Dimensional crossover of localization and delocalization in a quantum Hall bar

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The two- to one-dimensional crossover of the localization of electrons confined to a disordered quantum wire of finite width L_y is studied in a model of electrons moving in the potential of uncorrelated impurities. The localization length is derived as a function of the perpendicular magnetic field *B*, the wire width L_y, and the conductance parameter *g*. The analytical theory allows us to study the localization continuously from weak to strong magnetic fields. On the basis of these results, the scaling analysis of the quantum Hall effect in high Landau levels and the delocalization transition in a quantum Hall wire are reconsidered. We conclude that in quantum Hall bars, the quantum Hall transition is driven in all but the lowest two Landau bands by a noncritical dimensional crossover of the localization length.

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I. INTRODUCTION

The Hall conductance of a two-dimensional electron system in a strong magnetic field[1] is precisely quantized due to the trapping of electrons to localized states in the bulk of the system. Thereby, a change of electron density does not result in a change of the Hall conductance.[2–4] In the tail of the Landau bands the localization length ξ is small, on the order of the cyclotron length $l_{cyc}=\sqrt{2n+1}\,l_B$, where the magnetic length l_B is defined by $l_B^2=\hbar/(qB)$. The localization length increases towards the middle of the Landau bands, located at energies $E_n=\hbar\omega_c(n+1/2)$, where $\omega_c=qB/m$ is the cyclotron frequency, q being the electron charge, and $n=0,1,2,\ldots$. In an infinite system the localization length of an eigenstate with energy E is expected to diverge as $\xi\sim(E-E_n)^{-\nu}$. The exponent ν is found for the lowest two Landau bands, $n=0,1$, to be $\nu\approx2.3$ for spin split Landau levels, as supported by analytical,[5,6] numerical[7,8] and experimental studies,[9] reminiscent of a second-order transition from an insulator to a metal. Thus, for a finite system, there should exist in the middle of a disorder broadened Landau band, E_n, n being the Landau index, a band of states, which extend through the whole system of size L, with bandwidth $\Delta E=(l_{cyc}/L)^{1/\nu}\Gamma$, where $\Gamma=\hbar\sqrt{2/\pi}\sqrt{\omega_c/\tau}$. On the other hand, the localization length in two-dimensional systems with broken time-reversal symmetry is from the one-parameter scaling theory expected to depend exponentially on the conductance g as[10–13]

$$\xi\sim\exp(\pi^2g^2), \tag{1}$$

where g is the conductance parameter per spin channel. g exhibits the Shubnikov–de Haas oscillations as a function of the magnetic field, for $\omega_c>1/\tau$, where $1/\tau$ is the elastic-scattering rate. The maxima occur, when the Fermi energy is in the middle of the Landau band. Thus, the localization length is expected to increase strongly from the tails to the middle of the Landau bands, irrespective of the existence of the quantum critical point. For uncorrelated impurities, within self-consistent Born approximation,[14] one finds that the maxima in the longitudinal conductance are given by $g(E=E_n)=(1/\pi)(2n+1)=g_n$. Thus, one gets localization lengths $\xi_{2D}(E_n)=l_{\rm cyc}\exp(\pi^2g_n^2)$ in the middle of higher Landau levels, $n>1$, which are macroscopically large.[8,15] Thus, even when the wire width is smaller than $\xi_{2D}(E_n)$ there are wide quantum Hall plateaus, whose widths are determined by the number of states with localization length smaller than the wire width.

When the width of the wire L_y is smaller than the length scale ξ, the localization is expected to become quasi-one-dimensional; that is, the electrons in the middle of a Landau band can diffuse freely between the edges of the wire, but are localized along the wire. The quasi-one-dimensional localization length is known to depend only linearly on the conductance, and is, in a magnetic field, with broken time-reversal symmetry, given by[16–19]

$$\xi=2g(B)L_y. \tag{2}$$

Thus, for $L_y\ll\xi_{2D}(E_n)$ there is a crossover from two- to one-dimensional localization as the Fermi energy is moved from the tails into the middle of a Landau band. It is known from numerical[7] and analytical studies,[6] that in a finite quantum Hall bar, the criticality in the middle of the Landau band results in a finite localization length $\xi_{crit}\approx1.2L_y$, exceeding the wire width L_y. Comparing this value with the quasi-one-dimensional localization length for uncorrelated impurities in the middle of the Landau band, obtained from Eq. (2), $\xi_{n1D}=(2/\pi)(2n+1)L_y$, we see that this length scale exceeds the *critical* localization length ξ_{crit} in all but the lowest Landau level, $n=0$. Thus, the question arises, if the transitions between Hall plateaus in quantum Hall wires in higher Landau bands are at all sensitive to the critical point existing in an infinite system.

It is the aim of this paper to resolve this question and to derive analytically the dimensional crossover of the localization length ξ in a wire as a function of a perpendicular magnetic field. This might also help to identify the irrelevant scaling parameters observed in numerical studies of the integer quantum Hall transition in lower Landau bands.[7,8] Furthermore, these results suggest that in quantum Hall bars of finite width L_y, there exists at low temperatures a new phase, when the phase coherence length L_φ exceeds the quasi-one-dimensional localization length, in the middle of the Landau band, $L_\varphi > \xi_n$.[20] This phase may accordingly be called the mesoscopic quantum Hall phase, exhibiting plateaus in the Hall conductance, when the bulk localization length is smaller than the wire width, and the Hall conductance is carried by edge states, separated by regions in energy where all states are localized along the wire, and the conductance is zero. The quasi-one-dimensional localization along the quantum Hall bar has been speculated on before[21] and noted recently in a renormalization-group study of an effective quantum percolation model in Ref. 6. Here, this is confirmed by an explicit analytical calculation, and extended to the localization in higher Landau levels, on the assumption that the edge states do mix with the bulk states when the localization length exceeds the wire width.

While in previous treatments, limiting behaviors have been addressed, here a consistent theory for the dimensional crossover is provided. The nonlinear equations describing the dimensional crossover of the localization are derived and their solution is obtained. This theory allows us to study the localization length continuously from weak to strong magnetic field, and thus allows a more detailed evaluation of experimental and numerical studies of quantum wires in a magnetic field. On the technical side, we note that we obtained the magnetic-field dependence of the localization length by calculating the β function in second loop, the so-called Hikami boxes, as a function of the magnetic field.

Based on the nonlinear sigma model (NLSM) with topological term, which is rederived in Appendix D, allowing for an inhomogeneous conductivity and for a confining potential, it is shown and discussed in Sec. IV that the topological term, which is responsible for criticality in the middle of the Landau bands, does not become effective in higher Landau bands for uncorrelated disorder in quantum Hall wires of any finite width, since to become critical the NLSM must be two dimensional on length scales of the order of the noncritical localization length, which becomes exponentially large in higher Landau bands. Thus, we conclude that the plateau transition in high Landau bands is driven by the noncritical dimensional crossover of the localization length, and that there the topological term does not become effective. This has important consequences for the experimental and numerical scaling studies of plateau transitions in higher Landau bands.

Furthermore, we show that the noncritical crossover can qualitatively account for the irrelevant scaling observed in lower Landau bands. Section IV contains a short review of what is known on the critical theory of the quantum Hall transition. We review how information on critical parameters can be obtained from the dimensional crossover analysis of the conformal theory. There, an account is given on how the edge states do affect the critical behavior. The possible localization transition of these edges states is discussed.

