Charged two-dimensional quantum gas in a uniform magnetic field at finite temperature

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We present closed-form, analytical expressions for the thermodynamic properties of an ideal, two-dimensional (2D) charged Fermi or Bose gas in the presence of a uniform magnetic field of arbitrary strength. We consider both the homogeneous quantum gas (in which case our expressions are *exact*) and the inhomogeneous gas within the local-density approximation. Our results for the Fermi gas are relevant to the current-density-functional theory of low-dimensional electronic systems in magnetic fields. For a 2D charged Bose gas (CBG) in a homogeneous magnetic field, we show that the uniform system undergoes a sharp transition at a critical temperature $T_c^\star$, below which there is a macroscopic occupation of the lowest Landau level. An examination of the one-body density matrix, however, reveals the absence of long-range order, thereby indicating that the transition cannot be interpreted to a Bose-Einstein condensate. Nevertheless, for $T<T_c^\star$ and weak magnetic fields, the system still exhibits magnetic properties which are practically indistinguishable from those of a condensed, superconducting CBG. We therefore conclude that while a condensate is a sufficient condition for the ideal CBG to exhibit a superconducting state, it may not be a necessary condition.

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I. INTRODUCTION

In a series of recent papers, we have calculated closed-form, analytical expressions for the single-particle and nonlocal properties of a harmonically confined, ideal gas of fermions (bosons) at both zero and finite temperatures in arbitrary dimensions.[1–3] These analytical results have already provided some insight into the role of dimensionality, interactions, and the quantum statistics of ultracold Fermi and Bose gases, which have recently been created in the laboratory.[4,5] The central theoretical tool used in these studies is the Bloch density matrix, which is related to the one-body density matrix (1DM) through an inverse Laplace transform (ILT).[6] The ILT method yields, *in a single calculation*, the finite-temperature 1DM for both fermions and bosons without having to resort to the single-particle wave functions of the associated trapping potential; one only requires a knowledge of the Bloch density matrix, which is independent of the quantum statistics of the gas. Owing to the current topical interest in these systems [e.g., Bose-Einstein condensation (BEC) in the trapped Bose gas and the possibility of a BCS-BEC crossover in the trapped Fermi gas], our applications of the ILT method have focused exclusively on the harmonically trapped quantum gases.

It is also well known that the uniform quantum gases can exhibit remarkable physical properties, particularly in low dimensions, where the strong quantum confinement in one or more directions can result in new phenomena associated with the reduced dimensionality (see Secs. II and III for details). Motivated by the success of the ILT method in the trapped quantum gases, this paper then is devoted to applying the same technique to study the thermodynamic and magnetic properties of the uniform quantum gases. Specifically, the two systems that we wish to investigate are the uniform, ideal two-dimensional electron gas (2DEG) and the ideal charged Bose gas (CBG). While our approach is valid in any dimension, we will focus our attention on the case of 2D for the following reasons: (i) the 2D calculation is the most transparent application of the ILT technique and (ii) the low-dimensional systems display physical properties not found in their higher-dimensional counterparts. Moreover, the ILT method provides a powerful, alternative approach to what is usually found in the literature, and readily yields analytical results for both Fermi and Bose systems which are otherwise very difficult (if not impossible) to obtain using the standard wave-function based approaches (see, e.g., Ref. 7). Indeed, the beauty of the ILT method is that it simultaneously yields *universal*, finite-temperature results for both Fermi and Bose statistics with very little effort. Thus, many of the analytical expressions for the uniform quantum gases, found previously by other authors, will be seen to be special cases of our more general results, which to our knowledge, have not appeared in the literature.

The rest of our paper is organized as follows. In Sec. II A exact, closed-form expressions for the finite-temperature and zero-temperature 1DM are provided. Then, in Sec. II B, we make use of the local-density approximation (LDA) to extend the exact results of the uniform gas to inhomogeneous Fermi systems. Using the 1DM, we will provide in Sec. II C explicit energy density functionals for the 2DEG in a magnetic field, which should be useful in the current-density functional theory (CDFT) of inhomogeneous Fermi systems.[8,9] We conclude our investigation of the 2DEG by presenting some analytical results for the magnetic properties in Sec. II D.

The thermodynamic and magnetic properties of the 2D CBG are then considered in Sec. III. Sections III A and III B focus on the thermodynamics of the 2D CBG in both zero and finite magnetic fields, respectively. The finite-temperature magnetic properties of the 2D CBG are then considered in Sec. III C, with particular emphasis on the possibility of a superconducting transition at sufficiently low temperatures. Finally, in Sec. IV, we will present our closing remarks.

II. FERMI GAS

The 2DEG that we consider is made up of a system of N independent electrons of charge e and mass m in a uniform magnetic field perpendicular to the plane of confinement. The 2DEG is immersed in a background medium with a uniform positive charge, so that the total system is charge neutral. Consequently, we assume that screening effects render the Coulomb interactions short ranged, and the gas can be treated as being approximately noninteracting. The 2DEG itself is typically fabricated using the electronic properties of III-V semiconductor heterostructures (e.g., GaAs-AlGaAs). The use of a "layer-by-layer" molecular-beam epitaxial growth processes results in atomically smooth heterostructure interfaces which contain the 2D gas.

This low-dimensional Fermi system has associated with it a number of interesting phenomena, such as the integer and fractional quantum Hall effect,[10] de Haas–van Alphen oscillations,[11] and the fractal energy spectrum of the Hofstadter butterfly,[12] to name a few. Moreover, using highly controllable lithographic and/or metallic gate technologies, the 2DEG's dimensionality can be further lowered to quasi-1D (i.e., quantum wires) or quasi-0D (i.e., quantum dots), with additional dimensionally dependent properties manifesting themselves in the form of, e.g., collective excitations[13] and magnetoresitivity oscillations.[14]

We begin this section by evaluating the exact finite-temperature 1DM for the 2DEG subjected to an external, homogeneous magnetic field. From the 1DM, we will construct explicit energy density functionals which may then be used to formulate a state-of-the-art CDFT of low-dimensional Fermi systems. As all of our calculations are done at finite temperatures, the important low-temperature properties of the gas (i.e., $T=0$) are readily obtained as a special case of the finite-temperature results.

A. Uniform 2DEG

Without loss of generality, we choose the symmetric gauge, where the vector potential reads $\mathbf{A}=(-By/2,Bx/2,0)$. Under these conditions, the zero-temperature Bloch density matrix is known to be given by[15]

$$C_0(\mathbf{r}_1,\mathbf{r}_2;\beta)=\frac{m\omega_c}{4\pi\hbar}\frac{1}{\sinh(\hbar\omega_c\beta/2)}\times\exp\left(-i\frac{m\omega_c}{2\hbar}(x_2y_1-y_2x_1)\right)\exp\left(-\frac{m\omega_c}{4\hbar}\times\left|\mathbf{r}_1-\mathbf{r}_2\right|^2\coth(\hbar\omega_c\beta/2)\right), \tag{1}$$

where $\omega_c=eB/mc$ is the cyclotron frequency and β is to be interpreted as a mathematical variable which in general is taken to be complex, and *not* the inverse temperature $1/k_BT$. For later convenience, we introduce the center-of-mass and relative coordinates:

$$\mathbf{r}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2),\quad \mathbf{s}=\mathbf{r}_1-\mathbf{r}_2, \tag{2}$$

which allow us to write the Bloch density matrix as

$$C_0\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};\beta\right)=\frac{m\omega_c}{4\pi\hbar}\frac{1}{\sinh(\hbar\omega_c\beta/2)}\times\exp\left(-i\frac{m\omega_c}{2\hbar}(x_2y_1-y_2x_1)\right)\times\exp\left(-\frac{m\omega_c}{4\hbar}s^2\coth(\hbar\omega_c\beta/2)\right). \tag{3}$$

