $\boldsymbol{F}$ only via $\mu_x = E - \boldsymbol{F}\cdot\boldsymbol{x}$.

In order to obtain an equation for the Green function we use the generalized translation invariance property (2.5b) and the time-reversal invariance. Due to these properties the Green function in the presence of the electric field can be written in the form[23]

$$G_\alpha(\boldsymbol{x},\boldsymbol{x}'|E;\omega) = \int \frac{d\boldsymbol{k}}{(2\pi)^d}e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \times g_\alpha(\boldsymbol{k}|E-\boldsymbol{F}\cdot(\boldsymbol{x}+\boldsymbol{x}')/2;\omega), \tag{3.3}$$

where $g_\alpha(\boldsymbol{k}|E;\omega)$ is a function that depends only quadratically on the field $\boldsymbol{F}$. If we work to linear order in the field, the function $g_\alpha(\boldsymbol{k}|E;\omega)$ thus reduces to the equilibrium Green function,

$$g_\alpha(\boldsymbol{k}|E;\omega) \approx \frac{1}{i\omega_\alpha + E - \epsilon_{\boldsymbol{k}} + \frac{i}{2\tau}\tilde{Q}_\alpha(E)}, \tag{3.4}$$

where $\epsilon_{\boldsymbol{k}} = \boldsymbol{k}^2/(2m)$. The only difference to the situation in the absence of the field is then that the quantity $\mu_x$, which has replaced the chemical potential, goes to zero at the classical turning point, where $\boldsymbol{F}\cdot\boldsymbol{x} = E$. We will nevertheless consider $|\mu_x|$ large compared to $1/(2\tau)$ in calculating the integral in Eq. (3.3), i.e., we consider only the region far from the classical turning point. In this approximation we obtain

$$Q_\alpha(\boldsymbol{x}) = \operatorname{sgn}\omega_\alpha. \tag{3.5}$$

In writing down Eq. (3.5) we have ignored the imaginary part of $Q_\alpha$, which only leads to a weakly field-dependent renormalization of the chemical potential. The saddle-point Green function obtained in this way agrees with that derived in Ref. 23 in a self-consistent Born-approximation.

We note that the simple structure of the above saddle-point solution is a result of the constant density of states in our model. If the density of states were energy dependent, the saddle-point field would pick up an energy and field dependent contribution from the density of states, which would modify Eq. (3.5).

### B. Gaussian fluctuations

We now consider the Gaussian fluctuations about our saddle-point solution. Substituting $Q = Q^{\rm SP} + \delta Q$ into Eq. (2.11), we find for the Gaussian part of the action

$$A^{(2)}[\delta Q] = \frac{\pi\nu}{8\tau}\int d\boldsymbol{x}d\boldsymbol{y}\sum_{\alpha\alpha'}\delta Q_{\alpha\alpha'}(\boldsymbol{x})\left[\delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{1}{2\pi\nu\tau}G_{\alpha'}(\boldsymbol{y},\boldsymbol{x}|E;\omega)G_\alpha(\boldsymbol{x},\boldsymbol{y}|E;\omega)\right]\delta Q_{\alpha'\alpha}(\boldsymbol{y}). \tag{3.6}$$

The properties of the Gaussian propagators depend on whether both Green functions in Eq. (3.6) are retarded or advanced, or whether one is retarded and the other advanced. In the former case, the propagator is massive, in the latter, soft. Setting $\omega = \Omega/2$, the soft propagator $P$ satisfies the equation

$$\int d\boldsymbol{y}\Gamma(\boldsymbol{x},\boldsymbol{y}|E;\Omega)P(\boldsymbol{y},\boldsymbol{x}'|E;\Omega) = \delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{3.7a}$$

where

$$\tau\Gamma(\boldsymbol{x},\boldsymbol{y}|E;\Omega) = \delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{1}{2\pi\nu\tau}G_{\alpha>0}(\boldsymbol{x},\boldsymbol{y}|E;\Omega/2) \times G_{\alpha'<0}(\boldsymbol{y},\boldsymbol{x}|E;\Omega/2) \tag{3.7b}$$

is the corresponding soft vertex. $P$ describes the relaxation of the particle number density.[9]

From Eqs. (2.5) we obtain the following properties of the vertex function $\Gamma$,

$$\Gamma(\boldsymbol{x},\boldsymbol{x}'|E;\Omega)=\Gamma(\boldsymbol{x}',\boldsymbol{x}|E;\Omega), \tag{3.8a}$$

and

$$\Gamma(\boldsymbol{x}+\boldsymbol{a},\boldsymbol{x}'+\boldsymbol{a}|E;\Omega)=\Gamma(\boldsymbol{x},\boldsymbol{x}'|E-\boldsymbol{F}\cdot\boldsymbol{a};\Omega). \tag{3.8b}$$

These relations are important for a gradient expansion of the vertex function. With their help, Eqs. (3.7) can be written to second order in the gradient operator and in the classically accessible region,

$$[\Omega-\boldsymbol{\nabla}\cdot D(\mu_x)\boldsymbol{\nabla}]P(\boldsymbol{x},\boldsymbol{x}'|E;\Omega)=\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{3.9}$$

Here

$$D(\mu_x)=\mu_x\tau/m \tag{3.10}$$

is the electron diffusion coefficient in a local equilibrium approximation. The details of the derivation are given in the Appendix. We see that, inside the classically accessible region, the soft propagator is governed by a generalized diffusion equation, as one would expect from the fact that physically, $P$ describes the relaxation of density fluctuations. Outside the classically accessible region, $\Gamma(\boldsymbol{x},\boldsymbol{x}'|E;\Omega)\approx\delta(\boldsymbol{x}-\boldsymbol{x}')$. Accordingly, only the modes inside the classically accessible region are generalized diffusion modes, again in agreement with what one would expect. Equation (3.9) is very similar to the equation derived in Ref. 17 by kinetic theory methods.

