Ballistic electron motion in a random magnetic field

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Using a scheme of the derivation of the nonlinear σ model we consider the electron motion in a random magnetic field in two dimensions. The derivation is based on writing quasiclassical equations and representing their solutions in terms of a functional integral over supermatrices Q with the constraint $Q^2=1$. Contrary to the standard scheme, neither singling out slow modes nor saddle-point approximations are used. The σ model obtained is applicable at the length scale down to the electron wavelength. We show that this model differs from the model with a random potential. However, after averaging over fluctuations in the Lyapunov region the standard σ model is obtained leading to the conventional localization behavior.

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I. INTRODUCTION

Description of the two-dimensional (2D) electron motion in a random magnetic field (RMF) is of a considerable interest for both experimentalists and theoreticians. Two-dimensional electron systems in a random magnetic field were realized in a number of recent experiments when a high-mobility heterostructure was located under an overlayer with randomly pinned flux vortices in a type-II superconducting gate[1] or type-I superconducting grains[2] or a demagnetized ferromagnet.[3] From the theoretical point of view the RMF model is an example of a system with the interaction which is realized through an effective gauge field. In particular, this model arises in the theory of quantum Hall effect with a half-filled Landau level.[4] Another application of this model is a gauge-field description of the doped Mott insulators.[5]

One of the most important problems in the RMF models is the question about localization of electron states. This question has been studied in many numerical works and very different conclusions were drawn, from (a) all the states are localized, Refs. 6–8, to (b) there may be a band of delocalized states Refs. 9–14 and (c) all the states are localized except those with the precisely zero energy, Refs. 15,16. The problem of comparison of the results obtained in different numerical calculations is a quite complicated task partly because extended states and the states with very large localization length can very often be hardly distinguished from each other.

From the point of view of the generally accepted scaling theory of localization[17] the RMF model should not be different from the model describing the electron motion in a random potential in a homogeneous magnetic field. In both the cases all electron states are expected to be localized in two dimensions in an arbitrarily weak random potential. Using the supersymmetry technique[18] this prediction was checked in several works by deriving a proper σ model. The authors of Ref. 19 used the standard scheme of the derivation finding first the saddle point in the integral over supermatrices Q and expanding then in slow modes near this saddle point. As a result, they obtained a standard diffusive unitary σ model similar to what one has for the model with a random potential (RP) and the broken time-reversal symmetry. The long-range character of correlations of the random vector potential, which is possible even if the correlations of the magnetic field are short ranged, did not play any role.

A possibility of a new term in the σ model due to special character of the correlations of the vector potential was discussed later in Refs. 20–22. This was done by considering more carefully short distances. A ballistic σ model similar to that of Ref. 23 was derived in Refs. 20,22 and the calculations were checked by direct diagrammatic and path-integrals methods.[21] The final conclusion of these works was that the σ model maintained the standard form[19] corresponding to the unitary ensemble unless the correlations of the magnetic field were long ranged. This was considered, as usual, as the proof of the localization. An additional term in the σ model was still possible if the correlation of the magnetic field was proportional to q^{-2}, Ref. 22, where q is the momentum, and this could lead to antilocalization (see also Ref. 24). However, no possibility to obtain anything but the standard unitary σ model and, hence, the localization for any finite range correlations of the magnetic field, was seen finally from these works and no difference between the RMF model and the RP model with a magnetic field was found even in the ballistic case.

Nevertheless, the question about the localization in the RMF model in two-dimensional was raised again in a recent numerical work.[25] On the basis of the numerical study the author of Ref. 25 suggested quite a different scenario of the electron motion in the RMF model, arguing that there could be some ''hidden degrees of freedom'' that lead to essential deviations from the standard scaling description of disordered systems.

This result challenges the analytical results obtained on the basis of the σ-model description but it is fair to say that the previous analytical study was not complete. All calculations were carried out using the traditional form of the ballistic σ model,[22,23,26] with a conventional collision term. However, this form may be used for a long-range disorder at sufficiently long distances only. The derivation of such a σ model is based on finding a saddle point in the integral over the supermatrices Q and expanding in slow modes. This procedure fails at short (but still much exceeding the wave

length λ_F) distances. As a result, the form of ballistic σ model is not applicable at the lengths smaller than a characteristic length $l_L \gg \lambda_F$ and this puts doubts on some conclusions drawn previously.