At finite temperature, the (spin-averaged) one-body density matrix can be obtained from the zero-temperature Bloch density matrix by using the relation[6]

$$\rho_1(\mathbf{r}_1,\mathbf{r}_2;T)=\mathcal{L}_\mu^{-1}\left[\frac{2}{\beta}C_T(\mathbf{r}_1,\mathbf{r}_2;\beta)\right], \tag{4}$$

where

$$C_T(\mathbf{r}_1,\mathbf{r}_2;\beta)=C_0(\mathbf{r}_1,\mathbf{r}_2;\beta)\frac{\pi\beta k_BT}{\sin(\pi\beta k_BT)}\qquad\text{(fermions)} \tag{5}$$

is the finite-temperature Bloch density matrix, μ is the chemical potential, and k_B is the Boltzmann constant. In Eq. (4), the ILT is two sided, thereby allowing μ to take on negative values. Making use of the identity[2]

$$\exp\{-y\coth(\hbar\omega_c\beta/2)\}=\sum_{n=0}^{\infty}L_n(2y)e^{-y}\{e^{-n\hbar\omega_c\beta}-e^{-(n+1)\hbar\omega_c\beta}\}, \tag{6}$$

with $y\equiv m\omega_cs^2/4\hbar$, leads to the following expression for the zero-temperature Bloch density matrix:

$$C_0\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};\beta\right)=\frac{m\omega_c}{4\pi\hbar}\frac{1}{\sinh(\hbar\omega_c\beta/2)}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}\times\sum_{n=0}^{\infty}L_n(2y)e^{-y}\{e^{-n\hbar\omega_c\beta}-e^{-(n+1)\hbar\omega_c\beta}\}. \tag{7}$$

The finite-temperature 1DM is then given by taking the ILT as given by Eq. (4). Combining Eq. (7) with Eq. (5) gives

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};T\right)=\frac{m\omega_c}{2\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}\sum_{n=0}^{\infty}e^{-y}L_n(2y)\times\mathcal{L}_\mu^{-1}\left[\frac{(e^{-n\hbar\omega_c\beta}-e^{-(n+1)\hbar\omega_c\beta})}{\sinh(\hbar\omega_c\beta/2)}\frac{\pi k_BT}{\sin(\pi k_BT\beta)}\right]. \tag{8}$$

Making the change of variables $\tilde{\mu}=2\mu/\hbar\omega_c$, $\tilde{T}=2k_BT/\hbar\omega_c$, $\tilde{\beta}=\hbar\omega_c\beta/2$, and applying the convolution theorem for two-sided ILT's, we obtain

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};T\right)=\frac{m\omega_c}{2\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}\sum_{n=0}^{\infty} e^{-y}L_n(2y)\mathcal{L}_{\tilde{\mu}}^{-1}\left[\frac{(e^{-2n\tilde{\beta}}-e^{-2(n+1)\tilde{\beta}})}{\sinh(\tilde{\beta})}\frac{\pi\tilde{T}}{\sin(\pi\tilde{\beta}\tilde{T})}\right]$$

$$=\frac{m\omega_c}{\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}\sum_{n=0}^{\infty} e^{-y}L_n(2y)\sum_{\ell=0}^{\infty}\left\{\int_{-\infty}^{\infty}d\tau\delta(\tau-(2\ell+1)-2n)\frac{1}{e^{(\tau-\tilde{\mu})/\tilde{T}}+1}\right.$$

$$\left.-\int_{-\infty}^{\infty}d\tau\delta(\tau-(2\ell+1)-2(n+1))\frac{1}{e^{(\tau-\tilde{\mu})/\tilde{T}}+1}\right\}. \tag{9}$$

In going from the first to second line in Eq. (9), we have made use of the following two-sided ILT's:[2]

$$\mathcal{L}_{\eta}^{-1}\left[\frac{e^{-n\beta}}{\sinh(\beta)}\right]=2\sum_{k=0}^{\infty} \delta(\eta-(2k+1)-n)\Theta(\eta), \tag{10}$$

$$\mathcal{L}_{\mu}^{-1}\left[\frac{\pi k_BT}{\sin(\pi\beta k_BT)}\right]=\frac{1}{\left[\exp\left(-\frac{\mu}{k_BT}\right)+1\right]}. \tag{11}$$

Now, all but the first term arising from the $\ell=0$ sum give vanishing contributions to Eq. (9). Therefore, upon restoring the original units, the finite-temperature 1DM reduces to

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};T\right)=\frac{m\omega_c}{\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}e^{-(m\omega_c/4\hbar)s^2}\sum_{n=0}^{\infty} L_n\left(\frac{m\omega_c}{2\hbar}s^2\right)\frac{1}{e^{(\varepsilon_n-\mu)/k_BT}+1}$$

$$=\frac{m\omega_c}{\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}e^{-(m\omega_c/4\hbar)s^2}\sum_{n=0}^{\infty} F_n(\mu)L_n\left(\frac{m\omega_c}{2\hbar}s^2\right), \tag{12}$$

where $\varepsilon_n=\hbar\omega_c(n+1/2)$ are the discrete Landau-level energies arising from the applied magnetic field, n denotes the Landau-level index, and

$$F_n(\mu)=\frac{1}{e^{(\varepsilon_n-\mu)/k_BT}+1} \quad \text{(fermions)}. \tag{13}$$

Equation (12) represents the exact spin-averaged finite-temperature 1DM for a homogeneous 2DEG in the presence of a magnetic field of arbitrary strength. The single-particle density is immediately obtained by setting $s=0$. Note that, for higher dimensions, the functional form for the 1DM is identical to Eq. (12), but with the thermal factor $F_n(\mu)$, being replaced by a more complicated function which contains the dimensional and temperature dependencies. Thus, while Eq. (12) may appear to be rather obvious, i.e., it looks as if we have simply included by hand the Fermi-Dirac distribution function, this *is not* the case. Indeed, it is a highly nontrivial task to show that the finite-temperature 1DM has the form of Eq. (12) (in any dimension), if one starts from the single-particle wave functions.[2] The main difficulty lies in the fact that, unlike the Bose-Einstein distribution function, the Fermi-Dirac occupation factor cannot be expanded as a power series for arbitrary temperatures. Consequently, the summation over the Landau-level index is exceedingly difficult to execute in closed form.

It is instructive to also consider the $T\rightarrow 0$ limit of Eq. (12), in which case the Fermi distribution function goes over to the Heaviside step function, viz.,

$$\frac{1}{e^{(\varepsilon_n-\mu)/k_BT}+1}\to\Theta(\varepsilon_F-\varepsilon_n). \tag{14}$$

In a magnetic field at zero temperature, one has $n=0,1,2,\ldots,n_F-1$ Landau levels fully occupied with $\rho=m\omega_c/\pi\hbar$ electrons; the n_F level will only be partially occupied with $\lambda\rho$ electrons $(0<\lambda<1)$. If we focus on the case of closed shells, we then obtain

$$\varepsilon_F=\hbar\omega_c(n_F-1+1/2)=\hbar\omega_c(n_F-1/2) \tag{15}$$

for the Fermi energy. As a result, Eq. (12) in the zero-temperature limit reduces to the simple expression

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2}\right)=\frac{m\omega_c}{\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}e^{-(m\omega_c/4\hbar)s^2}\sum_{n=0}^{n_F-1}L_n\left(\frac{m\omega_c}{2\hbar}s^2\right)$$
$$=\frac{m\omega_c}{\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}e^{-(m\omega_c/4\hbar)s^2}L^{(1)}_{n_F-1}\left(\frac{m\omega_c}{2\hbar}s^2\right). \tag{16}$$

The last line in Eq. (16) has been obtained by making use of the summation relation $\Sigma_{n=0}^{m}L_n^{(\alpha)}(x)=L_m^{(\alpha+1)}(x)$. Note that while the 1DM depends on the chosen gauge, all physical observables are manifestly gauge invariant. We point out that Ghosh and Dhara,[16] have also obtained an expression identical to Eq. (16), but through an entirely different approach [see their Eq. (13)]. Indeed, their zero-temperature result is seen to be a special case of our more general, finite-temperature 1DM given by Eq. (12).

