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Russian doll renormalization group and superconductivity

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We show that a simple extension of the standard BCS Hamiltonian leads to an infinite number of BCS eigenstates with different energy gaps and self-similar properties, described by a cyclic renormalization group flow of the BCS coupling constant which returns to its original value after a finite renormalization group time.

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The renormalization group (RG) continues to be one of the most important tools for studying the qualitative and quantitative properties of quantum field theories and many-body problems in condensed matter physics. The emphasis so far has been mainly on flows toward fixed points in the ultraviolet or the infrared. Recently, an entirely novel kind of RG flow has been discovered in a number of systems wherein the RG exhibits a cyclic behavior: after a *finite* RG transformation the couplings return to their original values and the cycle repeats itself. Thus if one decreases the size of the system by a specific factor that depends on the coupling constants, one recovers the initial system, much like a Russian doll, or quantum version of the Mandelbrot set. Bedaque, Hammer, and van Kolck observed this behavior in a three-body Hamiltonian of interest in nuclear physics.[1] This motivated Glazek and Wilson to define a very simple quantum-mechanical Hamiltonian with similar properties.[2] In the meantime such behavior was proposed for a certain regime of anisotropic current-current interactions in two-dimensional quantum field theory.[3]

The models in Refs. 1 and 2 are problems in zero-dimensional quantum mechanics, and are thus considerably simpler than the quantum-field theory in Ref. 3 In the latter, standard quantum field-theory methods of the renormalization group were used, however knowledge of the β function to all orders was necessary to observe the cyclic flow. What is somewhat surprising is that the model considered in Ref. 3 is not very exotic, and is in fact a well-known theory that arises in many physical problems: at one loop it is nothing more than the famous Kosterlitz-Thouless RG flow, where the cyclic regime corresponds to the usual crossover regime. This motivated us to find a simpler many-body problem that captures the essential features of the cyclic RG behavior. We found that a simple extension of the BCS Hamiltonian has the desirable properties, namely, our model is based on the BCS Hamiltonian with scattering potential $V_{jj'}$ equal to $g+ih$ for $\varepsilon_j>\varepsilon_{j'}$ and $g-ih$ for $\varepsilon_j<\varepsilon_{j'}$ in units of the energy spacing δ.

The main features of the spectrum are the following. For large system size, there are an infinite number of BCS eigenstates, each characterized by an energy gap Δ_n which depends on g and h. The role of these many eigenstates becomes clearer when we investigate the RG properties. As in the models considered in Refs. 1–3, the RG flow possesses jumps from $g=+\infty$ to $g=-\infty$ and a new cycle begins. Let $L=e^{-s}L_0$ denotes the RG scale, which in our problem corresponds to N the number of unperturbed energy levels, and λ the period of RG cycle: $g(e^{-\lambda}L)=g(L)$. We show that $\lambda=\pi/h$.

The model we shall consider is an extension of the reduced BCS model used to describe ultrasmall superconducting grains,[4,5] although our results are valid for more general cases. Let $c^\dagger_{j,\pm}$ $(c_{j,\pm})$ denote creation-annihilation operators for electrons in time-reversal states $|\pm\rangle$. The index $j=1,\ldots,N$ refers to N equally spaced energy levels ε_j with $-\omega<\varepsilon_j<\omega$. The energy ε_j represents the energy of a pair of electrons in a given level. The single-particle level spacing will be denoted by δ, i.e., $\varepsilon_{j+1}-\varepsilon_j=2\delta$, so that $\omega=N\delta$ is twice the Debye energy. Let $b_j=c_{j,-}c_{j,+}$, $b^\dagger_j=c^\dagger_{j,+}c^\dagger_{j,-}$ denote the usual annihilation and creation Cooper-pair operators. Our model is defined by the reduced BCS Hamiltonian

$$H=\sum_{j=1}^{N}\varepsilon_j b^\dagger_j b_j-\sum_{j,j'=1}^{N}V_{jj'}b^\dagger_j b_{j'}, \tag{1}$$

where $V_{jj'}$ is the scattering potential. In the usual BCS model, $V_{jj'}$ is taken to be a constant. Here we add an imaginary part which breaks time reversal,

$$V_{jj'}=\begin{cases}(g+ih)\delta & \text{if} \quad \varepsilon_j>\varepsilon_{j'}\\ g\delta & \text{if} \quad \varepsilon_j=\varepsilon_{j'}\\ (g-ih)\delta & \text{if} \quad \varepsilon_j<\varepsilon_{j'}.\end{cases} \tag{2}$$

This Hamiltonian is Hermitian since $V^*_{jj'}=V_{j'j}$. We consider the positive dimensionless couplings g and h. This is the simplest possible extension of the BCS model that breaks time-reversal symmetry of the pairing interaction. Universality arguments suggest it should arise in a number of ways.