For later reference we note that, in a schematic notation, the Gaussian propagator $P$ has the form

$$P=1/[\Omega-D\boldsymbol{\nabla}^2+(D/E)\boldsymbol{F}\cdot\boldsymbol{\nabla}]. \tag{3.11}$$

Accordingly, there is a frequency scale

$$\Omega^*=D\boldsymbol{F}^2/E^2, \tag{3.12}$$

or a time scale $t^*=1/\Omega^*$ that separates diffusive behavior at $\Omega\gg\Omega^*$ from field-dominated drift behavior at $\Omega\ll\Omega^*$.

In deriving the generalized diffusion equation we have taken into account only terms linear in $F$. An estimate of the terms quadratic in $F$ shows that they are small compared to the ones kept provided that $|\boldsymbol{F}|\ell/\mu\ll 1$, with $\ell$ the mean-free path.

### C. Density relaxation

The generalized diffusion equation, Eq. (3.9), differs from the ordinary diffusion equation by the real-space dependence of $D$, which produces a term linear in the gradient. This introduces a new singularity into the differential equation. As a result, the solution of this equation for long times differs strongly from the ordinary Gaussian one. Let the field point in $x$ direction, $\boldsymbol{F}=(F,0)$, put $\boldsymbol{x}=(x,y)$ and $\boldsymbol{x}'=(x',0)$,[30] and perform a Fourier transform with respect to the direction transverse to the field,

$$P(x,k;x'|E;\Omega)=\int dy\, e^{iky}P(\boldsymbol{x},\boldsymbol{x}'|E;\Omega) \tag{3.13}$$

We then find that the solution of Eq. (3.9) takes the form

$$P(x,k;x'|E;\Omega)=\mathcal{P}(\mu_x,|k|;\mu'_x|E;\Omega), \tag{3.14a}$$

where

$$\begin{aligned}\mathcal{P}(\mu,p;\mu'|E;\Omega)&\\ =\frac{1}{D'F}e^{-p(\mu+\mu')/F}\{&\Theta(\mu-\mu')\\ &\times M[\tfrac{1}{2}(1+\Omega/D'Fp),1,2p\mu'/F]\\ &\times U[\tfrac{1}{2}(1+\Omega/D'Fp),1,2p\mu/F]+\Theta(\mu'-\mu)\\ &\times U[\tfrac{1}{2}(1+\Omega/D'Fp),1,2p\mu'/F]\\ &\times M[\tfrac{1}{2}(1+\Omega/D'Fp),1,2p\mu/F]\}\\ &\times\Gamma[\tfrac{1}{2}(1+\Omega/D'Fp)].\end{aligned} \tag{3.14b}$$

Here $D'=D(E)/E=\tau/m$. The functions $U$ and $M$ in Eq. (3.14b) are the confluent hypergeometric functions.[31] In order to obtain this solution we have required that $P$ vanishes at infinity in the classically accessible region and that the probability current vanishes at the turning point, so that the particles cannot penetrate into the classically forbidden region.

To understand the nature of this solution we recall that $P$ describes the relaxation of the particle number density. Let us consider, at time $t=0$, an ensemble of particles with energy $E=\mu$, located at $x=0$, and with a homogeneous density in the direction perpendicular to the electric field. Accordingly, the initial number density is given by

$$n_0(\boldsymbol{x},E)=\frac{2\pi}{L}\delta(E-\mu)\delta(x), \tag{3.15}$$

where $L$ is the linear dimension of the system in $y$ direction. The evolution of this initial density is governed by the propagator $P$ for $k=0$. In this limit $\mathcal{P}$ takes the form

$$\begin{aligned}\mathcal{P}(\mu,0;\mu'|E;\Omega)&\\ =\frac{2}{D'F}\Bigg[&\Theta(\mu-\mu')K_0\left(\sqrt{\frac{4\Omega\mu}{D'F^2}}\right)I_0\left(\sqrt{\frac{4\Omega\mu'}{D'F^2}}\right)\\ &+\Theta(\mu'-\mu)K_0\left(\sqrt{\frac{4\Omega\mu'}{D'F^2}}\right)I_0\left(\sqrt{\frac{4\Omega\mu}{D'F^2}}\right)\Bigg].\end{aligned} \tag{3.16}$$

Here $I_0$ and $K_0$ are the modified Bessel functions. The inverse Laplace transformation yields

$$P(x,0;x'|E;t)=\frac{\exp\left(-\dfrac{\mu_x+\mu'_x}{D'F^2t}\right)}{D'Ft}I_0\left(\frac{2\sqrt{\mu_x\mu'_x}}{D'F^2t}\right). \tag{3.17}$$