The zero-temperature single-particle density is readily obtained by setting $s=0$ in Eq. (16), giving

$$\rho=\frac{m\omega_c}{\pi\hbar}n_F=\frac{m\omega_c}{\pi\hbar}\left[\frac{\varepsilon_F}{\hbar\omega_c}+\frac{1}{2}\right], \tag{17}$$

where in the last line of Eq. (17), $[\cdot]$ denotes taking the integer part, with a floor of unity. In particular, as $\omega_c\to\infty$, $\rho\to m\omega_c/\pi\hbar$ [see also Eq. (19) below].

It is not difficult to examine the two limiting cases of the $T=0$ 1DM, viz., the vanishing and high-field limits. In the former case, we note that as $\omega_c\to 0$, $n_F\approx\varepsilon_F/\hbar\omega_c\to\infty$. Therefore, for a vanishing magnetic field, Eq. (16) behaves asymptotically as

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2}\right)\sim\frac{m\varepsilon_F}{\pi\hbar^2}\frac{1}{n_F}L^{(1)}_{n_F}\left(\frac{s^2}{n_F}\frac{m\varepsilon_F}{2\hbar^2}\right)\quad(n_F\to\infty)$$
$$=\frac{1}{\pi s}\sqrt{\frac{2m\varepsilon_F}{\hbar^2}}J_1\left(\sqrt{\frac{2m\varepsilon_F}{\hbar^2}}s\right), \tag{18}$$

where $J_1(x)$ is a cylindrical Bessel function. Equation (18) is just the 1DM for a uniform 2DEG with $B=0$. For extremely high magnetic fields, only the lowest Landau level is occupied, and we immediately get from Eq. (16) (i.e., retaining only the $n=0$ term)

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2}\right)=\frac{m\omega_c}{\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}e^{-(m\omega_c/4\hbar)s^2}. \tag{19}$$

Thus, at low-temperatures and high magnetic fields, the Fermi gas "condenses" into the lowest Landau level, $n=0$. However, it is important to realize that this is not a condensate in the sense of the phase transition that takes place in the trapped or uniform Bose gases. The easiest way to see this is to note that the highly degenerate lowest Landau level does not represent a *unique quantum state* into which all of the fermions have condensed. Indeed, it is well known since the early work of Penrose and Onsager[17] that for a homogeneous system, the phenomenon of a condensate is intimately related to the presence of long-range order (LRO) in the 1DM, namely,

$$\lim_{s\to\infty}\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2}\right)=\frac{N_0}{V}, \tag{20}$$

where V is the volume of the system and N_0 is the number of particles in the ground state. Thus, we see immediately from Eq. (19) that even at $T=0$, there is no LRO, and the system continues to behave as a normal Fermi gas. We will come back to the issue of LRO in Sec. III during our discussion of BEC in the 2D CBG.

B. Inhomogeneous 2DEG: Local-density-approximation

In the case where the 2DEG is further confined by a one-body potential $V(\mathbf{r})$ (i.e., the potential energy determined by a self-consistent field theory), one can make use of the LDA in order to obtain the Bloch density matrix for the inhomogeneous gas. The essential simplification is that the same free electron wave functions are used locally so that the energy levels are all shifted uniformly by $V(\mathbf{r})$. As a result, the zero-temperature Bloch density matrix within the LDA is simply given by[6]

$$C_0^{\mathrm{LDA}}\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};\beta\right)=C_0\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};\beta\right)\exp[-\beta V(\mathbf{r})], \tag{21}$$

where C_0 is given by Eq. (7). As in the uniform case, an ILT of Eq. (21) yields the LDA to the 1DM, namely,

$$\rho_1^{\mathrm{LDA}}\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};T\right)$$
$$=\frac{m\omega_c}{\pi\hbar}e^{-i(m\omega_c/2\hbar)(x_2y_1-y_2x_1)}e^{-(m\omega_c/4\hbar)s^2}$$
$$\times\sum_{n=0}^{\infty}L_n\left(\frac{m\omega_c}{2\hbar}s^2\right)\frac{1}{e^{[\varepsilon_n-\mu(\mathbf{r})]/k_BT}+1}. \quad (22)$$

As expected, the only modification from Eq. (12) is that the chemical potential is replaced by $\mu\to\mu(\mathbf{r})=\mu-V(\mathbf{r})$. It should be pointed out that Pfalzner and March[18] have previously attempted to find analytical, closed-form expressions for the 1DM of the inhomogeneous electron gas in a magnetic field within the LDA (i.e., Thomas-Fermi approximation) at *zero temperature*. However, they were unable to explicitly perform the ILT of the Bloch density matrix for arbitrary magnetic fields, and in contrast to the present work, their analysis was limited to a numerical implementation of the ILT method.

C. Energy density functionals

1. Exchange energy density

Armed with the knowledge of the 1DM it is now possible to evaluate in closed form the exchange energy density $\varepsilon_{\mathrm{ex}}(\mathbf{r})$ at any temperature and magnetic-field strength. Since the calculation of $\varepsilon_{\mathrm{ex}}(\mathbf{r})$ is essentially the same as in the LDA and uniform gas limits, we will focus our attention on the uniform 2DEG, where our analytical expressions are exact.

For simplicity, let us first consider the $T=0$ case for which the exact 1DM is given by Eq. (16). The exchange energy is then given by

$$\varepsilon_{\mathrm{ex}}=-\frac{e^2}{4}\int\frac{1}{s}\left|\rho\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right|^2d\mathbf{s}$$
$$=-e^2\frac{\pi}{2}\left(\frac{m\omega_c}{\pi\hbar}\right)^2\int e^{-(m\omega_c/2\hbar)s^2}\left[L_{n_F-1}^{(1)}\left(\frac{m\omega_c}{2\hbar}s^2\right)\right]^2ds$$
$$=-\frac{e^2}{\pi}\left(\frac{m\omega_c}{2\hbar}\right)^{3/2}\int x^{-1/2}e^{-x}[L_{n_F-1}^{(1)}(x)]^2dx, \quad (23)$$

where in the last line, we have changed over to the variable $x=m\omega_cs^2/2\hbar$. We now make use of the integral[3]

$$I_{m,n}(\alpha,\beta,\gamma)=\int_0^\infty x^\alpha e^{-x}L_m^\beta(x)L_n^\gamma(x)dx$$
$$=\frac{\Gamma(1+\alpha)\Gamma(n+\gamma+1)\Gamma(\beta-\alpha+m)}{\Gamma(m+1)\Gamma(n+1)\Gamma(1+\gamma)\Gamma(\beta-\alpha)}$$
$$\times{}_3F_2[1+\alpha-\beta,-n,1+\alpha;1+\gamma,1+\alpha-\beta-m;1], \quad (24)$$

where ${}_pF_q[a,b,c;d,e;z]$ is the generalized hypergeometric function,[19] to obtain

$$\varepsilon_{\mathrm{ex}}=-\frac{e^2}{\sqrt{2}\pi}\left(\frac{m\omega_c}{\hbar}\right)^{3/2}n_F\frac{\Gamma(n_F+1/2)}{\Gamma(n_F)}$$
$$\times{}_3F_2\left[-\frac{1}{2},1-n_F,\frac{1}{2};2,\frac{1}{2}-n_F;1\right]. \quad (25)$$

Once again, we point out that the authors of Ref. 16 have also calculated the exchange energy density at $T=0$ (see also Refs. 20–22 for earlier related work on this problem). However, a direct comparison of our Eq. (25) with their Eq. (19) suggests that the two expressions are not in agreement. This is quite surprising since our respective expressions for the $T=0$ 1DM were found to be mathematically identical. Under the assumption that none of us have made any trivial algebra mistakes, the apparent discrepancy between the two results must be a consequence of our different analytical approaches. In fact, this is entirely the case, and we have resolved the issue by proving the following exact finite summation relation:[23]

$$\sum_{k=0}^{n_F-1}\{\Gamma(n_F+k+1)\Gamma(2k+\tfrac{1}{2})/[\Gamma(n_F-k)\Gamma(2k+2)$$
$$\times\Gamma(k+2)\Gamma(k+1)]\}\,{}_2F_1[-n_F+k+1,2k+\tfrac{1}{2};2k+2;2]$$
$$=2\frac{\Gamma(n_F+\frac{1}{2})}{\Gamma(n_F)}\,{}_3F_2\left[-\frac{1}{2},1-n_F,\frac{1}{2};2,\frac{1}{2}-n_F;1\right]. \quad (26)$$

Using this identity in Eq. (19) of Ref. 16 establishes that our zero-temperature expressions for the exchange energy density are completely equivalent. A plot of the zero-temperature exchange energy can be found in Fig. 1 of Ref. 16.