In the following section the crossover between one-dimensional and two-dimensional localization is studied. In Sec. III the localization length is derived as a function of magnetic field. In Sec. IV, the scaling theory of the integer quantum Hall effect is considered. It is shown that the irrelevant scaling parameter and the scaling function are in qualitative agreement with those obtained from the noncritical theory. Furthermore it is shown that in higher Landau bands the transition between quantized Hall plateaus is driven by the dimensional crossover of the noncritical localization length. In the last section, a summary is given, and implications for experimental investigations of quantum wires are pointed out. The derivations are given in Appendixes A (orthogonal localization length as a function of wire width L_y), B (unitary localization as a function of wire width L_y), and C [orthogonal to unitary crossover of two-dimensonal (2D) localization length]. In Appendix D a generalized derivation of the field theory including an edge state potential and the topological term in the presence of a magnetic field is given.

II. DIMENSIONAL CROSSOVER OF LOCALIZATION

In the following, we study the localization length ξ of electrons confined to a disordered wire of finite width L_y.

Orthogonal regime. First, let us consider the problem without magnetic field, $B=0$, the orthogonal regime.

An estimate of the localization length ξ can be obtained by performing a perturbative renormalization of the dimensionless conductance g, which appears as the coupling constant of the action in the nonperturbative theory of disordered electrons. The bare conductance g is obtained in self-consistent Born approximation.[14] The vanishing of the renormalized conductance $\tilde{g}\to 0$ as one increases the wavelength of renormalization signals the localization, and can be used to obtain an estimate of the localization length ξ, as done in Appendix A.

The first order in $1/g$ of the perturbative renormalization corresponds to the weak localization correction to the conductivity. Thus, one can estimate the localization length ξ at zero magnetic field $B=0$ as done in Appendix A. There, the renormalization is performed for arbitrary finite widths L_y of the wire. One thus gets the following equation for the localization length ξ:

$$\xi = gL_y - \frac{L_y}{\pi}\ln\left[2\frac{k_0 L_y/(2\pi)}{1+\sqrt{1+[L_y/(2\pi\xi)]^2}}\right], \tag{3}$$

where $k_0 = 2\pi/l = \pi k_F/g$, and we assumed that the wire width is diffusive, $L_y > l$.

In the quasi-1D limit, $\xi \gg L_y$, we find a logarithmic correction to the expected quasi-1- D result, $\xi = gL_y$:

$$\xi = gL_y - \frac{L_y}{\pi}\ln(k_F L_y/g). \tag{4}$$

In the opposite limit, $\xi \ll L_y$, the nonlinear equation for the localization length simplifies to

$$\xi = l\exp(\pi g)\exp\left(-\pi\frac{\xi}{L_y}\right). \tag{5}$$

The solution of this equation can be written in closed form in terms of the Lambert-W-function $W_0(z)$:[22]

$$\xi = \frac{L_y}{\pi}W_0\left(\frac{2\pi}{L_y}\frac{g}{k_F}\exp(\pi g)\right), \tag{6}$$

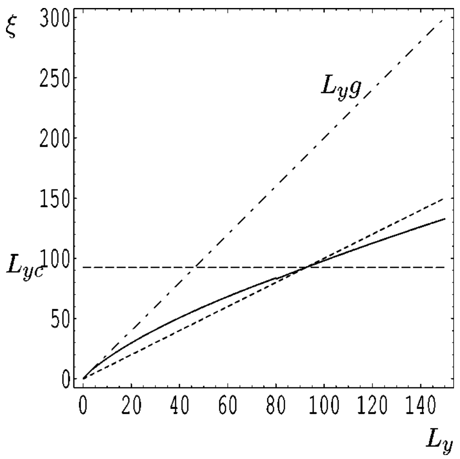

FIG. 1. The width dependence of the localization length ξ at fixed conductance $g=2$ without magnetic field, $B=0$, is shown as the full line. For comparison, the quasi-one-dimensional limit $L_y g$ is drawn as the dashed-dotted line. The crossover occurs at $\xi = L_{yc} = \exp(-\pi)(2g/k_F)\exp(\pi g)$.

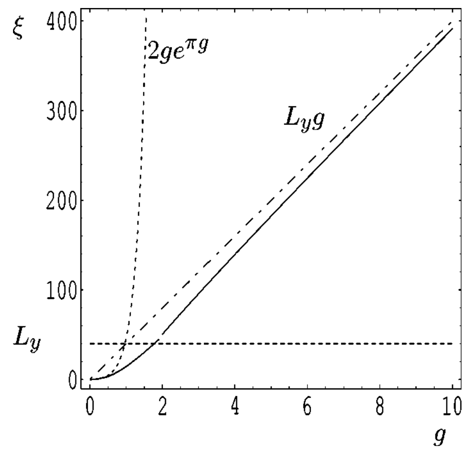

FIG. 2. The localization length ξ (full line) in units of k_F^{-1} as a function of the conductance g at fixed width $L_y=40$ and without magnetic field, $B=0$. It is plotted using Eq. (6) for $\xi<L_y$, and Eq. (4) for $\xi>L_y$. For comparison, we show the quasi-two-dimensional $\xi_{2D}=(2g/k_F)\exp(\pi g)$ (dashed line) and the quasi-one-dimensional $L_y g$ (dashed-dotted line) limiting functions.

where we substituted $l=2g/k_F$. The Lambert-W-function $W_0(x)$ is defined as the solution of the nonlinear equation[22]

$$z=a\exp(-bz), \tag{7}$$

given by

$$z=\frac{1}{b}W_0(ab). \tag{8}$$

Thus, the localization length is found to increase linearly with the width as

$$\xi|_{\xi\gg L_y}=gL_y, \tag{9}$$

when the localization length exceeds the width L_y, $\xi\gg L_y$. It logarithmically deviates from this behavior when the width L_y is on the order of $L_{yc}=\exp(-\pi)l\exp(\pi g)$. For larger widths, it slowly saturates towards the width independent 2D-localization length

$$\xi|_{\xi\ll L_y}=\frac{2g}{k_F}\exp(\pi g), \tag{10}$$

as seen in Fig. 1.

For intermediate widths there is a wide regime where the localization lengths deviate strongly from both 1D, Eq. (9), and 2D, Eq. (10), which would yield $\xi_{2D}(g=2)=2142$ on the scale $1/k_F$, used in Fig. 1.

As one fixes the width L_y and increases the dimensionless conductance g, a crossover from 2D to 1D localization is observed, as shown in Fig. 2. There, Eq. (6) is plotted as a function of g, for $\xi<L_y$. For $\xi>L_y$, the localization length is plotted using Eq. (4). In the intermediate regime, the solution of the general equation, Eq. (3), deviates only little from these asymptotic solutions. Note that the validity of the derivation is limited to $g>1$, while the deviations from 2D localization behavior occur for the chosen width L_y already close to $g=1$. The derivation given above has been done for wires of diffusive width, $l<L_y$, or $g<L_y k_F/2$, corresponding to $g<20$ in Fig. 2.

Unitary regime. At moderately strong magnetic field, the time-reversal symmetry is broken, and the so-called unitary regime is reached, when the localization length exceeds the magnetic diffusion length,[23] $\xi>L_B$ [where $L_B=l_B$, when $l_B<L_y$ and $L_B=(3)^{1/2}l_B^2/L_y$, when $l_B>L_y>l$]. Then, the first-order, weak localization correction vanishes and one needs to do the perturbative renormalization to second order in $1/g$. Thus, we have to calculate all diagrams contributing to this order, the so-called Hikami boxes,[12] to study the dimensional crossover in a magnetic field. An efficient way to do this is to start from the supersymmetric nonlinear sigma model and do an expansion around its classical point, as done in Ref. 16 for the pure orthogonal and unitary regimes in two dimensions.