Equation (3.17) explicitly displays the characteristic time

$$t^*=E/D'F^2, \tag{3.18}$$

which was apparent already in Eq. (3.9) [see Eqs. (3.11) and (3.12)], and which serves as the boundary between the short-time and the long-time behavior. In order to determine the structure of the generalized diffusion propagator for $t \ll t^*$, we use Eq. (3.14a). For $t \ll t^*$ the spread of the initial $\delta$ package in $x$ direction is small, and we can use $Fx/E, Fx'/E \ll 1$, in addition to the asymptotic expansion of $I_0$ for large arguments. If we expand the exponent with respect to $x-x'$, we find

$$P(x,0;x'|E;t) = \frac{1}{\sqrt{4\pi D(E)t}} e^{-(x-x')^2/4D(E)t}. \tag{3.19}$$

As expected, the relaxation of the initial density perturbation is diffusive in this time regime. Note that in writing Eq. (3.19) we have ignored the first moment of the generalized diffusion propagator, which is nonzero. While the first moment is crucial for calculating currents, it is irrelevant for our current discussion.

In the opposite limit, $t \gg t^*$, the behavior is very different. In this case the width of the particle packet becomes very large and the asymmetry of the particle packet, which is small initially, is getting considerable. The Bessel function in Eq. (3.17) approaches unity for asymptotically long times, so that

$$P(x,0;x'|E,t) = \frac{1}{D'Ft} \exp\left( -\frac{\mu_x + \mu_x'}{D'F^2 t} \right) \tag{3.20}$$

for $\sqrt{\mu_x \mu_x'}/D'F^2 t \ll 1$. For very large $|x|$, which satisfy the requirement $\sqrt{\mu_x \mu_x'}/D'F^2 t \gg 1$, we obtain

$$P(x,0;x'|E,t) = \frac{1}{D'Ft} \exp\left( -\frac{(\sqrt{\mu_x} - \sqrt{\mu_x'})^2}{D'F^2 t} \right). \tag{3.21}$$

These results show that in this regime the dynamics is no longer diffusive.

If, instead of the distribution function (3.15), we consider a $\delta$ pulse in both the longitudinal and transverse directions, we have to investigate the function $P$ for nonzero values of $k$. We have not been able to do so exactly, and have resorted to a WKB approximation instead. The results obtained in this way are in qualitative agreement with the case analyzed above.

### D. Transport and heating

So far we have considered the relaxation of density perturbations. We now turn to the questions of transport and Joule heating. To this end we consider, instead of Eq. (3.15), an initial density distribution

$$n_0(\boldsymbol{x},E) = \frac{2\pi N}{V} \delta(E - \mu - \boldsymbol{F}\cdot\boldsymbol{x}). \tag{3.22}$$

Here $N$ is the total particle number and $V$ is the system volume. Note that Eq. (3.22) describes a uniform number density in the classically accessible region, since

$$n_0(\boldsymbol{x}) = \int \frac{dE}{2\pi} n_0(\boldsymbol{x},E) = \Theta(\mu_x) N/V. \tag{3.23}$$

To see how such a density distribution evolves, we realize that $\mathcal{P}$, Eq. (3.17), gives the probability to find a particle at energy $\mu'$ at time $t$, if it had the energy $\mu$ at time $t=0$. In the limit of long times, $t \gg t^*$, the mean particle energy therefore increases linearly with time,

$$\mu(t) = \frac{1}{F} \int d\mu\, \mu \mathcal{P}(\mu,0;\mu'|t) = \mu + D'F^2 t. \tag{3.24}$$

Note that $P$ and $\mathcal{P}$ were normalized with respect to an integration over $x$. Changing the integration variable to $\mu$ results in the additional factor $1/F$ in Eq. (3.24). Equation (3.24) shows that the energy fed into the system increases the kinetic energy of the charge carriers.

According to the generalized diffusion equation, the current-density distribution is given by

$$\boldsymbol{j}(\boldsymbol{x},E|\Omega) = -D(\mu_x) \nabla n(x,E|\Omega), \tag{3.25}$$

where $n(\boldsymbol{x},E|\Omega)$ is to be calculated from $n_0$ by means of the propagator $P$. The volume averaged current density takes the form

$$\boldsymbol{j}(\Omega) = \frac{1}{V} \int \frac{dE}{2\pi} d\boldsymbol{x} \boldsymbol{j}(\boldsymbol{x},E|\Omega). \tag{3.26}$$

For our spatially uniform charge-density distribution, $n(\boldsymbol{x},E|\Omega) = n(\mu_x|\Omega)$, and we obtain

$$\boldsymbol{j}(\Omega) = -\boldsymbol{F} \int \frac{d\mu}{2\pi} \frac{dD(\mu)}{d\mu} n(\mu|\Omega). \tag{3.27}$$

In our Gaussian approximation, the derivative of the diffusion constant is independent of $\mu$, and thus can be taken out of the integral. We finally obtain

$$\boldsymbol{j}(\Omega) = -\frac{\tau N}{mV} \frac{\boldsymbol{F}}{\Omega}. \tag{3.28}$$

The Gaussian theory thus yields an Ohmic current that leads to Joule heating.