The finite-temperature calculation is analogous to the $T=0$ case, with the 1DM now being given by Eq. (12). The central difference between the zero- and finite-temperature calculations is that we have to evaluate the integral $I_{n,k}(-1/2,0,0)$. From Eq. (24), we readily obtain the finite-temperature exchange energy density, viz.,

$$\varepsilon_{\mathrm{ex}}(T)=-\frac{e^2}{\pi}\left(\frac{m\omega_c}{2\hbar}\right)^{3/2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}F_n(\mu)F_k(\mu)\frac{\Gamma(n+1/2)}{\Gamma(n+1)}$$
$$\times{}_3F_2\left[\frac{1}{2},-k,\frac{1}{2};1,\frac{1}{2}-n;1\right]. \quad (27)$$

To our knowledge, Eq. (27) is a new result, which is valid at all temperatures and magnetic-field strengths. It is well known from numerical investigations that the sharp "sawtooth" oscillations found in the $T=0$ thermodynamic properties of the 2DEG in a magnetic field are smoothed out at finite temperatures.[6]

2. Kinetic-energy density

The kinetic-energy density at zero or finite temperature is also readily evaluated from the 1DM. Specifically, in the symmetric gauge, the kinetic-energy density is evaluated from the expression

$$\varepsilon_{\rm kin}=\frac{1}{2m}\left\{\left(-i\hbar\frac{\partial}{\partial x}+\frac{eBy}{2c}\right)^2+\left(-i\hbar\frac{\partial}{\partial y}-\frac{eBx}{2c}\right)^2\right\}\rho_1(\mathbf{r}_1,\mathbf{r}_2;T)\Big|_{\mathbf{r}_1=\mathbf{r}_2}. \qquad (28)$$

At low temperature, the magnetic-field-dependent oscillations (associated with the filling of the Landau levels) are most pronounced. Thus, the zero-temperature kinetic energy is of particular interest, and from Eq. (28), we immediately obtain

$$\varepsilon_{\rm kin}=\frac{m\omega_c^2}{2\pi}n_F^2. \qquad (29)$$

Using Eq. (17) for the single-particle density, we obtain the zero-temperature kinetic-energy density functional

$$\varepsilon_{\rm kin}[\rho]=\pi\frac{\hbar^2}{2m}\rho^2. \qquad (30)$$

The above kinetic-energy density has the same functional form as that of the free 2DEG in the absence of a magnetic field, but here, the magnetic-field dependence is implicitly contained in the density ρ. This interesting result is unique to 2D, since in 3D, the form of the kinetic-energy functional depends explicitly on the strength of the magnetic field. For example, in the strong-field regime one has[24]

$$\varepsilon_{\rm kin}^{3D}[\rho]=\frac{\hbar\omega_c}{2}\rho+\frac{2\pi^4\hbar^4}{3m^3\omega_c^2}\rho^3, \qquad (31)$$

which is very different from the zero-field kinetic-energy functional.[6]

Note that the functionals derived here, which are formally exact in the uniform case, can also be used in the CDFT of inhomogeneous electron systems (i.e., within the LDA). The outline for such an implementation can be found in Refs. 9 and 16.

D. Finite-temperature magnetization: Landau diamagnetism

To close our investigation of the 2DEG, we now briefly consider some analytical results for the finite-temperature magnetization. In keeping with our previous calculations, we do not consider the Pauli paramagnetism associated with the electron spin interacting with the magnetic field in this section.

The magnetization is evaluated from the thermodynamic identity,

$$M(B,T)=-\frac{1}{V}\left(\frac{\partial\Omega(B)}{\partial B}\right)_{\mu,T}=-\frac{e}{mcV}\left(\frac{\partial\Omega(\omega_c)}{\partial\omega_c}\right)_{\mu,T}, \qquad (32)$$

where the grand canonical partition function (spin averaged, per unit volume) for the 2DEG is given by the well-known expression

$$\frac{\Omega(B)}{V}=-k_BT\frac{m\omega_c}{\pi\hbar^2}\sum_{n=0}^{\infty}\ln\{1+\exp[(\mu-\varepsilon_n)/k_BT]\}. \qquad (33)$$

In the above, ω_c and ε_n have the same meaning as before. From Eq. (32), we immediately obtain

$$M=\frac{e}{\pi c\hbar^2}k_BT\left[\sum_{n=0}^{\infty}\ln(1+e^{(\mu-\varepsilon_n)/k_BT})-\frac{1}{k_BT}\sum_{n=0}^{\infty}\varepsilon_n\frac{1}{\exp[(\varepsilon_n-\mu)/k_BT]+1}\right]. \qquad (34)$$

Equation (34) is an exact result. At zero temperature we readily find that it reduces to

$$M(B,T=0)=\frac{e}{\pi c\hbar^2}k_BT\left[\sum_{n=0}^{n_F-1}\frac{(\varepsilon_F-\varepsilon_n)}{k_BT}-\frac{1}{k_BT}\sum_{n=0}^{n_F-1}\varepsilon_n\right]=\frac{e}{\pi c\hbar^2}[n_F\varepsilon_F-n_F^2\hbar\omega_c]. \qquad (35)$$

Equation (35) is in agreement with the results found in Ref. 25 [i.e., see their Eq. (32b)], where the magnetic properties of the 2DEG were considered only at zero temperature. At temperatures for which $\mu<\varepsilon_0=\hbar\omega_c/2$, the Fermi-Dirac occupation factor can be expanded as a power series, and the density of fermions can be written as

$$\rho=\frac{m\omega_c}{2\pi\hbar}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{e^{j\mu/k_BT}}{\sinh(j\hbar\omega_c/2k_BT)}\quad(\mu<\varepsilon_0). \qquad (36)$$

Equation (36) is related to the grand canonical partition function through the thermodynamic identity,

$$\rho=-\frac{1}{V}\frac{\partial\Omega(B)}{\partial\mu}. \qquad (37)$$

An integration of Eq. (37) with respect to μ then yields (again, only for $\mu<\varepsilon_0$)

$$\frac{\Omega(B)}{V}=-\frac{m\omega_c}{2\pi\hbar}k_BT\sum_{j=1}^{\infty}(-1)^{j+1}\frac{e^{j\mu/k_BT}}{j\sinh(j\hbar\omega_c/2k_BT)}=-\frac{m\omega_c}{\pi\hbar}k_BT\left\{\ln(1+e^{(\mu-\varepsilon_0)/k_BT})+\frac{1}{2}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{e^{j(\mu-\hbar\omega_c)/k_BT}}{j\sinh(j\hbar\omega_c/2k_BT)}\right\}. \qquad (38)$$

A direct application of Eq. (32) gives

$$M=\frac{e}{\pi c\hbar}k_BT\Bigg[\ln(1+e^{(\mu-\hbar\omega_c/2)/k_BT})+\frac{1}{2}\sum_{j=1}^{\infty}(-1)^{j+1}\times\frac{e^{j(\mu-\hbar\omega_c)/k_BT}}{j\sinh(j\hbar\omega_c/2k_BT)}\Bigg]+\frac{e\omega_c}{2\pi c}\Bigg[-\frac{z}{1+z}-\sum_{j=1}^{\infty}(-1)^{j+1}\times\Bigg\{\frac{e^{j(\mu-\hbar\omega_c)/k_BT}}{\sinh(j\hbar\omega_c/2k_BT)}+\frac{1}{2}\times\frac{e^{j(\mu-\hbar\omega_c)/k_BT}\cosh(j\hbar\omega_c/2k_BT)}{[\sinh(j\hbar\omega_c/2k_BT)]^2}\Bigg\}\Bigg], \tag{39}$$

where $z\equiv\exp[(\mu-\varepsilon_0)/k_BT)]$. To our knowledge, this expression for $M(B,T)$ has not appeared in the literature. Equation (39) will prove to be useful in our discussion of the magnetic properties of the 2D CBG.