Performing this renormalization for wires of finite width L_y in the unitary regime, we obtain in Appendix B that the localization length ξ satisfies in the unitary regime the equation,

$$\xi^2=L_y^2\left(\Sigma^2(4g^2-1)+\frac{2}{\pi^2}\ln\left[\frac{1+[\xi/(2\pi L_y)]^2}{1+(\xi k_0)^2}\right]\right). \tag{11}$$

Here, the length scale $2\pi/k_0$ is the short-distance cutoff of the nonperturbative theory, being $l=2g(B=0)/k_F$ at moderate magnetic fields, when $\omega_c\tau<1$, and crossing over to the cyclotron length l_{cyc} at strong magnetic fields when $\omega_c\tau>1$.

Here, $\Sigma=1$, unless the spin degeneracy is broken, and the energy levels are mixed by spin flip as by scattering from magnetic impurities, then $\Sigma=2$. Note that the above results confirm the result for the localization length in the quasi-1D limit,[17,18]

$$\xi|_{\xi\gg L_y}=\Sigma\beta gL_y, \tag{12}$$

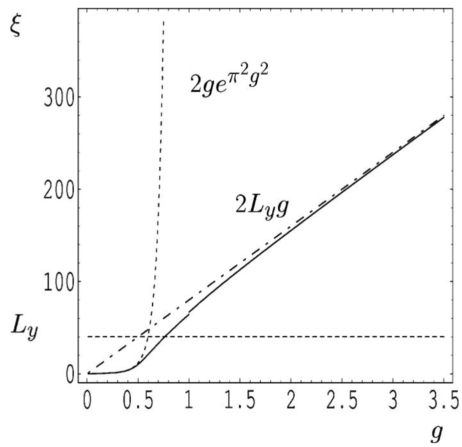

FIG. 3. The localization length ξ as a function of conductance g for large enough magnetic fields such that $\xi > L_B$ at fixed width L_y. For comparison, the quasi-two-dimensional behavior $\xi_{2D} = 2(g/k_F)\exp(\pi^2 g^2)$ (dotted line) and the quasi-one-dimensional limit $L_y g$ (dashed-dotted line). The width L_y is indicated by the dashed line.

where $\beta = 1,2$ with, without time-reversal symmetry. We will set $\Sigma = 1$ in the following. We can solve Eq. (11) in two limiting cases. When $L_y \ll \xi$, the quasi-one-dimensional localization length is obtained with a logarithmic correction,

$$\xi|_{\xi \gg L_y} = 2L_y g \left[1 - \frac{1}{\pi^2 g^2} \ln \sqrt{1 + [L_y k_0/(2\pi)]^2} \right]^{1/2}. \tag{13}$$

In the limit of two-dimensional localization, $\xi \ll L_y$, the localization length is found to satisfy the equation

$$\xi|_{\xi \ll L_y} = \frac{2\pi}{k_0} \exp\left[\frac{\pi^2}{4} \left(4g^2 - \frac{\xi^2}{L_y^2} \right) \right]. \tag{14}$$

Its solution can be written in terms of the Lambert-W-function[22] as

$$\xi = \frac{\sqrt{2} L_y}{\pi} W_0^{1/2} \left[\frac{2\pi^4}{k_0^2 L_y^2} \exp(2\pi^2 g^2) \right]. \tag{15}$$

Fixing the width L_y the dimensional crossover is seen in Fig. 3, where the localization length is plotted as a function of the dimensionless conductance g, using Eq. (13) for $\xi > L_y$ and Eq. (15) for $\xi < L_y$.

III. THE MAGNETIC-FIELD DEPENDENCE OF THE LOCALIZATION LENGTH

Weak magnetic field. In disordered quantum wires without strong spin-orbit scattering or magnetic impurities, the electron localization length is enhanced by a weak magnetic field. If the localization is quasi-one-dimensional, $\xi > L_y$, then the magnetic field results in a doubling, Eq. (12), $\xi = \beta g L_y$, where $\beta = 1,2$, corresponding to no magnetic field and finite magnetic field, respectively.[17,18] Recently, such a doubling of the localization length was observed experimentally.[24] That doubling is governed by the magnetic diffusion length $L_B = (D\tau_B)^{1/2}$, where τ_B is the magnetic phase shifting time, which is a function of magnetic length l_B, mean free path l and width of the wire L_y.[23] Thus, the localization length crosses over to $\xi = 2gL_y$, when the magnetic diffusion length becomes smaller than the localization length, $L_B < \xi$.

This crossover has recently been derived analytically with a nonperturbative method, based on the exact solution of the transfer matrix equation of the one-dimensional NLSM.[19,23] The quasi-1D localization length, valid for $\xi > L_y$, is obtained to be given by

$$\xi(B) = 2f(L_B/\xi(0)) g L_y. \tag{16}$$

Here,

$$f(x) = 2/(2 + \sqrt{49 + 64X^2} - \sqrt{25 + 64X^2}). \tag{17}$$

For a wire of diffusive width $W > l$, one finds $X = \xi(0)/L_B = \pi^2 3^{-1/2} N^2 b$, where $N_y = k_F L_y/\pi$ is the number of transverse channels and $b = \omega_c \tau = eB\tau/m$ the dimensionless magnetic-field parameter. The factor 64 was not noticed in a previous publication,[23] and is a consequence of the particular properties of the autocorrelation function of spectral determinants, and its relation to the actual localization length which was used to derive Eqs. (16) and (17).[23] When the localization is two-dimensional, the localization length ξ becomes exponentially enhanced. In 2D the transfer matrix equation cannot be solved, and we have to employ the loop expansion used in the preceding section. The crossover between the orthogonal, Eq. (10), and unitary, Eq. (1), localization lengths is governed by the parameter $X = \xi/L_B$. In previous studies[25,26] it has been argued that when integrating out modes on length scales smaller than L_B, one can use the one-loop expansion in the orthogonal regime, corresponding to vanishing magnetic field. It was argued in Ref. 26 further that for length scales larger than L_B one can use the two-loop β function in the unitary regime. Here, we note that the β function itself depends continuously on the magnetic field. Therefore we extend the perturbative renormalization by explicitly calculating the two-loop diagrams, the Hikami boxes as a function of magnetic field as outlined in Appendix B. (We note that the magnetic-field induced crossover of the weak localization corrections to the conductivity, had been considered to second order in $1/g$ also in Ref. 29, but using the complex matrix model.) One obtains the following equation for the 2D-localization length:

$$\xi = l \exp(\pi g)(1 + X^2)^{1/2 - 1/(2\pi g)}|_{X = \xi/L_B}. \tag{18}$$

Thus, when the time-reversal symmetry is broken, $\xi > L_B$, the localization is given by Eq. (13) for $\xi > L_y$ and Eq. (15) for $\xi < L_y$. The effective short-distance cutoff of the renormalization is a function of the magnetic field itself. We get $2\pi/k_0 = l$ at small magnetic fields, $b < 1$, and $2\pi/k_0 = l_{cyc}$ at large fields from the renormalization of the diffusive nonlinear sigma model. But, we should note that an analysis of the nonlinear sigma model on ballistic scales is needed to get more reliable information on $1/k_0$.

Strong magnetic field. In the following, we consider a quantum wire of disordered electrons in a strong magnetic field, which is expected to exhibit the quantum Hall effect, when its width L_y and the mean free path l do exceed the cyclotron length l_c. We consider quantum Hall wires of diffusive width, where the mean free path is smaller than the wire width, $l < L_y$. In the opposite limit, $l > L_y$, the ballistic motion between the edges of the wire leads to anomalous magnetoresistance phenomena due to classical commensurability effects of cyclotron orbits with the confinement potential of the wire.[27,28] This will not be pursued here, since these effects are not related to disorder induced quantum localization, which is the focus of this paper. $\sigma_{xx}(B)$ is the conductivity in self-consistent Born approximation. For weak magnetic field, $b<1$, it is identical to the Drude result $\sigma_{xx} = ne^2\tau/(1+b^2)$. With the electron density $n_e = Em/(2\pi\hbar^2)$ this can be rewritten as $g(b)=\sigma_{xx}/\sigma_0 = g/(1+b^2)$, where $g=E\tau/\hbar$ per spin channel. Fixing the electron density n_e, the Fermi energy depends on magnetic field $E_F(B)$ for $b>1$. In the following, we will fix the Fermi energy E_F, instead.