## IV. NONLINEAR $\sigma$ MODEL

### A. Effective action

The derivation of a matrix field theory in Sec. III has proceeded in analogy to the case without an electric field, and the result was structurally very similar to the latter. In particular, matrix elements that correspond to products of retarded and advance degrees of freedom are soft, while those corresponding to products of two advanced or two retarded degrees of freedom are massive. The chief difference is that, in the presence of an electric field, the soft modes in Gaussian approximation are not diffusive, but rather obey the more complicated differential Eq. (3.9).

It is obvious from these observations that one can derive an effective field theory for the soft modes by repeating the procedure that leads to a nonlinear $\sigma$ model in the zero-field case.[29,32] As expected, the only change is that the Laplace operator in the $\sigma$ model is replaced by the differential operator from Eq. (3.9). We thus find

$$A_{\rm eff}[\hat{Q}] = -\frac{\pi\nu}{8}\int d\boldsymbol{x}\,{\rm tr}[\hat{Q}(\boldsymbol{x})[\boldsymbol{\nabla}\cdot D(\mu_x)\boldsymbol{\nabla}]\hat{Q}(\boldsymbol{x})] - \frac{\pi\nu}{2}\Omega\int d\boldsymbol{x}\,{\rm tr}[\Lambda\hat{Q}(\boldsymbol{x})]. \tag{4.1}$$

Here $\Lambda$ is a diagonal matrix whose elements have been given after Eq. (2.9). The matrix elements of $\hat{Q}$ are elements of the homogeneous space $O(n,n)/O(n)\times O(n)$, and $\hat{Q}$ is subject to the constraints

$${\rm tr}\,\hat{Q}(\boldsymbol{x}) = 0, \tag{4.2a}$$

$$\hat{Q}^2(\boldsymbol{x}) = 1, \tag{4.2b}$$

$$\hat{Q}^T(\boldsymbol{x}) = \Lambda\hat{Q}(\boldsymbol{x})\Lambda. \tag{4.2c}$$

These can be incorporated in a parametrization in terms of $n\times n$ matrices $q$,

$$\hat{Q} = \begin{pmatrix} \sqrt{1+qq^T} & q \\ -q^T & -\sqrt{1+q^Tq} \end{pmatrix}. \tag{4.3}$$

Roughly speaking, the field $\hat{Q}$ contains the soft parts of the field $Q$ of the preceding section.

An expansion of the action, Eq. (4.1), to quartic order in $q$ reads

$$A_{\rm eff} = A^{(2)} + A^{(4)}, \tag{4.4a}$$

with

$$A^{(2)} = -\frac{\pi\nu}{4}\int d\boldsymbol{x}\sum_{\alpha\alpha'} q_{\alpha\alpha'}(\boldsymbol{x})[\Omega - \boldsymbol{\nabla}\cdot D(\mu_x)\boldsymbol{\nabla}]q_{\alpha\alpha'}(\boldsymbol{x}), \tag{4.4b}$$

$$A^{(4)} = \frac{\pi\nu}{32}\sum_{\{\alpha_i\}}\int d\boldsymbol{x}\{q_{\alpha_1\alpha_2}(\boldsymbol{x})q_{\alpha_3\alpha_2}(\boldsymbol{x})[\Omega - \boldsymbol{\nabla}\cdot D(\mu_x)\boldsymbol{\nabla}]q_{\alpha_3\alpha_4}(\boldsymbol{x})q_{\alpha_1\alpha_4}(\boldsymbol{x}) + q_{\alpha_2\alpha_1}(\boldsymbol{x})q_{\alpha_2\alpha_3}(\boldsymbol{x}) \times[\Omega - \boldsymbol{\nabla}\cdot D(\mu_x)\boldsymbol{\nabla}]q_{\alpha_4\alpha_3}(\boldsymbol{x})q_{\alpha_4\alpha_1}(\boldsymbol{x})\}. \tag{4.4c}$$

### B. One-loop theory for the diffusivity

The expansion of the effective action in powers of $q$ allows for a systematic loop expansion. To one-loop order we find the following result for the diffusivity $D$,

$$D^{(1)}(\mu_x|\Omega) = D(\mu_x)\left[1 - \frac{1}{\pi\nu}P(\boldsymbol{x},\boldsymbol{x}|E;\Omega)\right], \tag{4.5}$$

which has the same structure as in the absence of the electric field. The only difference is that $P(\boldsymbol{x},\boldsymbol{x}|E,\Omega)$ is calculated from Eq. (3.14b), according to the relationship

$$P(\boldsymbol{x},\boldsymbol{x}|E,\Omega) = \int_0^{\lambda}\frac{dp}{\pi}\mathcal{P}(\mu_x,p;\mu_x|E,\Omega), \tag{4.6}$$

where $\lambda$ is an ultraviolet cutoff. In the absence of the electric field this integral is infrared divergent if $\Omega = 0$. In the presence of the electric field we obtain, for $\Omega = 0$, and in the limit $\lambda\mu_x/F \gg 1$,

$$D^{(1)}(\mu_x) = D(\mu_x)\left[1 - \frac{1}{2\pi^2\nu D(\mu_x)}\ln(\lambda\mu_x/F)\right]. \tag{4.7}$$

The static one-loop diffusivity is thus finite in two dimensions, indicating that the electric field destroys the mechanism that gives rise to localization in $d = 2$. This was already obvious from Eq. (3.11), which is less infrared divergent than a diffusion propagator.