III. CHARGED BOSE GAS

The Bose analog of the uniform 2DEG is the 2D CBG. In its simplest incarnation, the CBG consists of a gas of spinless, charged bosons, coupled to an external, homogeneous magnetic field. In analogy with the 2DEG, the bosons are assumed to have a charge *e*, mass *m*, and are noninteracting. The 2D CBG may be realized from a 3D system with a small thickness δ_z. If δ_z is much smaller than the thermal wavelength $\delta_z \ll (2mk_BT/4\pi\hbar^2)^{-1/2}$, the k_z momentum will be frozen in the ground state $k_z=0$ and the system may be considered as a 2D Bose gas. In this light, it has recently been suggested that the 2D CBG may have some relevance to the theory of superconductivity in the high-temperature cuprates where preformed electron pairs (i.e., composite, spinless, charged bosons) are conjectured to exist.[26]

The CBG was first investigated in 3D by Osborne,[27] and then substantially improved upon by Schafroth,[28] who showed that it *does not* have a BEC at any finite temperature in the presence of a homogeneous magnetic field, although the system does exhibit the essential equilibrium features of a superconductor [e.g, the Meissner-Ochsenfeld (M-O) effect]. Following this work, May[29] then considered the superconductivity of the 2D CBG, and showed that even though the system exhibits an essentially perfect M-O effect (as in 3D), it does not undergo a BEC phase transition at any finite temperature. Some time later, May[30] further generalized Schafroth's 3D results to arbitrary dimensions, and pointed out that BEC in the CBG can take place only if $d\geqslant 5$. Toms[31] has subsequently argued that BEC cannot occur in the CBG in any spatial dimension *d*, whereas Rojas[32] has suggested that BEC may occur, although the transition is diffuse (i.e., there is no sharp critical temperature at which condensation begins). Daicic and Frankel[33] have also examined the statistical mechanics of the 2D CBG within the context of Mellin integral transforms, thereby confirming and extending the earlier work of May.[29] More recently, Bayindir and Tanatar[34] have used a semiclassical approach to concluded that BEC can take place in the CBG in a magnetic field, but only in the presence of a crossed electric field. Thus, irrespective of its current experimental feasibility, the issue of BEC in the CBG is still clearly an interesting problem in its own right, and is certainly worthy of further investigation.

Before we go on to discuss the issue of BEC in the 2D CBG, we would first like to demonstrate how the ILT method describes the uniform, ideal 2D Bose gas when the applied magnetic field is switched off. After this brief orientation, we will in Sec. III B extend these results to the case of a nonzero magnetic field. Then, in Sec. III C, we will consider the magnetic properties of the system, with particular emphasis on the possibility of a superconducting phase transition below some critical temperature $T_c^\star$. We once again stress that other dimensions are readily studied by using the same procedure outlined below.

A. 2D CBG: $B=0$

When the magnetic field is absent, the ideal CBG is formally identical to the well-known neutral, ideal Bose gas. The zero-temperature Bloch density matrix for the translationally invariant 2D system is obtained as follows:

$$C_0\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};\beta\right)=\frac{1}{V}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{s}}e^{-\beta\hbar^2k^2/2m}=\left(\frac{1}{2\pi}\right)^2\int_0^\infty 2\pi k e^{i\mathbf{k}\cdot\mathbf{s}}e^{-\beta\hbar^2k^2/2m}dk=\frac{m}{2\pi\hbar^2\beta}\exp\left(-\frac{m}{2\pi\hbar^2\beta}s^2\right). \tag{40}$$

The quantum statistics of the gas are encoded in the thermal Bloch density matrix, which for bosons reads [compare with Eq. (5)]

$$C_T(\mathbf{r}_1,\mathbf{r}_2;\beta)=C_0(\mathbf{r}_1,\mathbf{r}_2;\beta)\frac{-\pi\beta k_BT}{\tan(\pi\beta k_BT)}\quad\text{(bosons)}. \tag{41}$$

It is important to realize here that in obtaining Eq. (40), we are assigning *zero weight* to the $k=0$ term (i.e., the ground state). Therefore, any finite-temperature properties we derive from Eq. (41) will only describe the thermally excited (i.e., normal) state of the gas; the ground state must be treated separately. Using Eq. (4) (without the factor of 2 since the bosons are taken to be spinless), along with the following two-sided ILT's (Ref. 35)

$$\mathcal{L}_\varepsilon^{-1}\left[\frac{e^{-k/\beta}}{\beta}\right]=J_0(2\sqrt{k\varepsilon})\Theta(\varepsilon), \tag{42}$$

$$\mathcal{L}_\mu^{-1}\left[-\frac{\pi k_BT}{\tan(\pi\beta k_BT)}\right]=\frac{1}{\left[\exp\left(-\frac{\mu}{k_BT}\right)-1\right]}, \tag{43}$$

we obtain for the thermal part of the 1DM

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};T\right)=\frac{m}{2\pi\hbar^2}\int_0^\infty J_0\left(\sqrt{\frac{2m\varepsilon}{\hbar^2}}s\right)\frac{1}{\exp\left(\frac{(\varepsilon-\mu)}{k_BT}\right)-1}d\varepsilon=\frac{m}{2\pi\hbar^2}\sum_{j=1}^{\infty}e^{j\mu/k_BT}\int_0^\infty J_0\left(\sqrt{\frac{2m\varepsilon}{\hbar^2}}s\right)e^{-j\varepsilon/k_BT}d\varepsilon. \tag{44}$$

Making the change of variables $\varepsilon=\hbar^2x^2/(2ms^2)$ and $\gamma=j\hbar^2\beta/(2ms^2)$, we obtain

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};T\right)=\frac{1}{2\pi s^2}\sum_{j=1}^{\infty}e^{j\mu/k_BT}\int_0^\infty xJ_0(x)e^{-\gamma x^2}dx=k_BT\frac{m}{2\pi\hbar^2}\sum_{j=1}^{\infty}\frac{e^{j\mu/k_BT}}{j}e^{-k_BTms^2/(2j\hbar^2)}. \tag{45}$$

Setting $s=0$ in Eq. (45) then yields the normal-state density of particles, which we denote by $\rho_>$, viz.,

$$\rho_>(T)=\frac{m}{2\pi\hbar^2}k_BT\sum_{j=1}^{\infty}\frac{e^{j\mu/k_BT}}{j}=-\frac{m}{2\pi\hbar^2}k_BT\ln(1-e^{\mu/k_BT}). \tag{46}$$

At high temperatures, Eq. (46) describes the density of *all* of the particles in the gas. However, as the gas is cooled, it may happen that at some critical temperature $T_c^{(0)}$, the excited states cannot accommodate all of the particles. We have used the superscript on the critical temperature to emphasize that we are in the field-free case. In this scenario, the ground state of the system must become populated, and the system undergoes what is known as a BEC phase transition. The critical temperature is then formally defined as the highest temperature at which the macroscopic occupation of the lowest-energy state appears. In the homogeneous Bose gas, this corresponds to $\mu(T\leqslant T_c^{(0)})=0$, and from Eq. (46) we find

$$T_c^{(0)}=\lim_{\mu\to 0^-}\frac{-2\pi\hbar^2}{mk_B}\frac{\rho}{\ln(1-e^{\mu/k_BT})}=0. \tag{47}$$

Thus, for the uniform 2D Bose gas, there is no BEC transition at any finite temperature. In other words, for $T\neq 0$, Eq. (46) can accommodate all of the bosons, and we do not have to invoke the population of the ground state.[36] Of course, at identically $T=0$, Eq. (46) vanishes, implying that all of the particles are in the ground state.