The conductivity in self-consistent Born approximation (SCBA) for $b>1$, when the cyclotron length l_c becomes smaller than the mean free path l, or $\omega_c > 1/\tau$, and disregarding the overlap between Landau bands, is given by

$$g(B)=\frac{1}{\pi}(2n+1)[1-(E_F-E_n)^2/\Gamma^2] \qquad (19)$$

for $|E-E_n|<\Gamma$, where $\Gamma^2=(2/\pi)\hbar^2\omega_c/\tau$ for $\Gamma<\hbar\omega_c$.

One obtains thus the localization length for $b>1$ and $|\epsilon/b-n-1/2|<1$, by substituting the expression for the dimensionless conductance, Eq. (19), into Eq. (13) for $\xi > L_y$, and Eq. (15) for $\xi < L_y$, respectively. Thus, the localization length is found to oscillate between maximal values in the middle of the Landau bands and minimal values on the order of the cyclotron length l_{cyc} in the tail of the Landau bands, as seen in Fig. 4.

The localization is quasi-one-dimensional as long as $\xi(W)>L_y$. We see that for $n>1$ this is, for uncorrelated disorder potential, practically always the case in the middle of the Landau bands, with a logarithmic correction as given by Eq. (10), yielding

$$\xi_n=\frac{2}{\pi}(2n+1)L_y\left[1-\frac{1}{(2n+1)^2}\ln\sqrt{1+\left(\frac{L_y}{l_{\mathrm{cyc}}}\right)^2}\,\right]^{1/2}. \qquad (20)$$

We note that we have assumed in the derivation of Eqs. (18) and (20) that the conductivity is homogeneous. In a strong magnetic field, the formation of edge states can result in strongly inhomogeneous and anisotropic conductivity, thereby preventing the mixing of edge states with the bulk states.[31,32] The consequences of these effects will be discussed in more detail, in the following section and in a forthcoming paper, including a numerical analysis.[20]

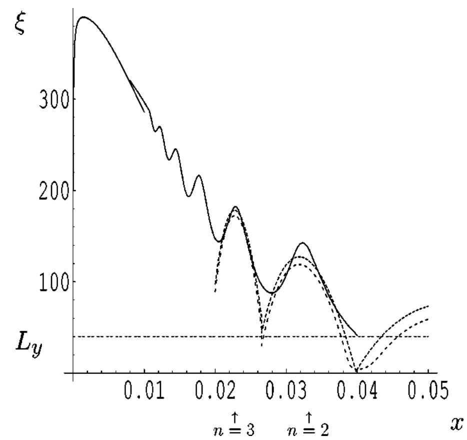

FIG. 4. The localization length as a function of magnetic flux through a unit cell of area $a^2=1/k_F^2$, $x=a^2/2\pi l_B^2$, with conductance parameter $g(B=0)=10$. For weak magnetic field, Eq. (16) is used. For strong magnetic fields the localization length is plotted by inserting $g(B)$ in the second-order Born approximation in the formula for the quasi-one-dimensional localization length, Eq. (2), including a summation over all Landau levels (full line). The short dashed line is obtained by using the self-consistent Born approximation (SCBA) for $g(B)$ for one Landau band, Eq. (20), and inserting it into the formula for the quasi-one-dimensional localization length, Eq. (2). The long dashed curve denotes the corresponding result using the SCBA conductance and inserting it in the crossover formula, Eq. (15). The width of the wire $L_y = 40a$ is indicated as the horizontal dashed line.

IV. THE QUANTUM HALL TRANSITION

In the above analysis of a disordered wire in a magnetic field we disregarded the effect of the topological term, which appears in a derivation of the nonlinear sigma model,[30,34,35] see Eq. (D13). It is known that in the two-dimensional limit this topological term is needed in order that the field theory becomes critical in the middle of Landau bands, and the quantum-Hall-transition from localized states in the Landau band tails to a critical state in the middle of the Landau bands can be described.[30,34–38]

Both in numerical calculations[7] and experiments one needs to perform a finite-size scaling analysis in order to extract the critical divergence of the localization length, $\xi \sim (E-E_n)^{-\nu}$, when approaching the middle of a Landau band, E_n. The procedure is to find numerically the scaling function $\Lambda=\xi/L_y=\Lambda(L_y/\tilde{\xi}(E))$, rescaling with the critical localization length, which diverges according to $\tilde{\xi}(E)\sim(E-E_n)^{-\nu}$ and does not depend on width L_y. Then, one can determine ν by optimizing the accuracy of scaling. The scaling function is not known *a priori*. It is clear that $\Lambda(x)\to 1/x$ for $x\gg 1$, since in the tails of the Landau band $\xi \ll L_y$, and ξ becomes independent of L_y, approaching $\tilde{\xi}$.

For the higher Landau bands, $n>0$, the single-parameter scaling is not accurate, and it is important to include irrelevant scaling parameters,[39] apart from the relevant parameter, $L_y/\tilde{\xi}(E)$.[7,40] So far the irrelevant scaling parameters have been included in the numerical scaling analysis on a phenomenological ground, without a precise knowledge of their physical origin. It has been observed, however, that the irrelevant scaling length increases by several orders of magnitude in higher Landau bands for uncorrelated disorder.[7]

Therefore, it seems worthwhile to first analyze, if the noncritical width dependence of $\xi(B)$, derived above, Eq. (3), can yield analytical knowledge on the scaling function $\Lambda(x)$ and moreover account for the observed irrelevant scaling parameter in higher Landau bands.

According to Eq. (23), the ratio Λ scales with the large length scale ξ_{2Dunit}, which is a huge length scale in the middle of higher Landau bands, where $g \gg 1$. Therefore, it is natural to expect that ξ_{unit} can be identified with the irrelevant length scale l_{irr}, and to compare the scaling function, Eqs. (22) and (23), with the one obtained numerically, Eq. (24). Furthermore, the noncritical localization length as a function of magnetic field, $\xi(B)$ and the respective scaling function may dominate the transition between Hall plateaus in quantum wires of finite width. Therefore we consider first the noncritical scaling function in the following section.

A. The noncritical quantum Hall transition

Let us rewrite the equations for the localization length in the unitary regime, Eqs. (13) and (15), as a function of $x = L_y/\xi_{2Dunit}$, where ξ_{2Dunit} is the 2D limiting value of the unitary localization length,

$$\xi_{2Dunit} = \frac{2\pi}{k_0}\exp(\pi^2 g^2). \tag{21}$$

Thus, Eqs. (13) and (15) become

$$\Lambda = \frac{2}{\pi}[-\ln\sqrt{x^2+(2\pi/k_0\xi_{2Dunit})^2}]^{1/2} \tag{22}$$

for $x = L_y/\xi_{2Dunit} < \exp(-\pi^2/4) \approx 0.085$, while

$$\Lambda = \frac{\sqrt{2}}{\pi}\sqrt{W_0\left(\frac{\pi^2}{2}\frac{1}{x^2}\right)}, \tag{23}$$

for $x = L_y/\xi_{2Dunit} > \exp(-\pi^2/4)$.

This noncritical scaling function is plotted in Fig. 5, where $g = 7/\pi$ has been chosen, corresponding to the unrenormalized conductance in the Landau band, $n=3$. We note that the derivation is only valid for $g \gg 1$, so that we are able to compare this function only with the scaling function in higher Landau bands, where $g_n \gg 1$. It is expected that this noncritical scaling function is accurate as long as $x = L_y/\xi_{2Dunit} < 1$.