The saddle-point approximation is equivalent to the self-consistent Born approximation (SCBA) and cannot be good for a long-range disorder. At the same time, even short-range correlations of the magnetic field correspond to long-range correlations of the vector potential and this problem is inevitably encountered in the RMF model. The diagrammatic expansion of Ref. 21 also starts with the SCBA for one-particle Green functions and one encounters the same problem.

In order to circumvent the problem related to the use of the saddle-point approximation and the expansion in the slow modes we suggested recently another scheme.[27] This method is based on equations for quasiclassical Green functions and resembles the phenomenological approach of Ref. 23. However, in contrast to the latter, we do not average over disorder in the beginning of the calculations and do not decouple an effective interaction by integration over an auxiliary field. Our approach is exact in the quasiclassical limit and a resulting ballistic σmodel is applicable at all distances exceeding the wave length λ_F. It can be reduced to the conventional ballistic σ model after a coarse-graining procedure and the latter is applicable at distances exceeding a Lyapunov length l_L introduced in Ref. 28. At distances smaller than l_L the form of the term due to disorder is different from the standard collision term.

In Ref. 27 we derived the ballistic σ model for the RP models and now we present an analogous derivation for the RMF models. It turns out that the terms in the ballistic σ models describing the disorder in the RP and RMF models differ from each other. They can become similar only after carrying out the coarse-graining procedure. We show that this procedure can be performed in the same way as for the RP problem, which leads to a similar reduced σ model.

The paper is organized as follows: In Sec. II we introduce a partition function generating correlation functions of interest in terms of a functional integral over supervectors ψ. We derive equations for Green function and simplify them using a quasiclassical approximation. Introducing quasiclassical Green functions we rewrite the equations in a gauge invariant form. The solution of the equations is found in terms of an integral over supermatrices Q with the constraint $Q^2=1$, which allows us to average over the RMF.

In Sec. III we integrate over fluctuations in the Lyapunov region and come to a reduced σ model with a collision term.

In the Appendix we consider the problem of the correlation of two particles moving in a RMF and find the characteristic time of this correlation.

II. FORMULATION OF THE PROBLEM. QUASICLASSICAL APPROXIMATION AND DERIVATION OF THE σ MODEL

In the present work we follow the method of derivation of the σ model suggested in our previous work.[27] In order to make the presentation self-contained we repeat the main steps of the derivation.

We start our consideration with the introduction of the partition function $Z[\hat{a}]$,

$$Z[\hat{a}]=\int \exp(-L_a[\psi])D\psi, \tag{2.1}$$

$$L_a[\psi]=-i\int \overline{\psi}(\mathbf{r})\left(\hat{H}(\mathbf{r})-\varepsilon+\frac{\omega}{2}+\frac{\omega+i\delta}{2}\Lambda\right)\psi(\mathbf{r})d\mathbf{r}$$

$$+i\int \overline{\psi}(\mathbf{r})\hat{a}(\mathbf{r})\psi(\mathbf{r})d\mathbf{r},$$

where ψ are eight-component supervectors[18] and the Hamiltonian $\hat{H}(\mathbf{r})$ in Eq. (2.1) is taken in the form

$$\hat{H}(\mathbf{r})=\left(-i\nabla_r-\frac{e}{c}\hat{\tau}_3\mathbf{A}(\mathbf{r})\right)^2/2m-\varepsilon_F+u(\mathbf{r}). \tag{2.2}$$

The last term in Eq. (2.1) contains a source function $\hat{a}(\mathbf{r})$. Choosing this function in a proper form and taking derivatives in its elements one can obtain correlation functions. For example, the level-level correlation function $R(\omega)$ can be written as

$$R(\omega)=\frac{1}{2}-\frac{1}{2(\pi\nu V)^2}\lim_{\alpha_1=\alpha_2=0}\mathrm{Re}\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}Z[\hat{a}], \tag{2.3}$$

where the source $\hat{a}(\mathbf{r})$ is the following matrix:

$$\hat{a}(\mathbf{r})=\begin{pmatrix}\hat{\alpha}_1 & 0\\ 0 & -\hat{\alpha}_2\end{pmatrix},\qquad \hat{\alpha}_{1,2}=\frac{\alpha_{1,2}}{2}(1-k). \tag{2.4}$$

Here k is the diagonal matrix with elements ± 1 in fermionic and bosonic blocks, respectively.[18]