For a large system size N the Hamiltonian (1) can be diagonalized by the grand-canonical BCS variational ansatz

$$|\psi_{\rm BCS}\rangle=\prod_{j=1}^{N}(u_j+v_j b^\dagger_j)|0\rangle. \tag{3}$$

The mean-field method yields the well-known equations

$$u_j^2=\frac{1}{2}\left(1+\frac{\xi_j}{E_j}\right),\quad v_j^2=\frac{1}{2}e^{2i\phi_j}\left(1-\frac{\xi_j}{E_j}\right),$$

$$E_j=\sqrt{\xi_j^2+\Delta_j^2},\quad \xi_j=\varepsilon_j-\mu-V_{jj}, \tag{4}$$

where Δ_j and ϕ_j are the modulus and the phase of the BCS order parameter, $\tilde\Delta_j=\Delta_j e^{i\phi_j}$, which satisfy the gap equation:

$$\tilde\Delta_j=\sum_{j'\neq j}V_{jj'}\frac{\tilde\Delta_{j'}}{E_{j'}},\quad \tilde\Delta_j\equiv\Delta_j e^{i\phi_j}. \tag{5}$$

We shall solve this equation at half filling where the number of electron pairs is half the number of energy levels. Due to particle-hole symmetry the chemical potential equation is satisfied with $\mu=0$. In the thermodynamic limit, $N\to\infty$ and $\delta\to 0$, with fixed $\omega=N\delta$, the sums over ε_j become the integrals $\int_{-\omega}^{\omega} d\varepsilon/2\delta$. The gap equation turns into

$$\tilde{\Delta}(\varepsilon)=g\int_{-\omega}^{\omega}\frac{d\varepsilon'}{2}\frac{\tilde{\Delta}(\varepsilon')}{E(\varepsilon')}+ih\left[\int_{-\omega}^{\varepsilon}-\int_{\varepsilon}^{\omega}\right]\frac{d\varepsilon'}{2}\frac{\tilde{\Delta}(\varepsilon')}{E(\varepsilon')}, \tag{6}$$

where $\tilde{\Delta}(\varepsilon)=\Delta(\varepsilon)e^{i\phi(\varepsilon)}$.

Differentiating Eq. (6) with respect to ε yields

$$\frac{d\phi}{d\varepsilon}=\frac{h}{E(\varepsilon)}, \tag{7}$$

and the condition that $\Delta(\varepsilon)=\Delta$ is independent of ε as for the standard BCS model which corresponds to the case $h=0$. The solution to Eq. (7) can be taken to be

$$\phi(\varepsilon)=h\sinh^{-1}\frac{\varepsilon}{\Delta}, \tag{8}$$

showing that the superconducting order parameter acquires an odd energy dependence phase which is logarithmic, i.e., $\ln(2|\varepsilon|/\Delta)$ for $|\varepsilon|\gg\Delta$.

Using Eq. (8) in the gap equation (6) gives

$$1=\int_0^{\phi(\omega)}\frac{d\phi}{h}(g\cos\phi+h\sin\phi)\Rightarrow\tan\phi(\omega)=\frac{h}{g}. \tag{9}$$