Close to the critical point, $x = L_y/\xi_{2Dunit} \gg 1$, the scaling function is from the numerical analysis obtained to be in the middle of the Landau band,

$$\Lambda - \Lambda_c = \frac{\xi}{L_y} = c\left(\frac{L_y}{l_{irr}}\right)^{-\gamma}, \tag{24}$$

where $\Lambda_c = 1.2$, and the irrelevant critical scaling exponent is numerically found to be $\gamma = 0.38 \pm 0.04$, and c is a constant.

In Fig. 6, we plot $\Lambda - \Lambda_c$, Eq. (24) using $\gamma = 0.3$ and $c = 0.5$, and compare it with Λ as obtained from the result of

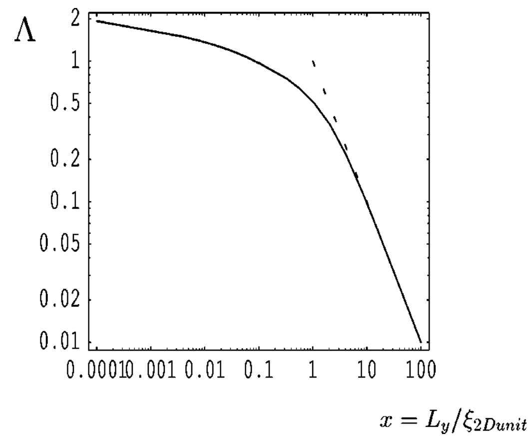

FIG. 5. The noncritical scaling function obtained analytically (full line), Eqs. (22) and (23), for the Landau band, $n=3$, with $g_n = 7/\pi$. The dashed line, $1/x$, is approached in the 2D limit, when the localization length becomes equal to ξ_{2Dunit}.

the noncritical analysis, Eq. (23). Note that this function converges to zero as $L_y \to \infty$, corresponding to $\Lambda_{crit} = 0$, since it was obtained from the noncritical field theory, disregarding the topological term. Thus, as expected, the form of the noncritical irrelevant scaling function defers from the results of the numerical analysis as seen in Fig. 6, but is in some quantitative agrement. In spite of this, it is expected that the scaling function itself is changed by the presence of the critical point in the middle of the Landau band, and the similarity to the noncritical scaling function derived above is only of qualitative nature. Since in higher Landau levels one is in the study of wires of finite width L_y for uncorrelated disorder always far away from the critical point, however, we conclude that this noncritical scaling function is important in order to enable one to analyze the quantum Hall transition in higher Landau bands, $n > 1$. We can estimate the region of criticality by the condition that $\tilde{\xi}(E)/\xi_{2Dunit} > 1$. Thereby we find for the interval of criticality around E_n

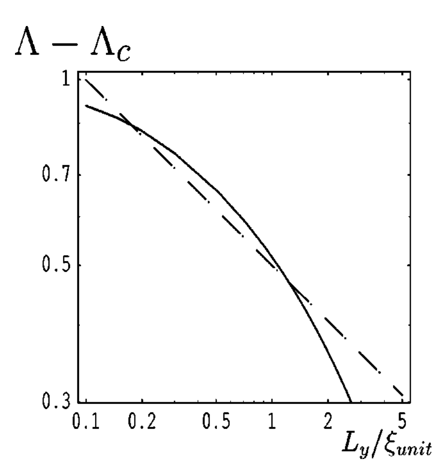

FIG. 6. The irrelevant scaling function obtained analytically for the Landau band $n=1$, $g = 3/\pi$ (full line) as compared with the function, Eq. (24) (dashed line) for $\gamma = 0.3$ and $c = 0.5$, double logarithmic scale.

$$\Delta E = \Gamma \exp\left(-\frac{(2n+1)^2}{\nu}\right), \tag{25}$$

which yields $\Delta E/\Gamma = 0.65, 0.02, 2\times10^{-5}, \ldots,$ for $n = 0,1,2, \ldots$. Thus, we conclude that criticality can for uncorrelated disorder only be observed in the lowest two Landau levels. Since the width of the quantum Hall plateaus is determined by the condition $\xi(E) = L_y$, there are nevertheless wide plateaus between higher Landau bands, and we conclude that criticality is not essential to observe the quantum Hall effect. Indeed, the transition between Hall plateaus in higher Landau bands is driven solely by the noncritical dimensional crossover of the localization length.

B. Towards the theory of the critical quantum Hall transition

Next, let us consider the effect of the topological term in the derivation of the scaling function. At small length scales, in high Landau bands, the dimensionless conductance σ_{xx} is large, and the instanton approximation can be used. To this end, one finds solutions which minimize the action of the NLSM, Eq. (D13),

$$F = \frac{h}{16e^2}\int d\mathbf{x} \sum_{i=x,y} \sigma(\omega=0)_{ii}(\mathbf{x}) S\,\mathrm{Tr}([\boldsymbol{\nabla}_i Q(\mathbf{x})]^2)$$
$$-\frac{1}{16}\frac{h}{e^2}\int d\mathbf{x}\sigma(\omega=0)_{xy}(\mathbf{x}) S\,\mathrm{Tr}(Q\partial_x Q\partial_y Q). \tag{26}$$

Here $\sigma(\omega=0)_{xy} = \sigma^I(\omega=0)_{xy}(\mathbf{x}) + \sigma^{II}(\omega=0)_{xy}$ where $\sigma^I(\omega=0)_{xy}$ is the dissipative part of the Hall conductivity,[14,34,33] and $\sigma^{II}(\omega=0)_{xy} = -e\, dn/dB$, n is the particle density, which yields a finite contribution at the boundary of the wire in the presence of a confinement potential, from the edge states.[41,33,16]

Disregarding the spatial variation of the coupling functions $\sigma_{ij}(\mathbf{x})$ in Eq. (26), and assuming isotropy, one finds in the two-dimensional limit that there are instantons with nonzero topological charge q, which are identical to the skyrmions of the compact $O(3)$ NLSM, as obtained form the compact part of the supersymmetric NLSM.[34,16] Their action is given by

$$F_q = 2\pi|q|\sigma_{xx} + 2\pi i q \sigma_{xy}, \tag{27}$$

where $\sigma_{xx} = \sigma_{yy}$ and σ_{xy} are the spatially averaged conductivities. Now, we can repeat the derivation of the scaling function by integrating out Gaussian fluctuations around these instantons. It is clear, however, that the contribution from instantons with $q \neq 0$ is negligible, as long as $\sigma_{xx} > 1$. Within the validity of the $1/g$ expansion one does not find a sizable influence of the topological term on the scaling function $\Lambda = \xi/Ł_y$. Still, the tendency is seen that at $\sigma_{xy} = 1/2$ the renormalization of the longitudinal conductance is slowed down and one may conclude from this observation the two-parameter scaling diagram with a critical state of finite conductance $0 < \sigma^* < 1$.[42,34] Furthermore, it is seen explicitly that in order that the instanton solutions with nonzero topological charge do exist the system must exceed the noncritical localization length ξ_{2Dunit}, when the assumption of uniform coupling paramters σ_{ij} is made.

Taking into account the spatial variation of $\sigma_{ij}(\mathbf{x})$, there are extended regions where $\sigma_{yy}(\mathbf{x}) \to 0$, indicating the decoupling of the edge states from the bulk states.[31] Thus, the free energy for spatial variations of Q is reduced in these regions. Thereby, one can find instantons with nonzero topological charge q, whose spatial variations are restricted to these edge regions with vanishing real part of the free energy: $F_{q\ \mathrm{edge}} = i2\pi\sigma_{xy}(\mathrm{edge})$, where $\sigma_{xy}(\mathrm{edge})$ is the Hall conductance of the edge states, which is quantized to integer values. Thus, we conclude that the renormalization and thereby the scaling function of the bulk, $\Lambda = \xi/L_y$, is not influenced noticeably by the presence of the edge states for $g > 1$.