For $\Omega \gg 1/t^*$, with $t^*$ from Eq. (3.18), the function $P$ reduces to the conventional diffusion propagator, as discussed in Sec. III. Consequently, in this regime the corrections to the bare diffusion coefficient take the same form as in the conventional weak-localization theory in the absence of the field.

### C. Scaling analysis

We now perform a scaling analysis of the generalized nonlinear $\sigma$ model. Our procedure is analogous to the one in Ref. 33. To this end, we deviate from our restriction to $d = 2$ and consider the model in $d \geq 2$, ignoring the complications that arise from the chemical-potential dependence of the density of states in $d > 2$. We first write the effective action in a schematic form that leaves out everything that is not necessary for power-counting purposes,

$$A_{\rm eff} = \frac{1}{G_2}\int d\boldsymbol{x}\boldsymbol{\nabla}^2q^2 + H\int d\boldsymbol{x}\Omega q^2 + \frac{F}{G_2}\int d\boldsymbol{x}\boldsymbol{x}\boldsymbol{\nabla}^2q^2 + \frac{1}{G_4}\int d\boldsymbol{x}\boldsymbol{\nabla}^2q^4 + \cdots. \tag{4.8}$$

Here $H \propto \nu$, $1/G_2 \propto \nu\tau$, etc. We now assign a scale dimension $[L] = -1$ to lengths $L$. For $F = 0$, this action contains two fixed points, namely, a stable Gaussian one that describes the diffusive phase, and a critical one that describes the Anderson transition.

#### *1. Gaussian fixed point*

The Gaussian fixed point that describes diffusion in the absence of an electric field $F = 0$ is characterized by a scale dimension of the field $q$ equal to

$$[q]_{\rm diff} = (d-2)/2. \tag{4.9a}$$

Frequencies must scale like wave numbers squared in a diffusive phase, so we also require

$$[\Omega]_{\rm diff} = 2. \tag{4.9b}$$

$G_2$ and $H$ are then dimensionless,

$$[G_2]_{\rm diff} = [H]_{\rm diff} = 0, \tag{4.9c}$$

and all higher-order terms are irrelevant, with the least irrelevant couplings having a scale dimension

$$[u]_{\text{diff}}=-(d-2), \tag{4.9d}$$

where $u$ denotes a generic irrelevant operator. $1/G_4$ is an example of such a least irrelevant operator. Adding the electric field, we see that $F$ is relevant with respect to the diffusive fixed point,

$$[F]_{\text{diff}}=1. \tag{4.10}$$

The frequency dependent diffusivity, whose bare value is $D = 1/G_2H$, therefore obeys a scaling law

$$D(\Omega,F,u)=D(\Omega b^2,F\,b,u\,b^{-(d-2)}). \tag{4.11}$$

At $\Omega=0$, and for small $F$, we conclude that the diffusivity has the structure

$$D(F)\propto \text{const}+F^{d-2}, \tag{4.12}$$

and in $d=2$ one expects a logarithmic dependence of $D$ on $F$. This is in agreement with the explicit perturbative result in the preceding section.

For $\Omega\neq 0$, $D$ is a function of $F^2/\Omega$, in agreement with the explicit Gaussian theory in Sec. III. There are two scaling regimes. For small $\Omega$, the scaling of $D$ is governed by the electric field, and for large $\Omega$, the scaling is governed by the frequency. The crossover between these two scaling regimes is at a frequency $\Omega=\Omega^*$, Eq. (3.12). In an experiment, the frequency $\Omega$ is effectively replaced by $1/\tau_\phi$, where $\tau_\phi$ is the phase relaxation time.[34] The weak-localization physics can therefore only be observed if the temperature is large compared to a crossover temperature $T^*$, which is given by

$$1/\tau_\phi(T^*)=D\boldsymbol{F}^2/E^2. \tag{4.13}$$

We will further discuss this condition in Sec. V below.

#### *2. Critical fixed point*

We now turn to the critical fixed point that describes an Anderson transition at $F=0$. Here we choose the field $q$ to be dimensionless, and the scale dimension of the frequency to be $d$,

$$[q]_{\text{c}}=0, \ \ [\Omega]_{\text{c}}=d. \tag{4.14}$$

The bare scale dimension of $G_2$ is then $[G_2]_{\text{c}}=[G_4]_{\text{c}}=\cdots \equiv[G]_{\text{c}}=2-d=-\epsilon$. An explicit renormalization-group calculation shows that $H$ is not renormalized, while the renormalized counterpart of $G$, $g$, has a fixed point value $g^* = O(\epsilon)$. The deviations of $g$ from $g^*$ constitute the relevant operator at the critical fixed point, whose scale dimension determines the correlation length exponent $\nu$. To one-loop order[4]

$$\nu=1/\epsilon+O(1). \tag{4.15}$$

The scale dimension of $F$ is also given by a loop expansion, but the leading term can again be determined just by power counting,

$$[F]_{\text{c}}=1+O(\epsilon). \tag{4.16}$$

The electric field is thus a relevant operator with respect to the critical Anderson fixed point. In particular, an arbitrarily weak field $F$ destroys the usual localization in $d=2$.