Solving Eq. (9) for the gap yields an infinite number of solutions Δ_n. They can be parametrized as follows:

$$\Delta_n=\frac{\omega}{\sinh t_n},\quad t_n=t_0+\frac{n\pi}{h},\quad n=0,1,2,\ldots, \tag{10}$$

where t_0 is the principal solution to the equation

$$\tan(ht_0)=\frac{h}{g},\quad 0<t_0<\frac{\pi}{2h}. \tag{11}$$

The gaps satisfy $\Delta_0>\Delta_1>\cdots$. Each gap Δ_n represents a different BCS eigenstate $|\psi_{\mathrm{BCS}}^{(n)}\rangle$. One can show that $|\langle\psi_{\mathrm{BCS}}^{(n)}|\psi_{\mathrm{BCS}}^{(n')}\rangle|<\exp[-N(\Delta_n-\Delta_{n'})^2/8\omega^2]$, in the limit where $\Delta_n\ll\omega$. Thus in the large-N limit, these eigenstates are orthogonal and should all appear in the spectrum, together with the usual quasiparticle excitations above them. At zero temperature and no external fields the system will be in its ground state $|\psi_{\mathrm{BCS}}^{(0)}\rangle$, and $|\psi_{\mathrm{BCS}}^{(n>0)}\rangle$ will be excited states. In the limit $h\to 0$ the gaps $\Delta_{n>0}\to 0$, and since $t_0=1/g$, $\Delta_0\sim 2\omega e^{-1/g}$, in the weak-coupling regime, recovering the standard BCS result.

For weak-coupling models $\Delta_n\ll\omega$, all the gaps are related by a scale transformation $\Delta_n\sim 2N\delta e^{-t_0-n\pi/h}$. Therefore, defining the condensation energy of the nth BCS eigenstate as $E_C^{(n)}=\langle\psi_{\mathrm{BCS}}^{(n)}|H|\psi_{\mathrm{BCS}}^{(n)}\rangle-E_{FS}$ we get

$$E_C^{(n)}\sim-\frac{\Delta_n^2}{8\delta}\Rightarrow E_C^{(n)}\sim-\tfrac{1}{2}\delta N^2 e^{-2t_0-2n\pi/h}. \tag{12}$$

Thus the spectrum of condensation energies reflects the scaling behavior of the gaps. At the microscopic level the scaling relation $\Delta_n\sim e^{-n\pi/h}\Delta_0$ implies that the size of the Cooper pairs of the nth solution, given by the correlation length, scales as $\xi_n\sim e^{n\pi/h}\xi_0$.

Next we derive the RG equations for our model. Let g_N, h_N denote the couplings for the Hamiltonian H_N with N energy levels. The idea behind the RG method is to derive an effective Hamiltonian H_{N-1} depending on renormalized couplings g_{N-1}, h_{N-1} by integrating out the highest energy levels ε_N or ε_1. This can be accomplished by a canonical transformation.[6]

We perform the calculation for general couplings $V_{jj'}$. The integration of the level ε_N yields

$$V_{jj'}^{(N-1)}=V_{jj'}^{(N)}+\frac{1}{2}V_{jN}^{(N)}V_{Nj'}^{(N)}\left(\frac{1}{\xi_N-\xi_j}+\frac{1}{\xi_N-\xi_{j'}}\right), \tag{13}$$

where $\xi_j=\varepsilon_j-V_{jj}$. Integration of the level ε_1 gives the same Eq. (13) with the replacement $\xi_N-\xi_j\to-\xi_1+\xi_j$.

Specializing to the potential, Eq. (2) and approximating $\varepsilon_N-\varepsilon_j$ or $-\varepsilon_1+\varepsilon_j$ by $\omega=N\delta$, the above equation implies

$$g_{N-1}=g_N+\frac{1}{N}(g_N^2+h_N^2),\quad h_{N-1}=h_N. \tag{14}$$

Thus h is unrenormalized.

In the large-N limit one can define a variable $s=\ln N_0/N$, where N_0 is the initial size of the system. Then the β function reads

$$\frac{dg}{ds}=(g^2+h^2),\quad s\equiv\ln\frac{N_0}{N}. \tag{15}$$

The solution to the above equation is

$$g(s)=h\tan\left[hs+\tan^{-1}\left(\frac{g_0}{h}\right)\right],\quad g_0=g(N_0). \tag{16}$$

The main features of this RG flow are the cyclicity

$$g(s+\lambda)=g(s)\Leftrightarrow g(e^{-\lambda}N)=g(N),\quad \lambda\equiv\frac{\pi}{h}, \tag{17}$$

and the jumps from $+\infty$ to $-\infty$, when reducing the size.