Closer to the critical point, the NLSM, Eq. (26), cannot be used to derive further information, since that theory flows to strong coupling, $g < 1$. It has been established numerically that the quantum Hall criticality is not sensitive to the type of disorder. This observation found further support by the proof that the Hamiltonian of a chain of antiferromagnetically interacting superspins can be derived both from the nonlinear sigma model for short-ranged disorder at the critical point $\sigma_{xy} = 1/2$,[43] and from the Chalker–Coddington model,[44] which is the reduced version of the quantum percolating network model of unidirectional (chiral) drifting modes along equipotential lines of a slowly varying disorder potential.[45] It has been shown by numerical solution of a finite number of antiferromagnetically coupled super spins that this theory is critical. So far, no analytical information has been obtained for the critical parameters, such as the localization exponent ν and the critical value Λ_c. However, building on this model of a superspin chain, supersymmetric conformal field theories have been suggested, which ultimately should yield the critical parameters of the quantum Hall transition.[46–48] The critical value of the scaling function Λ_c has been related to the free parameters of a class of conformal field theories.[47] Restricting this theory to quasi-1D,[49] by choosing a finite width L_y, on the order of ξ_{2Dunit}, which serves as the ultraviolet cutoff of the conformal field theory, one finds that the critical value of the scaling function Λ_c is inversely proportional to the gap between the lowest two eigenvalues of the Laplace-Beltrami operator of this reduced class of supersymmetric conformal field theories. Furthermore, it was concluded that the critical wave-function amplitudes are widely, namely, lognormally distributed, corresponding to a parabolic distribution of multicritical exponents around a value α_0 which was argued to be related to Λ_c as $\Lambda_c = 1/\pi(\alpha_0 - 2)$.[47] These assertions have found precise numerical confirmation.[50,51] Thus, it seems that the critical value Λ_c of the scaling function at the critical point can be obtained from the dimensional crossover of the conformal field theory. So far, the critical exponent of the localization length ν could only be derived for special classes of systems showing a type of quantum-Hall transition.[52] For the critical point of the integer quantum-Hall transition, however, ν has not been derived analytically from the conformal field theory, nor from the theory of superspin chains. Therefore, it is so far not possible to give an analytical derivation of the scaling function close to criticality.

V. CONCLUSIONS

In disordered quantum wires the electrons are localized due to quantum interference along the wire with a localization length which scales linearly with the wire width, as long as the electrons can diffuse freely across the wire width. For wires, which would classically be good metals, as characterized by a large dimensionless conductance $g=k_F l \gg 1$, the 2D quantum localization limit is never reached, but rather a slow crossover between quasi-1D and -2D localization occurs as a function of the wire width. Therefore, we think that the crossover function derived here can be relevant for the study of strong localization in weak magnetic fields in disordered quantum wires. These have been studied mainly by means of activated transport measurements.[24] Recently, the scanning of the local density of states has become possible, by means of the scanning tunneling microscopy,[53] which has a resolution corresponding to few eigenstates. Thus this will allow us to study the magnetic-field dependence of the localization length most directly. Furthermore, low-temperature capacitance measurements would yield the localization length directly. Since insulators are dielectrics, with dipole moments proportional to their localization length, the metallic divergence of the dielectrical constant is cutoff, $\epsilon(q \rightarrow 0) \sim \xi^2$. In general, for an insulator one obtains for $T \ll \Delta_c = 1/\xi^d \nu_d$,[54]

$$\epsilon(q \rightarrow 0, \omega=0) = 4\pi e^2 \frac{dn}{d\mu} \xi^2. \tag{28}$$

Thus, the measurement of the dielectrical constant has been used to study the metal–insulator transition,[55] where the localization length and thus the dielectrical constant is diverging.[56] For a quasi-one-dimensional wire one obtains

$$\epsilon(q \rightarrow 0, \omega=0) = 32\zeta(3) e^2 \nu_d \xi^2, \tag{29}$$

where ζ is the Riemann zeta-function.[17] Measuring the magnetocapacitance, $C(B) = \epsilon_0 \epsilon(B) S/L$, where S is the cross section and L the length of the wire, one would expect an enhancement of the dielectrical constant $\epsilon(B)$ by a factor 4 as the magnetic field is turned on. To our knowledge this positive magnetocapacitance in a wire has not yet been experimentally observed, and would be a means to study the dimensional crossover of localization directly.

In a strong magnetic field, the kinetic energy is quenched, resulting in enhanced localization. While in the tails of the Landau bands the localization length is small, on the order of the cyclotron length, it increases towards the center of the Landau bands, due to an increased classical conductance. For wires of finite width, this results in a dimensional crossover of localization form two- to one- dimensional behavior. The noncritical crossover function derived above is relevant for localization in higher Landau bands, where the noncritical 2D-localization length is exponentially large, dominating its behavior since the critical point in the middle of the Landau band becomes relevant only in the 2D limit. Thus, we conclude that the transition between quantized Hall plateaus in higher Landau bands is due to the noncritical dimensional crossover of the localization length derived above.

In the tails of the Landau bands, extended edge states exist due to the edge confinement potential of the wires, which can carry a quantized Hall current. When the dimensional crossover of the localization of the bulk states occurs, the edge states are expected to mix and become localized along the wire. In order to study this localization transition, the edge states have to be taken into account explicitly, by accounting for a strongly inhomogeneous and anisotropic conductivity. The ballistic length scales of the edge states exceeding the elastic mean free path in the bulk do have to be taken into account explicitly, in the derivation of the field theory of localization, as outlined in Appendix D. A full analysis of this theory, including the edge states and the topological term, in deriving dimensional crossover of localization remains to be done, as well as a numerical analysis of the metal-insulator transition of the edge states in quantum Hall wires.[20]

ACKNOWLEDGMENTS

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APPENDIX A:

Information on the dimensional crossover in a wire of finite width L_y can be obtained from the renormalization of the action of the nonperturbative theory of disordered electrons, the nonlinear sigma model, Eq. (A1).[11]

First, let us consider the problem without magnetic field, $B=0$. The coupling parameter is the conductance per spin channel, $g=\sigma_{xx}/\sigma_0$ in the action for $B=0$, which is given by

$$F = \frac{g}{16} \int d\mathbf{x} S \, \mathrm{Tr}([\nabla Q(\mathbf{x})]^2). \tag{A1}$$

Going to momentum representation, one performs successive integration over modes with momenta within the interval $k_0/b^l < |\mathbf{k}| < k_0/b^{l-1}$, where $k_0 \sim 1/l$ is the high momentum cutoff of the diffusive NLSM, Eq. (A1). $b>1$ is the renormalization parameter. Rescaling the coupling parameter g after each renormalization step l, integer, one obtains in one-loop approximation

$$g \rightarrow \tilde{g} = g\left(1 - \frac{2}{g} \int_{0<|\mathbf{k}|<\mathbf{k}_0} \frac{d\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2 + \lambda^2}\right), \tag{A2}$$

where λ is the low momentum cutoff. The first-order term in the perturbative renormalization in $1/g$ corresponds to the weak localization correction to the conductivity. One can es-

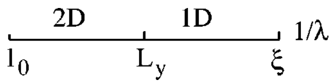

FIG. 7. Crossover in dimensionality as momentum λ of renormalization is changed.

timate the localization length ξ by the fact that the conductivity of a wire of length ξ is unity, $\tilde{g}\to 1$, when $\lambda=1/\xi$. Noting that

$$\int_{0<|\mathbf{k}|<k_0}\frac{d\mathbf{k}}{(2\pi)^2}\to\frac{1}{L_y}\sum_{n_y}\int\frac{dk_x}{2\pi},$$

for a wire of finite width L_y, with $k_y=2\pi n_y/L_y$, where n_y is an integer, one finds that the localization length in a wire of finite width L_y satisfies the equation

$$\xi=gW-\frac{2}{\pi^2}L_y\sum_{n=1}^{N_0}\frac{1}{\sqrt{n^2+[N_0/(k_0\xi)]^2}}\times\arctan\left(\frac{N_0}{\sqrt{n^2+[N_0/(k_0\xi)]^2}}\right), \tag{A3}$$

where $N_0=k_0L_y/(2\pi)$. For $N_0\gg 1$ this equation can be approximated by Eq. (3).