The above discussion shows only that the $F=0$ fixed point is unstable, and does not tell what happens instead. The perturbative result, Eq. (4.7), suggests that there is a metallic phase in $d=2$ for $F\neq 0$. We note that the result, Eq. (4.16), is different from the popular scaling argument which assumes that $F\xi$, with $\xi$ the correlation length, represents the critical energy or frequency scale, which yields $[F]=d+1$.[18] We will come back to this discrepancy in Sec. V below.

## V. DISCUSSION

In summary, we have used field-theoretic methods to investigate the impact of an electric field on the localization of noninteracting electrons, mostly in two dimensions. We have found that there is a characteristic temperature $T^*$ that separates a regime where the physics is dominated by the field from one where it is not. For $T<T^*$ the physics is dominated by the electric field, which directly affects the structure of the localization corrections. In this regime the density relaxation is strongly nondiffusive, and the usual weak-localization corrections to observables are replaced by logarithmic dependences on the field. The latter have the same structure as those derived in Refs. 20 and 22. Our treatment shows that the applicability of these results is restricted to $T<T^*$. For $T>T^*$ the electric field does not significantly affect the diffusion of a particle packet. Consequently, the weak-localization corrections to the diffusion coefficient in this regime are the same as in equilibrium and independent of the electric field. In this regime the approaches of Refs. 14 and 16 are well founded.

Let us estimate the value of $T^*$ for parameter values that are representative of a typical weak-localization experiment.[35] With $D\approx 14$ cm$^2$/s, $F\approx 1.6\times 10^{-2}$ eV/cm, and $E\approx 0.6$ meV, one has $1/\tau_\phi(T^*)\approx 10^4$ Hz. At low temperatures, $\tau_\phi$ is dominated by the electron-electron interaction and inversely proportional to the temperature,[36] $\tau_\phi = c/T$. For the data of Ref. 35 we obtain $c\approx 10^{11}$ K$^{-1}$s$^{-1}$. This yields $T^*\approx 10^{-7}$ K. We conclude that the crossover between the field-dominated regime and the usual weak-localization regime occurs at unobservably low temperatures. This explains why no field-dominated scaling corrections were observed in the experiments of Refs. 26 and 27, and it shows that these observations are not at odds with the notion that an arbitrarily weak electric field does indeed destroy the weak localization.

We now come back to the scale dimension of the field $F$ with respect to the critical fixed point that describes the metal-insulator transition. As mentioned in Sec. IV, a scaling argument given by Sondhi *et al.*[18] assumes that $F\xi$ scales like the critical energy scale. For the scale dimension of $F$ this yields $[F]_{\text{c}}=z+1$, with $z$ the dynamical critical exponent. Schematically, this corresponds to a critical propagator (in the case of an Anderson transition)

$$\frac{1}{\Omega+D(\boldsymbol{\nabla})\nabla^2+F/\boldsymbol{\nabla}}. \tag{5.1}$$

This is not consistent with perturbation theory, however, at least not for the model studied here. From Eq. (3.11) we see that the actual critical propagator has the structure

$$\frac{1}{\Omega+D(\boldsymbol{\nabla})\nabla^2+FD(\boldsymbol{\nabla})\boldsymbol{\nabla}}. \tag{5.2}$$

The difference between these two expressions accounts for the difference between the present result, $[F]_{\rm c}=1$, and that of Ref. 18. The reason for the additional gradient squared in Eq. (5.2) compared to Eq. (5.1) is the Ward identity[4] that is closely related to particle number conservation. Equation (5.1) implies that the electric field breaks particle number conservation, which it does not. The same point can be made at the level of a fermionic action,[5] which can be seen as follows. The $\boldsymbol{F}\cdot\boldsymbol{x}$ term in the Hamiltonian, Eq. (2.2), corresponds to a term

$$S_F=\int d\boldsymbol{x}\int_0^{1/T} d\tau(\boldsymbol{F}\cdot\boldsymbol{x})n(\boldsymbol{x},\tau) \tag{5.3}$$

in the action, with $n$ the electron number density field. The latter corresponds to $n=\mathrm{tr}\,Q$ in the $Q$-matrix formulation of the field theory, and in the nonlinear $\sigma$ model this corresponds to $\mathrm{tr}\,\hat{Q}=0$. This shows that the leading coupling of $\boldsymbol{F}$ to the electronic soft modes vanishes for symmetry reasons. The leading nonvanishing term carries an additional gradient squared. This is the reason why one cannot obtain the nonlinear $\sigma$ model for electrons in an electric field by simply replacing the coupling term in the fermionic action by its $Q$-field counterpart. In contrast, an external magnetic field *does* break a symmetry (viz., spin rotational invariance) and gives some soft modes (viz., the spin diffusions) a mass, and replacing the spin density in the Zeeman term by its corresponding trace over a $Q$-field gives the correct answer. We conclude that, in general, one cannot extract values for the dynamical exponent $z$ from experimental data by assuming $[F]=z+1$, as was done in Ref. 28.

At this point we would like to note that our model does not take into account the electron-electron interaction and the resulting inelastic collisions. Therefore, our results are only valid for sample sizes smaller than the energy relaxation length. If inelastic collisions were taken into account, the electron system should be described by a distribution function with an effective temperature. If the energy-transfer rate within the electronic subsystem is larger than the one between the electron and the phonon system, the effective temperature $T_{\rm eff}$ would depend also on the strength of the applied electric field. Since in this case the temperature $T$ in the in the phase relaxation time $\tau_\phi(T)$ would be replaced by $T_{\rm eff}$, the field dependence of the effective temperature would also be reflected in the logarithmic corrections to the conductivity, as pointed out in Ref. 37. This can be mistaken for a direct impact of the electric field on the localization corrections in a regime where there is actually none. However, these effects depend on the ratio between the energy-transfer rate in the electronic subsystem and the cooling rate. They are therefore not universal, but dependent on the experimental conditions.