The cyclicity of the RG has some important implications for the spectrum. Let $\{E(g,h,N)\}$ denotes the energy spectrum of the Hamiltonian H_N. The RG analysis implies we can compute this spectrum using the Hamiltonian $H_{N'}(g(N'))$ if $g(N')$ is related to $g(N)$ according the RG equation (16). Moreover, if N' and N are related by one RG cycle, $N'=e^{-\lambda}N$, then $g(N')=g(N)$. Thus a plot of the spectrum $\{E(g,h,N)\}$ as a function of N but at *fixed* g, h is expected to reveal the cyclicity $\{E(g,h,e^{-\lambda}N)\}=\{E(g,h,N)\}$. Since our RG procedure is not exact, we expect to observe this signature within the range of our approximations, i.e., for $|E|\ll\omega$. Indeed this agrees with the

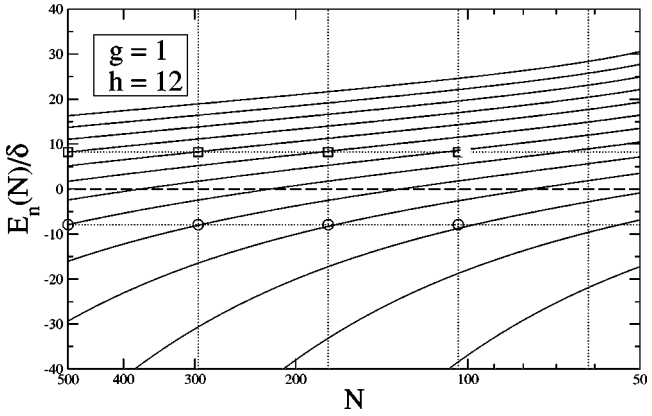

FIG. 1. Exact eigenstates of one-Cooper-pair Hamiltonian for N levels, from $N_0=500$ down to 50. We depict only the states nearest to zero. The vertical lines are at the values $N_n=e^{-n\lambda_1}N_0$. The dotted horizontal lines show the cyclicity of the spectrum.

result shown in Eq. (12). This can also be observed in Fig. 1 for the one-Cooper-pair case, with the cyclicity given by $\lambda_1=2\lambda$ (see below).

Eliminating g_0 in Eq. (16) in terms of the mean-field solution, Eqs. (10) and (11), we observe that the jumps in $g(s)$ from $+\infty$ to $-\infty$ occur at scales $s=t_n$. As N decreases, g increases steadily to $+\infty$ and then jumps to $-\infty$. At $g=+\infty$, $t_0=0$, $t_1=\pi/h,\ldots,$ whereas for $g=-\infty$, $t_0=\pi/h$, $t_1=2\pi/h,\ldots$. Plugging this into Eq. (10), one readily sees that

$$\Delta_0(g=+\infty)=\infty,$$

$$\Delta_{n+1}(g=+\infty)=\Delta_n(g=-\infty), \tag{18}$$

which indicates that at every jump the lowest condensate disappears from the spectrum, since $E_C^{(0)}(g=+\infty)=-\infty$. Equation (18) implies, for the remaining eigenstates, $E_C^{(n+1)}(g=+\infty)=E_C^{(n)}(g=-\infty)$. This result is in agreement with Eq. (12). Therefore, the eigenstate $|\psi_{\mathrm{BCS}}^{(n+1)}\rangle$ of one RG cycle plays the same role as $|\psi_{\mathrm{BCS}}^{(n)}\rangle$ of the next cycle. The blow up of Δ_0 and $E_C^{(0)}$ at $g=+\infty$ is an artifact of the RG scheme used here, since we can only trust the RG for energies below the cutoff ω. However the disappearance of bound states is correctly described by this RG (see the one-Cooper-pair problem for a more detailed discussion).