Finally, we observe from Eq. (45) that for any $T\neq 0$, we have

$$\lim_{s\to\infty}\rho_1(s;T)=0, \tag{48}$$

thereby reaffirming that at finite T, there is no BEC because we have no LRO.

B. 2D CBG: $B\neq 0$

Extending the results of Sec. III A to the case of a CBG coupled to an external magnetic field can be done in one of two ways. In the first instance, a brute force calculation paralleling the field-free case can be performed by replacing Eq. (40) with Eq. (1), and then evaluating all of the required ILT's. The second, more elegant approach (and the one which highlights the power of the ILT method) is to simply note that the Bose and Fermi calculations only differ by the temperature dependence introduced via the thermal Bloch density matrix. Specifically, we observe that the Bose statistics serve only to change the behavior of the function $F_n(\mu)$, viz.,

$$F_n(\mu)=\frac{1}{\left[\exp\left(\frac{\varepsilon_n-\mu}{k_BT}\right)-1\right]}\quad\text{(bosons)}, \tag{49}$$

which was introduced in Sec. II C in the context of the Fermi gas [see Eq. (13)]. As a result, *without any further calculation*, we can write down the finite-temperature 1DM for the charged (spinless) 2D Bose gas in a magnetic field:

$$\rho_1\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2};T\right)=\frac{m\omega_c}{2\pi\hbar}e^{-i\frac{m\omega_c}{2\hbar}(x_2y_1-y_2x_1)}e^{-\frac{m\omega_c}{4\hbar}s^2}\times\sum_{n=0}^{\infty}F_n(\mu)L_n\left(\frac{m\omega_c}{2\hbar}s^2\right)=\frac{m\omega_c}{4\pi\hbar}e^{-i\frac{m\omega_c}{2\hbar}(x_2y_1-y_2x_1)}e^{-\frac{m\omega_c}{4\hbar}s^2}\times\sum_{j=1}^{\infty}\frac{e^{j\mu/k_BT}}{\sinh(j\hbar\omega_c/2k_BT)}\times\exp\left[-\frac{m\omega_c}{2\hbar}\frac{s^2}{(e^{j\hbar\omega_c/k_BT}-1)}\right], \tag{50}$$

where in obtaining the last line of Eq. (50), we have used the identity[19]

$$\sum_{n=0}^{\infty}L_n(x)z^n=\frac{1}{1-z}\exp\left[\frac{xz}{z-1}\right],\quad |z|<1. \tag{51}$$

We reemphasize that Eq. (50) is not an obvious finite-temperature generalization of Eq. (12). In higher dimensions, $F_n(\mu)$ is a more complicated function of temperature and dimensionality, and the simple replacement of the Fermi factor with a Bose occupation factor will not give correct results.

Setting $s=0$ in Eq. (50) gives the finite-temperature density of the CBG in a magnetic field of arbitrary strength:

$$\rho(T)=\frac{m\omega_c}{4\pi\hbar}\sum_{j=1}^{\infty}\frac{e^{j\mu/k_BT}}{\sinh(j\hbar\omega_c/2k_BT)}$$
$$=\frac{m\omega_c}{2\pi\hbar}\sum_{n=0}^{\infty}\frac{1}{e^{(\varepsilon_n-\mu)/k_BT}-1}. \tag{52}$$

Note that as $\omega_c\rightarrow 0$, Eq. (52) reduces to Eq. (46). Thus, as in the field-free case, $\rho(T)$ above represents only the remaining bosons outside of the single ground state. Following Sec. III A, the critical temperature T_c is defined by $\mu\rightarrow\hbar\omega_c/2$. However, in this limit, the sum in Eq. (52) has no upper bound, and we once again find that the 2D CBG does not condense in a fixed homogeneous magnetic field.

We now wish to consider the temperature dependence of the number density of bosons in the presence of a finite magnetic field. In this section only, we take $\beta\equiv 1/(k_BT)$. Starting with the second line of Eq. (52), we have

$$\rho(T)=\frac{m\omega_c}{2\pi\hbar}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}e^{j\beta(\mu-\hbar\omega_c/2)}e^{-jn\beta\hbar\omega_c}$$
$$=\frac{m\omega_c}{2\pi\hbar}\sum_{j=1}^{\infty}\frac{e^{j\beta(\mu-\hbar\omega_c/2)}}{1-e^{-j\beta\hbar\omega_c}}, \tag{53}$$

Defining $z=\exp[\beta(\mu-\hbar\omega_c/2)]$ and $x=\exp(-\beta\hbar\omega_c)$ yields

$$\rho(T)=\frac{m\omega_c}{2\pi\hbar}\sum_{j=1}^{\infty}\frac{z^j}{1-x^j}=\frac{m\omega_c}{2\pi\hbar}\sum_{j=1}^{\infty}z^j\sum_{\ell=0}^{\infty}(x^j)^{\ell}$$
$$=\frac{m\omega_c}{2\pi\hbar}\left[\frac{z}{1-z}+\sum_{j=1}^{\infty}\frac{z^jx^j}{1-x^j}\right]$$
$$=\frac{m\omega_c}{2\pi\hbar}\left[\frac{z}{1-z}+\sum_{j=1}^{\infty}z^j\frac{e^{-j\beta\hbar\omega_c/2}}{2\sinh(j\hbar\omega_c\beta/2)}\right]. \tag{54}$$

Making the approximation $\beta\hbar\omega_c\ll 1$, we can now write $\rho(T)$ as

$$\rho(T)=\frac{m\omega_c}{2\pi\hbar}\left[\frac{z}{1-z}+\frac{k_BT}{\hbar\omega_c}\sum_{j=1}^{\infty}\frac{z^je^{-j\beta\hbar\omega_c/2}}{j}\right]$$
$$=\frac{m\omega_c}{2\pi\hbar}\left[\frac{z}{1-z}-\frac{k_BT}{\hbar\omega_c}\ln(1-ze^{-\beta\hbar\omega_c/2})\right]. \tag{55}$$

The first term in the square brackets of Eq. (55) is immediately recognized as the density of bosons, $\rho_0(T)$, in the lowest Landau level $n=0$. We can therefore define a critical temperature $T_c^{\star}$, at which the *lowest Landau level* becomes macroscopically populated. Putting $\rho_0(T_c^{\star})=0$ and $\mu=\hbar\omega_c/2$ in Eq. (55) gives

$$\rho(T_c)\equiv\rho_{>}=\frac{m\omega_c}{2\pi\hbar}\left[-\frac{k_BT_c^{\star}}{\hbar\omega_c}\ln(1-e^{-\beta_c\hbar\omega_c/2})\right]$$
$$\approx\frac{m\omega_c}{2\pi\hbar}\left[\frac{k_BT_c^{\star}}{\hbar\omega_c}\ln\left(\frac{2k_BT_c^{\star}}{\hbar\omega_c}\right)\right], \tag{56}$$

where, $\rho_{>}$ denotes the density of bosons outside of the $n=0$ state. The fractional density of particles in the lowest Landau level is then simply given by

$$\frac{\rho_0}{\rho}=1-\frac{T}{T_c^{\star}}\frac{\ln\left(\frac{2k_BT}{\hbar\omega_c}\right)}{\ln\left(\frac{2k_BT_c^{\star}}{\hbar\omega_c}\right)}=1-\frac{T}{T_c^{\star}}$$
$$(N,V\rightarrow\infty,\ \ N/V=\text{constant}). \tag{57}$$