APPENDIX B:

In a finite magnetic field, the first order in $1/g$ correction to the conductance is vanishing. An efficient way to do the perturbative renormalization to second order in $1/g$ is to start from the supersymmetric nonlinear sigma model and do an expansion around its classical point, as done in Ref. 16 for the pure unitary limit. Here we extend this derivation taking into account the finite wire width L_y. We note that the dimensionality changes as one integrates out the *Q modes* from large momenta, corresponding to the smallest length scales, which is l_0 in the unitary limit, to the largest length scale, which is the localization length ξ, see Fig. 7.

Integrating the renormalization flow from the smallest to the largest length scale, one finds for $\xi>L_y$:

$$4\int_{g(l_0)}^{g(\xi)}dg\,g=\left[16\lim_{\delta\to 0}\left(\frac{1}{2}-\frac{1}{2-\delta}\right)(I_{2DL_y}^2-l_{2Dl}^2)+16\left(\frac{1}{2}-1\right)(I_{1D\xi}^2-I_{1Dl}^2)\right]. \tag{B1}$$

Here,

$$I_{2Dx}=\int_{1/x<k<\infty}\frac{d\mathbf{k}}{(2\pi)^2}\frac{1}{k^2+\lambda^2}, \tag{B2}$$

and

$$I_{1Dx}=\int_{1/x<k_x<\infty}\frac{dk_x}{\pi L_y}\frac{1}{k_x^2+\lambda^2}=\frac{1}{\pi L_y\lambda}\left(\frac{\pi}{2}-\arctan\frac{1}{x\lambda}\right). \tag{B3}$$

Clearly, we cannot simply set $\delta=0$ in the first term of Eq. (B1), because of the logarithmic divergency of the integral I_{L_y,l_0}. Going to dimension $d=2-\delta$, and taking the limit $\delta\to 0$, the expression, one has to evaluate, is given by

$$K=\lim_{\delta\to 0}\left(\frac{1}{2}-\frac{1}{2-\delta}\right)\left(\int\frac{d\Omega_{2-\delta}}{(2\pi)^{2-\delta}}\right)^2\left(\int dk\frac{k^{1-\delta}}{k^2+\lambda^2}\right)^2, \tag{B4}$$

where $\int d\Omega_{2-\delta}$ is the angular integral in $2-\delta$ dimensions. By performing an analytical continuation,

$$\int_0^\infty dk\frac{k^{1-\delta}}{k^2+\lambda^2}=\frac{\pi i}{(\lambda)^\delta}\frac{1}{1-\exp(-2\pi i\delta)}, \tag{B5}$$

and using that $\lim_{\delta\to 0}[-(1/\delta)k^{-\delta}]=\ln k$, one finds

$$K=\frac{1}{8\pi^2}\ln\lambda. \tag{B6}$$

Thus, setting $\lambda=1/\xi$, we get that the localization length satisfies Eq. (11),

$$\xi^2=L_y^2\left(4(g^2-1)+\frac{2}{\pi^2}\ln\left[\frac{1+(2\pi\xi/L_y)^2}{(2\pi\xi/L_y)^2+(\xi k_0)^2}\right]\right). \tag{B7}$$

APPENDIX C:

Here, we extend the derivation of the localization length in the 2 D limit to the crossover in a magnetic field between the orthogonal and unitary limits.

One obtains in two-loop approxmiation,

$$\tilde{g}=g\left\{1-\frac{4}{g}\int_{0<k<k_0}\frac{d\mathbf{k}}{(2\pi)^2}\frac{1}{k^2+\lambda^2+1/D\tau_B}+\frac{16}{g^2}\left(\frac{1}{2}-\frac{1}{d}\right)\times\left[\left(\int_{0<|\mathbf{k}|<k_0}\frac{d\mathbf{k}}{(2\pi)^2}\frac{1}{k^2+\lambda^2}\right)^2\times\left(\int_{0<k<k_0}\frac{d\mathbf{k}}{(2\pi)^2}\frac{1}{k^2+\lambda^2+1/D\tau_B}\right)^2\right]\right\}, \tag{C1}$$

where λ is the low momentum cutoff. Setting the lower momentum cutoff equal to the inverse localization length, $\lambda=1/\xi$, we find

$$\tilde{g}=g\left\{1-\frac{2}{g\pi}\left[\ln\left(\frac{\xi}{l}\right)-\ln\sqrt{1+\frac{\xi^2}{D\tau_B}}\right]-\frac{1}{\pi^2g^2}\ln\sqrt{1+\frac{\xi^2}{D\tau_B}}\right\}\to 1. \tag{C2}$$

Thereby one obtains the equation for the localization length in a magnetic field, Eq. (18).

APPENDIX D:

In the following we review the nonperturbative theory of a disordered quantum wire in a magnetic field. The Hamiltonian of disordered noninteracting electrons is

$$H = [\mathbf{p} - q\mathbf{A}]^2/2m + V(\mathbf{x}) + V_0(\mathbf{x}), \tag{D1}$$

where q is the electron charge. $V(\mathbf{x})$ is taken to be a Gaussian distributed random function, with a distribution function

$$P(V) = \exp\left(- \int \frac{d\mathbf{x}}{\mathcal{V}} \frac{d\mathbf{x}'}{\mathcal{V}} J(\mathbf{x}-\mathbf{x}') V(\mathbf{x}) V(\mathbf{x}') \right). \tag{D2}$$

Impurity averaging is thus given by $\langle \dots \rangle_V = \int \Pi_{\mathbf{x}} dV P(V) \cdots$. We take

$$J(\mathbf{x}-\mathbf{x}') = \mathcal{V}\Delta \hbar/\tau \delta(\mathbf{x}-\mathbf{x}')$$

for uncorrelated impurities, where $1/\tau$ is the elastic-scattering rate and $\Delta = 1/(\nu\mathcal{V})$ the mean level spacing of the mesoscopic sample with volume $\mathcal{V}$. $V_0(\mathbf{x})$ is the electrostatic confinement potential defining the width of the wire L_y. The vector potential is used in the gauge $\mathbf{A} = (-By,0,0)$, where x is the coordinate along the wire of length L, y the one in the direction perpendicular both to the wire and to the magnetic field $\mathbf{B}$, which is directed perpendicular to the wire. The electron spin degree of freedom is not considered here.

While the disorder averaged electron wave-function amplitude decays on time scales on the order of the elastic-scattering time τ, information on quantum localization is contained in the impurity averaged evolution of the electron density $n(\mathbf{x},t) = \langle |\psi(\mathbf{x},t)|^2 \rangle$. Thus, nonperturbative averaging of products of retarded and advanced propagators, $\langle G^R(E) G^A(E') \rangle$ has to be performed to obtain information on quantum localization.