When $N=\infty$ the infinite number of BCS eigenstates are all expected to appear in the spectrum. However at finite N this is not possible since the Hilbert space is finite dimensional. One can use the RG to estimate the number of eigenstates n_C in the spectrum as a function of N. From the discussion above an eigenstate disappears from the spectrum for each RG cycle. Thus n_C should simply correspond to the number of cycles in $\ln N$:

$$n_C\sim\frac{h}{\pi}\ln N. \tag{19}$$

So far we have found a close relationship between the spectrum of our extended BCS Hamiltonian in the mean-field approximation and the RG flow of the coupling constants. In order to get a further confirmation of our results we should compute the spectrum for a finite-size system. However, it is very difficult to reach intermediate sizes for this model numerically, since the dimension of the Hilbert space grows as $2^N/N^{1/2}$. Fortunately, to this end, the similarities between the many-body case and the case of one Cooper pair, in the presence of the Fermi sea, are widely known.

For one Cooper pair in the presence of the Fermi sea, consider an eigenstate of the form $|\psi\rangle=\Sigma_j\psi_j b_j^\dagger|0\rangle$. The Schrodinger equation reads

$$\frac{1}{\delta}(\varepsilon_j-E)\psi_j=g\psi_j+(g+ih)\sum_{l<j}\psi_l+(g-ih)\sum_{l>j}\psi_l, \tag{20}$$

with $\varepsilon_j\in(0,\omega)$, i.e., the Fermi sea is not accessible for the pair. Then, in the large-N limit, the sums Σ_j are replaced by integrals $\int_0^\omega d\varepsilon/2\delta$, leading to

$$(\varepsilon-E)\psi(\varepsilon)=g\int_0^\omega\frac{d\varepsilon'}{2}\psi(\varepsilon')+ih\left[\int_0^\varepsilon-\int_\varepsilon^\omega\right]\frac{d\varepsilon'}{2}\psi(\varepsilon'). \tag{21}$$

Differentiating the above with respect to ε and integrating, one obtains

$$\psi(\varepsilon)\sim\frac{1}{\varepsilon-E}e^{ih\ln(\varepsilon-E)}. \tag{22}$$

This wave function does not have cuts in two cases: (i) $E<0$ and (ii) $E>\omega$. Case (i) corresponds to the usual Cooper-pair problem where one is looking for bound-state solutions (these are the solutions we claim to have a similar behavior to the many-body case). Plugging Eq. (22) back into Eq. (21) one finds

$$\frac{g+ih}{g-ih}=\left(1-\frac{\omega}{E}\right)^{ih}. \tag{23}$$

Equation (23) has an infinite number of solutions given by

$$E_n=-\frac{\omega}{e^{t_n}-1},\quad t_n=t_0+\frac{2\pi n}{h},\quad n\in\mathcal{Z}, \tag{24}$$

where t_0 is the principal solution to the equation

$$\tan\left(\frac{1}{2}ht_0\right)=\frac{h}{g},\quad 0<t_0<\frac{\pi}{h}. \tag{25}$$

The $n\geqslant 0$ solutions correspond to $E_n<0$, while those with $n<0$ yield $E_n>\omega$.

As for the many-body case the spectrum has a scaling behavior for weak-coupling systems, namely,

$$E_n\sim -N\delta e^{-t_0-2n\pi/h}. \tag{26}$$

A RG analysis similar to the one in the many-body problem leads to the equation

$$g_{N-2}=g_N+\frac{g_N^2+h_N^2}{N-1-g_N},\quad h_{N-1}=h_N, \tag{27}$$

which in the large-N limit becomes

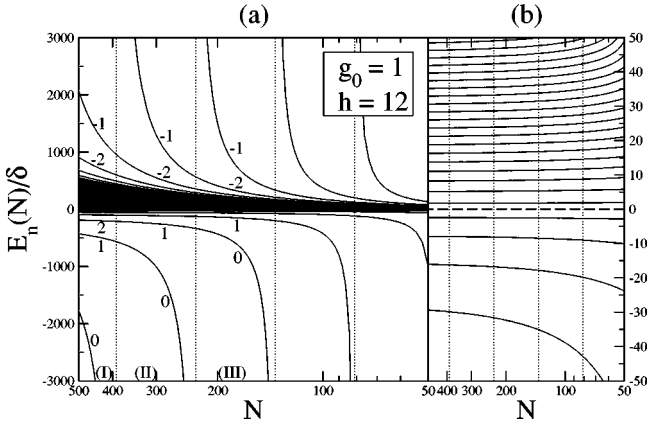

FIG. 2. Eigenstates of one-Cooper-pair Hamiltonian with g_N given by Eq. (27) with $g_0=1$ and $h=12$. The vertical lines denote the positions at which g jumps from $+\infty$ to $-\infty$.

$$\frac{dg}{ds}=\frac{1}{2}(g^2+h^2). \tag{28}$$

The solution to this equation is given by Eq. (16) just by replacing $hs\to hs/2$. This implies that the period of the cyclicity in $s=\ln N_0/N$ is $\lambda_1=2\pi/h$.