Let us now turn briefly to the BEC of a finite number of ideal bosons in a 1D harmonic trap. This system was recently discussed by Ketterle and van Druten (KvD),[37] where it was suggested that BEC exists, contrary to previous predictions.[38] Their criterion for BEC was the presence of a macroscopic occupation of the lowest oscillator state below some temperature $T_c^{(0)}$.[39] Specifically, they showed that the temperature dependence on the number of particles is given by

$$N=\frac{z}{1-z}-\frac{k_BT}{\hbar\omega_0}\ln\left[1-z\exp\left(-\frac{\hbar\omega_0}{2k_BT}\right)\right], \tag{58}$$

where ω_0 is the trap frequency, $z=\exp[(\mu-\hbar\omega_0/2)/k_BT]$, and $N_0=z/(1-z)$ are the number of bosons in the lowest-lying energy state. It is readily shown however, that there is no thermodynamic signature for the presence of a BEC phase transition (i.e., in the behavior of the specific heat) for the ideal 1D trapped Bose gas.[40] Therefore, the strict identification of $T_c^{(0)}$ with a BEC phase-transition temperature is incorrect. Nevertheless, Eq. (58) is equivalent to Eq. (55) under the replacement $\omega_c\rightarrow\omega_0$. In addition, we now see an obvious formal connection between the critical temperature $T_c^{(0)}$, of the 1D ideal harmonically trapped Bose gas, and the critical temperature $T_c^{\star}$, which defines the transition to a macroscopic occupation of the lowest Landau level in the 2D CBG. Similarly, the "condensate" fraction obtained by KvD is identical to our Eq. (57), provided we replace ω_c by ω_0. An illustration of the sharpness of the transition below $T_c^{\star}$ can be found in Fig. 4 of Ref. 37.

The clear similarities between our results for the 2D CBG and those of KvD are in fact not so surprising. Recall that the uniform 2D gas in the presence of a homogeneous magnetic field has a quantum Hamiltonian whose structure is formally identical to that of a 1D harmonic oscillator, with the "trap frequency" being identified with the cyclotron frequency ω_c.[41] Therefore, it is entirely reasonable that we should obtain analogous expressions for the fractional occupancy and critical temperature, since both systems have identical eigenvalue spectrums, viz., $\varepsilon_n=\hbar\omega(n+1/2)$, with $\omega=\omega_c$ or ω_0, in the 2D CBG and 1D harmonic trap, respectively.

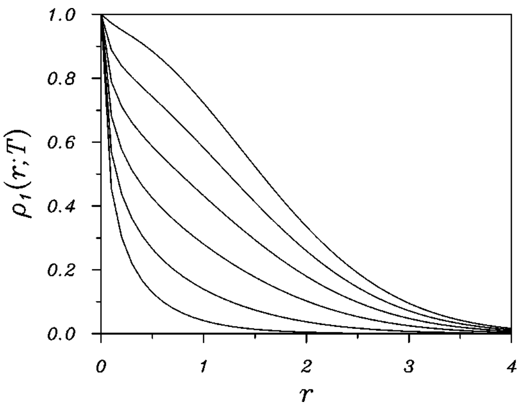

FIG. 1. The normalized 1DM [see Eq. (50) with $\mathbf{r}_2=0$] at various temperatures. From left to right, the curves correspond to $T/T_c^\star=1.1,0.9,0.7,0.5,0.3,0.1$. All lengths and energies have been scaled by $\sqrt{\hbar/m\omega_c}$ and $\hbar\omega_c$, respectively. Note that while there is a marked increase in the length scale over which the 1DM decays at low temperatures, the ideal 2D CBG clearly does not exhibit LRO as defined by Eq. (20).

In spite of the similarities, however, the interpretation that we must give to our results is quite different. Specifically, while KvD characterize their system as exhibiting a BEC phase for $T \ll T_c^{(0)}$, our condensation phenomenon is clearly *not* a BEC. This fact is made rigorous by examining the asymptotic spatial behavior of the 1DM. We can quite arbitrarily set $\mathbf{r}_2=0$ in Eq. (50) to obtain

$$\lim_{r\to\infty}\rho_1(\mathbf{r};T\leqslant T_c^\star)=\lim_{r\to\infty}\frac{m\omega_c}{4\pi\hbar}e^{-(m\omega_c/4\hbar)r^2}\times\sum_{j=1}^{\infty}\frac{e^{j\hbar\omega_c/2k_BT}}{\sinh(j\hbar\omega_c/2k_BT)}\times\exp\left[-\frac{m\omega_c}{2\hbar}\frac{r^2}{(e^{j\hbar\omega_c/k_BT}-1)}\right]=0. \tag{59}$$

Therefore, while we have a sharp, macroscopic occupation of the lowest Landau level for $T\leqslant T_c^\star$, the system exhibits no LRO in the Penrose-Onsager sense, meaning that we cannot call the statistical accumulation of bosons a BEC. Equation (59) also serves as an unambiguous definition of BEC in the 2D CBG, as it clearly distinguishes the macroscopic occupation of the $n=0$ state from the BEC phenomenon, which is associated with the condensation into a single quantum state and the presence of LRO. The absence of LRO at any finite temperature is illustrated in Fig. 1, where we present the normalized 1DM at various temperatures. The key point to be taken from this figure is that the 1DM decays rapidly to zero after only a few magnetic lengths, and thus the ideal 2D CBG does not possess LRO below $T_c^\star$.

C. Magnetization and the Meissner-Ochsenfeld effect

In Sec. III B above, we have argued that the 2D CBG does not undergo a transition to the BEC state for $T\leqslant T_c^\star$. Rather, this critical temperature reflects a sharp transition of the system to a state in which there is a macroscopic occupation of the lowest Landau level. Even though there is no finite-temperature condensation for the 2D CBG, it is still interesting to examine what effect the large occupation of the $n=0$ state has on the magnetic properties of the system. To this end, we will now consider the evaluation of the finite-temperature magnetization of the 2D CBG, and subsequently discuss its connection to the M-O effect (see also Ref. 33 for related work). Although the magnetization of the ideal 2D CBG has already been considered by May,[29] we feel that our derivation is more transparent in that it avoids the introduction of the "formal" temperatures and magnetic fields found in his earlier work. Moreover, our approach clearly highlights the role of the bosons in the lowest Landau level with respect to the magnetization of the system for $T<T_c^\star$.

We begin by considering the 2D CBG in an "acting" homogeneous magnetic field B', which is related to the applied external field B and the magnetization M by the relation

$$B'=B+2\pi M. \tag{60}$$

The acting field is then to be identified with the average microscopic field in the gas. As in the Fermi gas case, the magnetization is evaluated from Eq. (32) with the cyclotron frequency now given by $\omega_c=eB'/mc$. The grand canonical partition function is once again related to the density of the gas by Eq. (37), and we readily obtain

$$\frac{\Omega(B')}{V}=-k_BT\frac{m\omega_c}{4\pi\hbar}\sum_{j=1}^{\infty}\frac{e^{j\mu/k_BT}}{j\sinh(j\hbar\omega_c/2k_BT)}=-\frac{m\omega_c}{2\pi\hbar}k_BT\left\{-\ln(1-e^{(\mu-\hbar\omega_c/2)/k_BT})+\frac{1}{2}\sum_{j=1}^{\infty}\frac{e^{j(\mu-\hbar\omega_c)/k_BT}}{j\sinh(j\hbar\omega_c/2k_BT)}\right\}. \tag{61}$$

Using Eq. (32), we obtain for the finite-temperature magnetization

$$M=\frac{e}{2\pi\hbar c}k_BT\left[-\ln(1-e^{(\mu-\hbar\omega_c/2)/k_BT})+\frac{1}{2}\sum_{j=1}^{\infty}\frac{e^{j(\mu-\hbar\omega_c)/k_BT}}{j\,\sinh(j\hbar\omega_c/2k_BT)}\right]+\frac{e\omega_c}{4\pi c}\left[-\frac{z}{1-z}-\sum_{j=1}^{\infty}\left\{\frac{e^{j(\mu-\hbar\omega_c)/k_BT}}{\sinh(j\hbar\omega_c/2k_BT)}+\frac{1}{2}\frac{e^{j(\mu-\hbar\omega_c)/k_BT}\cosh(j\hbar\omega_c/2k_BT)}{[\sinh(j\hbar\omega_c/2k_BT)]^2}\right\}\right]. \tag{62}$$

Equation (62) is an exact result valid for all temperatures and magnetic-field strengths. Notice that Eq. (62) also bears a striking similarity to the Fermi gas result given by Eq. (39). Indeed, Eq. (62) could have been written down immediately from Eq. (38) (which is valid only for $\mu<\varepsilon_0$) by simply removing the $(-1)^{j+1}$ coefficient in the j sum of the Fermi grand canonical potential.