We finally discuss the relation of our theory to some previous work in the literature. We have already noted that our equation for the crucial Gaussian propagator, Eq. (3.9), is very similar to the one obtained by Kirkpatrick with different methods.[17] Indeed, our theory is in many respects a field-theoretic version of his kinetic theory. A different differential equation for the propagator was used in Refs. 21–23, which found that a critical field strength $F_{\rm c}$ is necessary to delocalize the states at the Fermi energy. Instead of Eq. (3.9), these authors used

$$[\Omega-D(E)\boldsymbol{\nabla}^2+D'(E)\boldsymbol{F}\cdot\boldsymbol{\nabla}]P(\boldsymbol{x},\boldsymbol{x}'|E;\Omega)=\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{5.4}$$

with $D'(E)=dD(E)/dE$. The use of this equation was based on the notion that there is a rapid mechanism for cooling, so that heating processes can be ignored. The resulting equation violates the properties expressed by Eqs. (3.8), which are based on time reversal symmetry and generalized translational invariance. Indeed, if there is a rapid mechanism for cooling that validates Eq. (5.4), then the electrons must obviously have already experienced inelastic collisions before the time at which Eq. (5.4) becomes valid. They are thus already in the diffusive regime $t>t^*$ where the electric field no longer provides the leading effect. In the opposite regime $t<t^*$ Eq. (5.4) is not valid, and Eq. (3.9) must be used instead.

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## APPENDIX: DERIVATION OF THE GENERALIZED DIFFUSION EQUATION

In order to obtain Eq. (3.9) we have to expand $\Gamma$ about the $\delta$ function. To this end we need to calculate the moments of $\Gamma$. We first consider the zeroth moment, which is given in terms of

$$M_0=\int d\boldsymbol{y}G_\alpha(\boldsymbol{x},\boldsymbol{y}|E;\omega)G_{-\alpha}(\boldsymbol{y},\boldsymbol{x}|E;\omega), \tag{A1}$$

with $\omega=\Omega/2$ and $\alpha>0$. In terms of the function $g_\alpha$, Eq. (3.4), we have

$$M_0=\int d\boldsymbol{y}\int_{\boldsymbol{k}}\int_{\boldsymbol{p}} g_\alpha[\boldsymbol{k}|E-\boldsymbol{F}\cdot(\boldsymbol{x}-\boldsymbol{y}/2);\omega]\times g_{-\alpha}[\boldsymbol{p}|E-\mathbf{F}\cdot(\boldsymbol{x}-\boldsymbol{y}/2);\omega]e^{i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{y}}, \tag{A2}$$

where $\int_{\boldsymbol{k}}=\int d\boldsymbol{k}/(2\pi)^2$. The terms $\boldsymbol{F}\cdot\boldsymbol{y}/2$ in Eq. (A2) lead to corrections of $O(F^2)$. If we omit them we obtain

$$M_0 = \int_{\boldsymbol{k}} g_\alpha(K|\mu_x;\omega) g_{-\alpha}(K|\mu_x;\omega)$$

$$= \nu \int_{-\mu_x}^{\infty} \frac{d\epsilon}{i\omega - \epsilon + \frac{i}{2\tau}\mathrm{sgn}(\omega)} \frac{1}{i\omega - \epsilon_k + \frac{i}{2\tau}\mathrm{sgn}(-\omega)}. \tag{A3}$$

In order to evaluate this integral we use the approximation already discussed in Sec. III: If $\mu_x > 0$ we replace $\mu_x$ by $\infty$, and if $\mu_x < 0$ we replace $\mu_x$ by $-\infty$. In the hydrodynamic limit we then obtain

$$M_0 = 2\pi\nu\tau\Theta(\mu_x)(1-\Omega\tau). \tag{A4}$$

The first moment $\boldsymbol{M}_1$ is defined as

$$\mathbf{M}_1 = \int d\mathbf{y}(\boldsymbol{x}-y) G_\alpha(\boldsymbol{x},\boldsymbol{y}|E;\omega) G_{-\alpha}(\boldsymbol{y},\boldsymbol{x}|E;\omega). \tag{A5}$$

In the same approximation as above we find

$$\boldsymbol{M}_1 = \frac{\boldsymbol{F}}{4} \frac{d}{dE} \int_{\boldsymbol{k}} \frac{\partial^2}{\partial \boldsymbol{\kappa}^2}\bigg|_{\boldsymbol{\kappa}=0} g_\alpha(\boldsymbol{k}+\boldsymbol{\kappa}|\mu_x;\omega) g_{-\alpha}(\boldsymbol{k}|\mu_x;\omega)$$

$$= \boldsymbol{F}\frac{d}{dE} M_2. \tag{A6}$$

Here $M_2$ is the second moment. It is defined as

$$M_2 = \frac{1}{4}\int d\boldsymbol{y}(\boldsymbol{y}-\boldsymbol{x})^2 G_\alpha(\boldsymbol{x},\boldsymbol{y}|E;\omega) G_{-\alpha}(\boldsymbol{y},\boldsymbol{x}|E;\omega). \tag{A7}$$