The factor 1/2 in the above formula, as compared to Eq. (15), comes from the nonaccessibility of the Cooper pair to the states below the Fermi level.

The discussion leading to Eq. (18) can be repeated for the one-Cooper-pair problem obtaining

$$E_0(g=+\infty)=-\infty,$$

$$E_{n+1}(g=+\infty)=E_n(g=-\infty). \tag{29}$$

Thus we expect that in each RG cycle a bound state will disappear. The analog of Eq. (19) is

$$n_B\sim\frac{h}{2\pi}\ln\frac{N}{2}, \tag{30}$$

where n_B is the number of bound states in the spectrum.

This again shows the agreement between the mean-field and the RG results. We confirm below this picture with numerical calculations. Figure 1 shows the numerical solution of Eq. (20) for $g=1$, $h=12$, and N ranging from 500 down to 50. For each N there are $n_B(N)$ bound states, $E_n<0$, where $n_B(N)$ is in good agreement with Eq. (30).

The spectrum shows the self-similarity found in the approaches above: scaling the system by a factor $e^{-\lambda_1}$, with $\lambda_1=2\pi/h$, one recovers the same spectrum for sufficiently small energies, i.e.,

$$E_{n+1}(N,g,h)=E_n(e^{-\lambda_1}N,g,h). \tag{31}$$

Figure 1 also shows the existence of critical values $N_{c,n}$, in the intervals $(e^{-n\lambda_1}N,e^{-(n+1)\lambda_1}N)$, where the bound state closest to the Fermi level disappears into the "continuum." This effect leads to the reshuffling of bound states, $n+1\to n$, observed in Eq. (31). The critical sizes are also related by scaling, i.e., $N_{c,n}/N_{c,n+1}=e^{-\lambda_1}$. All these phenomena are in good agreement with the RG interpretation we proposed, where the eigenstates disappear at scales where $g(s=t_n)=+\infty$. All these points are related by the scaling factor $e^{-\lambda_1}$.

The RG behavior is presented in Fig. 2, which shows the eigenvalues $E_n(N)$ of the one-Cooper-pair Hamiltonian, with g_N running under Eq. (27). The spectrum remains unchanged for $E_n(N)\ll N\delta$, as shown in Fig. 2(b). In Fig. 2(a) one observes that for the energies $E_n(N)\gtrsim N\delta$ the result of the RG is not reliable. Nevertheless the RG flow describes qualitatively the disappearance of the lowest bound state and the reshuffling of energy levels after a cycle, and furthermore at the predicted scales.

In summary, we have shown that adding to the standard BCS Hamiltonian a time-reversal breaking term, parametrized by a coupling constant h, generates an infinite number of BCS eigenstates with energy gaps Δ_n related, for weak BCS couplings g, by a scale factor $e^{-\lambda}$ with $\lambda=\pi/h$. This unusual spectrum is explained by the cyclic behavior of the RG flow of g, which reproduces itself after a finite RG time s equal to λ. We have also solved the finite-temperature BCS gap equation, obtaining a critical temperature $T_{c,n}$ for the nth eigenstate which is related to the zero-temperature gap $\Delta_n(0)$ exactly as in the BCS theory, i.e., $\Delta_n(0)/T_{c,n}\cong 3.52$ for weak couplings.

The results obtained in this Rapid Communication can be easily extended to a generic BCS model where the pairing transition amplitude has an imaginary term depending on the energy difference between the levels involved. In this sense the cyclicity of the RG and the existence of infinitely many BCS eigenstates are robust features of the proposed model.

An important issue is whether the h interaction has a microscopic origin. Assuming that the *ab initio* Hamiltonian is time-reversal invariant, this would require that time-reversal symmetry is broken explicitly or spontaneously. We hope to report our investigations in this direction in the near future.

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