In the limit $\hbar\omega_c/k_BT\ll 1$ (i.e., weak acting homogeneous magnetic field), we find that the second term in Eq. (62) exactly cancels the last term, leaving ($T<T_c^\star$)

$$M=-\frac{e\omega_c}{4\pi c}\frac{z}{1-z}+\frac{e}{2\pi\hbar c}k_BT\ln\left[\frac{1-e^{(\mu-\hbar\omega_c)/k_BT}}{1-e^{(\mu-\hbar\omega_c/2)/k_BT}}\right]. \tag{63}$$

The logarithmic term in Eq. (63) is negligible compared to the first term, and with the aid of Eq. (55), we may write

$$\begin{aligned}M&=-\mu_0\rho_0=-\mu_0\rho\left[1-\frac{T}{T_c^\star}\frac{\ln\left(\frac{2k_BT}{\hbar\omega_c}\right)}{\ln\left(\frac{2k_BT_c^\star}{\hbar\omega_c}\right)}\right]\\&=-\mu_0\rho\left(1-\frac{T}{T_c^\star}\right)\quad(N\to\infty,V\to\infty,\ \ N/V=\text{constant}),\end{aligned} \tag{64}$$

where $\mu_0=e\hbar/2mc$ is the Bohr magneton. May[29] has earlier obtained an expression similar to Eq. (64), but using a very different analysis. Our approach clearly illustrates that below $T_c^\star$, the macroscopic occupation of the lowest Landau level leads to a magnetization that is analogous to the result obtained in the condensed CBG, even though here, there is strictly no BEC. It is important to note that because there is no BEC phase, we have $M(B'\to 0)=0$, so that the system does not exhibit a spontaneous magnetization. The absence of a spontaneous magnetization is sufficient to show that the system does not exhibit a complete field expulsion for $T<T_c^\star$ (i.e., there is no perfect M-O effect). Nevertheless, at identically $T=0$, the gas does have a nonzero spontaneous magnetization and consequently, exhibits a perfect M-O effect, characterized by Eq. (64) (at $T=0$) but with ρ interpreted as the *condensate* density.

Returning now to the case of $T<T_c^\star$, we have from Eq. (64)

$$M(B')=-B'\frac{\mu_0 e}{2\pi\hbar c}\frac{z}{1-z}, \tag{65}$$

and we recall that the factor $z/(1-z)\gg 1$ in this regime. From Eq. (60), we immediately obtain

$$B'=\kappa B, \tag{66}$$

where

$$\kappa=\left(1+\mu_0\frac{e}{c\hbar}\frac{z}{1-z}\right)^{-1}. \tag{67}$$

Thus, for weak magnetic fields and $T<T_c^\star$, $\kappa\approx 0$, indicating that there is a nearly complete expulsion of the magnetic field from the gas. As a result, we can define a critical magnetic field

$$B_c^\star=2\pi\mu_0\rho\left(1-\frac{T}{T_c^\star}\right), \tag{68}$$

such that for $B>B_c^\star$, the applied field will penetrate the gas with a magnetic induction $B'\approx B-B_c^\star$. Below $B_c^\star$, the field is almost entirely expelled, which can be viewed as an "imperfect" M-O effect. In other words, for $T<T_c^\star$, the macroscopic occupation of the lowest Landau level leads to a near perfect M-O effect, and the magnetic properties of the 2D CBG are essentially those of a superconductor, in spite of the absence of a BEC phase transition. We reemphasize here that our results are for a noninteracting gas. Thus, it is surprising that one can have a (near perfect) M-O effect in the absence of a BEC. Although one could make the argument that the neutral 2D Bose gas can also exhibit a superfluid phase without the presence of a BEC, it should be recalled that in that system, the presence of interactions is crucial for the onset of the superfluidity (such a transition is sometimes called a *dynamical* phase transition). This is not the case for the ideal 2D CBG in a magnetic field, where the M-O effect has a purely kinematical origin associated with the statistical accumulation of bosons into the lowest Landau level (i.e., the M-O effect occurs even in the absence of interparticle interactions).

Finally, we wish to point out that the finite-T behavior of the magnetization of the 2D CBG above is not the same as in the case of a perfect diamagnet. Specifically, a perfect diamagnet in a *fixed* homogeneous magnetic field has no field expulsion as the system is cooled to lower temperatures. However, when the magnetic field changes in time, the induced currents in the metal generate a magnetic field that is directly opposed to the applied field, and one obtains perfect field expulsion (i.e., as dictated by Lenz's law for a perfect diamagnet).

IV. SUMMARY AND CONCLUSIONS

We have investigated the thermodynamic and magnetic properties of the ideal 2DEG and 2D CBG from the point of view of the ILT method, which is not widely used in the literature. Although the technique is valid in arbitrary dimensions, we have focused on $d=2$. In the case of the 2DEG, we were able to obtain a closed-form, analytical expression for the 1DM which is valid at any temperature and arbitrary magnetic-field strengths. The 1DM was then used to examine explicit energy density functionals for the 2DEG in a magnetic field, and extended to the inhomogeneous electron gas through the LDA. The zero-temperature analytical results obtained previously by other authors, e.g., Refs. 16,18 and 25, were shown to be special cases of our more general analysis.

When applied to the CBG, the ILT also gave an exact, closed-form expression for the 1DM at arbitrary temperatures and magnetic-field strengths. One particularly noteworthy consequence of the method is the *universal* functional form of the 1DM for both the Bose and Fermi gases. This result is nontrivial, and we have highlighted that it is very difficult to establish using the standard wave-function based approaches.

We have also illustrated that the thermodynamic properties of the 2D CBG in a magnetic field are formally identical to the ''BEC-like'' transition recently shown to take place in the ideal Bose gas in 1D harmonic trap.[37] In spite of the similarities, however, the 2D CBG does not condense below $T_c^\star$, but rather undergoes a transition to which there is a macroscopic occupation of the lowest Landau level. In this way, we were able to connect the critical temperature for BEC in the 1D harmonically confined Bose gas to the critical temperature $T_c^\star$ for the macroscopic occupation of the lowest Landau level in the 2D CBG. The absence of the BEC transition was made more rigorous by establishing the lack of LRO in the asymptotic spatial behavior of the 1DM. The macroscopic occupation of the lowest Landau level below $T_c^\star$ was subsequently shown to have a profound effect on the magnetic properties of the system. In particular, the large number of bosons in the $n=0$ state were unambiguously related to the onset of an (essentially) perfect M-O effect for $T<T_c^\star$, just as in the condensed, superconducting CBG.[28] Therefore, our results for the 2D CBG clearly show that there is a *sharp* transition (i.e., a well-defined critical temperature $T_c^\star$) below which the gas exhibits the essential equilibrium features of a superconductor. This naturally leads to the conclusion that while a condensed phase is a sufficient condition for the CBG to exhibit a superconducting state, it may not be a necessary condition (at least for weak homogeneous magnetic fields), as we have illustrated here in the case of a 2D CBG.

Of course, all of the above results for the CBG in a magnetic field have been obtained for the ideal case. It would be of great interest to also include the Coulomb interactions between the charged bosons and examine the possibility of LRO and BEC in this situation. To this end, we mention that Davoudi *et al.*[42] have recently considered the ground-state properties (i.e., $T=0$) of the 2D CBG interacting via a logarithmic potential, while Strepparola *et al.*[43] have investigated the 2D CBG interacting via a e^2/r potential at finite temperatures. In the latter study, the authors found strong evidence for quasi-LRO, characterized by the 1DM exhibiting an asymptotic algebraic power-law decay. However, neither Refs. 42 and 43 have considered the consequences of including an external magnetic field for the interacting 2D CBG. We plan on presenting the results of such an investigation in a future publication.

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