Here we ignore the fact that anisotropic terms can arise due to the electric field, since any such terms are of $O(F^2)$. Again using the same approximations as above, we obtain in the hydrodynamic limit

$$M_2 = \frac{\nu\mu_x}{2m}\int_{-\mu_x}^{\infty} d\epsilon \frac{1}{(1/4\tau^2+\epsilon^2)^2} = 2\pi\nu\tau D(\mu_x)\tau\theta(\mu_x). \tag{A8}$$

Collecting our results, we now have the following expression for $\Gamma$ in the hydrodynamic regime, and for $\boldsymbol{x}$ in the classically accessible region,

$$\Gamma(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})[\Omega - \boldsymbol{\nabla}\cdot D(\mu_x)\boldsymbol{\nabla}]. \tag{A9}$$

From this, Eq. (3.9) follows immediately.

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[1] N.F. Mott, *Metal-Insulator Transitions* (Taylor & Francis, London, 1990).

[2] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[3] This holds for noninteracting electrons. It is probably also true for interacting systems, at least for weakly interacting ones, but the situation in the presence of strong electron-electron interactions is not clear, see Ref. 28.

[4] F. Wegner, Z. Phys. B **35**, 207 (1979).

[5] K.B. Efetov, A.I. Larkin, and D.E. Khmelnitskii, Zh. Éksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)].

[6] K.B. Efetov, Zh. Éksp. Teor. Fiz. **82**, 872 (1982) [Sov. Phys. JETP **55**, 514 (1982)].

[7] K.B. Efetov, Zh. Éksp. Teor. Fiz. **83**, 833 (1982) [Sov. Phys. JETP **56**, 467 (1982)].

[8] W. Götze, Solid State Commun. **27**, 1393 (1978).

[9] D. Vollhardt and P. Wölfle, Phys. Rev. B **22**, 4666 (1980).

[10] D. Belitz, A. Gold, and W. Götze, Z. Phys. B: Condens. Matter **44**, 273 (1981).

[11] V.N. Prigodin, Zh. Éksp. Teor. Fiz. **79**, 2338 (1980) [Sov. Phys. JETP **52**, 1185 (1980)].

[12] V.N. Prigodin and B.L. Altshuler, Phys. Lett. A **137**, 301 (1989).

[13] F. Delyon, B. Simon, and B. Souillard, Phys. Rev. Lett. **52**, 2187 (1984).

[14] B.L. Altshuler, A.G. Aronov, and D.E. Khmelnitskii, Solid State Commun. **39**, 619 (1981).

[15] X.L. Lei and J. Cai, Phys. Rev. B **42**, 1574 (1990).

[16] S. Hershfield and V. Ambegaokar, Phys. Rev. B **34**, 2147 (1986).

[17] T.R. Kirkpatrick, Phys. Rev. B **33**, 780 (1986).

[18] S.L. Sondhi, S.M. Girvin, J.P. Carini, and D. Shahar, Rev. Mod. Phys. **69**, 315 (1990).

[19] T. Tsuzuki, Physica B **107**, 679 (1981).

[20] T. Tsuzuki, Prog. Theor. Phys. **67**, 68 (1982).

[21] Y.C. Lee, C.S. Chu, and E. Castano, Phys. Rev. B **27**, 6136 (1983).

[22] V.V. Bryksin, H. Schlegel, and P. Kleinert, Phys. Rev. B **49**, 13 697 (1994).

[23] O. Bleibaum, H. Böttger, V.V. Bryksin, and P. Kleinert, Phys. Rev. B **52**, 16 494 (1995).

[24] G.J. Dolan and D.D. Osheroff, Phys. Rev. Lett. **43**, 721 (1979).

[25] D.J. Bishop, D.C. Tsui, and R.C. Dynes, Phys. Rev. Lett. **44**, 1153 (1980).

[26] G. Bergmann, Z. Phys. B: Condens. Matter **49**, 133 (1982).

[27] Z. Ovadyahu, Phys. Rev. B **63**, 235403 (2001).

[28] E. Abrahams, S.V. Kravchenko, and M.P. Sarachik, Rev. Mod. Phys. **73**, 251 (2001).

[29] A.J. McKane and M. Stone, Ann. Phys. (N.Y.) **131**, 36 (1981).

[30] With the field in $x$ direction, one has translational invariance in $y$ direction, so this choice can be made without loss of generality.

[31] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[32] A.M.M. Pruisken and L. Schäfer, Nucl. Phys. B **200**, 20 (1982).

[33] D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **56**, 6513 (1997).

[34] E. Abrahams, P.W. Anderson, and T.V. Ramakrishnan, Philos. Mag. B **42**, 827 (1980).

[35] S.V. Kravchenko and T.M. Klapwijk, Phys. Rev. Lett. **84**, 2909 (2000).

[36] B.L. Altshuler, A.G. Aronov, and D.E. Khmel'nitskii, J. Phys. C **15**, 7367 (1982).

[37] P.W. Anderson, E. Abrahams, and T.V. Ramakrishnan, Phys. Rev. Lett. **43**, 718 (1979).