In useful analogy to the study of spin systems, the supersymmetry method contracts the information on localization into a theory of Goldstone modes Q, arising from the global symmetry of rotations between the retarded propagator (‘‘spin up’’) and the advanced propagator (‘‘spin down’’) in a representation of superfields (composed of scalar and Grassmann components). Spatial fluctuations of these modes contribute to the partition function

$$Z = \int \prod dQ_{4\times 4}(\mathbf{x}) \exp(-F[Q]), \tag{D3}$$

and are governed by the action

$$F[Q] = \frac{\pi}{4} \frac{\hbar}{\Delta \tau} \int \frac{d\mathbf{x}}{L_y L} \mathrm{Tr}(Q_{4\times 4}(\mathbf{x})^2) + \frac{1}{2} \int d\mathbf{x} \langle \mathbf{x} | \mathrm{Tr} \ln[G(\hat{x},\hat{p})] | \mathbf{x} \rangle, \tag{D4}$$

where

$$G(\hat{x},\hat{p}) = 1 \Big/ \left(\frac{1}{2}\omega\Lambda_3 - \frac{(\hat{p}-qA)^2}{2m} - V_0(\hat{x}) + i\frac{\hbar}{2\tau} Q_{4\times 4}(\hat{x}) \right). \tag{D5}$$

To summarize the notation, here, and in the following, Λ_i are the Pauli matrices in the subbasis of the retarded and advanced propagators. We used the notation $\hat{x}$, in order to stress that it is an operator and does not commute with the kinetic-energy term $H_0 = (\hat{p} - qA)^2/2m$. Here, $\omega = E - E'$ breaks the symmetry between the retarded and advanced sectors. The long-wavelength modes of Q do contain the nonperturbative information on the diffusion and cooperon modes, and thus on localization.

In order to consider the action of these long-wavelength modes governing the physics of diffusion and localization, one can now expand around the saddle-point solution of the action of Q, $\delta F = 0$, satisfying for $\omega = 0$,

$$Q = i/(\pi\nu) \langle \mathbf{x} | 1/[E - H_0 - V_0(\mathbf{x}) + i/(2\tau) Q] | \mathbf{x} \rangle. \tag{D6}$$

This saddle-point equation is found to be solved by $Q_0 = \Lambda_3 P$, which is the self-consistent Born approximation for the self-energy P. At $\omega = 0$ the rotations U, which leave Q in the supersymmetric space, yield the complete manifold of saddle-point solutions as $Q = \bar{U}\Lambda_3 P U$, where $U\bar{U} = 1$, with $Q^T C = CQ$. In general, in order to account for the ballistic motion of electrons along the edges, or to account for different sources of randomness, a directional dependence of the matrix $U = U(\mathbf{x}.\mathbf{n})$ where $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$ has to be considered.[57,58] The modes which leave Λ_3 invariant are surplus, and can be factorized out, leaving the saddle-point solutions to be elements of the semisimple supersymmetric space $Gl(2|2)/[Gl(1|1) \times Gl(1|1)]$.[59] In addition to these gapless modes there are massive longitudinal modes with $Q^2 \neq 1$, which can be integrated out,[16] and the partition function thereby reduces to a functional integral over the transverse modes U.

Now, the action of finite frequency ω and spatial fluctuations of Q around the saddle-point solution can be found by an expansion of the action F, Eq. (D4). Inserting $Q = \bar{U}\Lambda_3 P U$ into Eq. (D4), and performing the cyclic permutation of U under the trace Tr, yields,[34]

$$F = -\frac{1}{2} \int d\mathbf{x} \langle \mathbf{x} | \mathrm{Tr} \ln(G_0^{-1} - U[H_0, \bar{U}] + \omega U \Lambda \bar{U}) | \mathbf{x} \rangle, \tag{D7}$$

where

$$G_0^{-1} = E - H_0 - V_0(\mathbf{x}) + \frac{i\hbar}{2\tau} \Lambda P. \tag{D8}$$

Expansion to first order in the energy difference ω and to second order in the commutator $U[H_0, \bar{U}]$ yields

$$F[U]=-\frac{1}{2}\omega\int d\mathbf{x}\langle\mathbf{x}|\mathrm{Tr}\,G_{0E}U\Lambda\overline{U}|\mathbf{x}\rangle + \frac{1}{2}\int d\mathbf{x}\langle\mathbf{x}|\mathrm{Tr}\,G_{0E}U[H_0,\overline{U}]|\mathbf{x}\rangle + \frac{1}{4}\int d\mathbf{x}\langle\mathbf{x}|\mathrm{Tr}(G_{0E}U[H_0,\overline{U}])^2|\mathbf{x}\rangle. \quad \text{(D9)}$$

The first-order term in $U[H_0,\overline{U}]$ is proportional to the local current, and found to be finite only at the edge of the wire in a strong magnetic field, due to the chiral edge currents. It can be rewritten as

$$F_{xyII}=-\frac{1}{8}\int dx\,dy\,\frac{\sigma_{xy}^{II}(\mathbf{x})}{e^2/h}S\,\mathrm{Tr}\,Q\partial_x Q\partial_y Q, \quad \text{(D10)}$$

where the prefactor is the nondissipative term in the Hall conductivity in self-consistent Born approximation:[33]

$$\sigma_{xy}^{II}(\mathbf{x})=-\frac{1}{\pi}\frac{\hbar e^2}{m^2}\langle\mathbf{x}|(x\pi_y-y\pi_x)\mathrm{Im}\,G_E^R|\mathbf{r}\rangle. \quad \text{(D11)}$$

One can separate the physics on different length scales, noting that the physics of diffusion and localization is governed by spatial variations of U on length scales larger than the mean free path l. The smaller length scale physics is then included in the correlation function of Green's functions, being related to the conductivity by the Kubo-Greenwood formula,

$$\sigma_{\alpha\beta}(\omega,\mathbf{x})=\frac{\hbar}{\pi SL}\langle\mathbf{r}|\pi_\alpha G_{0E}^R\boldsymbol{\pi}_{\boldsymbol{\beta}}G_{0E+\omega}^A|\mathbf{r}\rangle, \quad \text{(D12)}$$

where $\boldsymbol{\pi}=(\hbar/i)\boldsymbol{\nabla}-q\mathbf{A}$. The remaining averaged correlators involve products $G_{0E}^R G_{0E+\omega}^R$ and $G_{0E}^A G_{0E+\omega}^A$ and are therefore by a factor $\hbar/(\tau E)$ smaller than the conductivity, and can be disregarded for small disorder $\hbar/\tau\ll E$. In order to insert the Kubo-Greenwood formula in the saddle-point expansion of the nonlinear sigma model, it is convenient to rewrite the propagator in F as

$$G_{0E}=\tfrac{1}{2}G_{0E}^R(1+\Lambda)+\tfrac{1}{2}G_{0E}^A(1-\Lambda).$$

Then, we can use that

$$\mathrm{Tr}\left[\sum_{\alpha=1}^{d}\sum_{s=\pm}(1+s\Lambda)U(\boldsymbol{\nabla}_\alpha\overline{U})(1-s\Lambda)U(\boldsymbol{\nabla}_\alpha\overline{U})\right] = -\mathrm{Tr}[(\boldsymbol{\nabla}Q)^2].$$

Using the Kubo formula, Eq. (D12), this functional of Q simplifies to

$$F=\frac{h}{16e^2}\int d\mathbf{x}\sum_{i=x,y}\sigma(\omega=0)_{ii}(\mathbf{x})S\,\mathrm{Tr}([\boldsymbol{\nabla}_i Q(\mathbf{x})]^2) - \frac{1}{8}\frac{h}{e^2}\int d\mathbf{x}\sigma(\omega=0)_{xy}(\mathbf{x})S\,\mathrm{Tr}(Q\partial_x Q\partial_y Q), \quad \text{(D13)}$$

where $\sigma(\omega=0)_{xy}=\sigma^I(\omega=0)_{xy}(\mathbf{x})+\sigma^{II}(\omega=0)_{xy}$ where $\sigma^I(\omega=0)_{xy}$ is the dissipative part of the Hall conductivity.[14,34,33]

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