The Hamiltonian $H(\mathbf{r})$, Eq. (2.2), contains both scalar and vector potentials $u(\mathbf{r})$, $\mathbf{A}(\mathbf{r})$ that are assumed to be random functions of the space coordinates distributed according to the Gauss law, $\hat{\tau}_3$ is the third Pauli matrix in the particle-hole space. Below we consider a general case when the scalar potential $u(\mathbf{r})$ contains both the short-range $u_s(\mathbf{r})$ and long range $u_l(\mathbf{r})$ parts with the characteristic correlation lengths of the order and larger than the Fermi wavelength $\lambda_F=(2\pi p_F)^{-1}$, respectively. Their statistics are determined by the pair-correlation functions

$$\langle u_s(\mathbf{r})u_s(\mathbf{r}')\rangle=\frac{1}{2\pi\nu\tau_s}\delta(\mathbf{r}-\mathbf{r}'), \tag{2.5}$$

$$\langle u_l(\mathbf{r})u_l(\mathbf{r}')\rangle=W(\mathbf{r}-\mathbf{r}'), \tag{2.6}$$

where the function $W(\mathbf{r}-\mathbf{r}')$ is assumed to fall off over a length $d\gg\lambda_F$. Statistics of the magnetic field will be introduced later. Although the main goal of this paper is to study the RMF model, we add the scalar potential into the Hamiltonian for a more explicit comparison between the RMF and RP models.

Following the standard approach of Ref. 18 one would average the partition function $Z[\hat{a}]$, Eq. (2.1), over the random external fields and then, singling out fluctuations slowly varying in space and integrating over an auxiliary smooth

matrix field Q, decouple the interaction term $(\psi\overline{\psi})^2$ that appears after the averaging. This method was recently used, e.g., in Ref. 29 in a derivation of the ballistic σ model for quantum billiards and in Refs. 20, 22, where the two-dimensional electron gas was considered in a random magnetic field. As it has been mentioned in Sec. I the latter problem is rather specific because the vector potential $\mathbf{A}(\mathbf{r})$ can have long-range correlations even if correlations of the magnetic field are short ranged.

The singling out of slow modes with the subsequent decoupling of the interaction by integrating over an auxiliary smooth matrix Q is not a rigorous procedure because some part of the interaction is assumed to be irrelevant and is neglected. Although this assumption works well for short-range impurities, it is not justified for long-range correlations. Below we use another method based on the Green function and quasiclassical approximation of Ref. 27. This method allows one to derive a σ model applicable down to the length scale of the order of the wavelength λ_F.

Following Ref. 27 we average over the short-range potential $u_s(\mathbf{r})$, decouple the interaction term appearing after this averaging using the standard integration over an auxiliary smooth matrix field $M(\mathbf{r})$, and finally rewrite the partition function as follows:

$$Z[\hat{a}]=\int Z_1[J]\exp\left(-\frac{\pi\nu}{8\tau_s}\int \mathrm{Str}M^2(\mathbf{r})d\mathbf{r}\right)DM, \quad (2.7)$$

where

$$Z_1[J]=\int \exp(-L_J[\psi])D\psi. \quad (2.8)$$

The Lagrangian $L_J[\psi]$ coincides with $L_a[\psi]$, Eq. (2.1), provided the substitutions $u_s(\mathbf{r})=0$ and $i\hat{a}(\mathbf{r})\to J(\mathbf{r})=i\hat{a}(\mathbf{r})+M(\mathbf{r})/2\tau_s$ are made in the Lagrangian $L_a[\psi]$, Eq. (2.1). The structure of the matrix $M(\mathbf{r})$ can be found in, Ref. 18. It is important that $M(\mathbf{r})$ is self-conjugate: $\overline{M}(\mathbf{r})=M(\mathbf{r})$ where the bar means the ''charge conjugation''

$$\overline{M}(\mathbf{r})=CM^T(\mathbf{r})C^T$$

$$C=\Lambda\otimes\begin{pmatrix}c_1 & 0\\ 0 & c_2\end{pmatrix},\quad c_1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad c_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

(see also Ref. 18).

Following Refs. 23, 27 we introduce the Green function $G(\mathbf{r},\mathbf{r}')$,

$$G^{\alpha\beta}(\mathbf{r},\mathbf{r}')=Z_1^{-1}[J]\int \psi^{\alpha}(\mathbf{r})\overline{\psi}^{\beta}(\mathbf{r}')e^{-L_J[\psi]}D\psi. \quad (2.9)$$

For the most correlation functions of interest the source function $\hat{a}(\mathbf{r})$ can be chosen to be self-conjugate. If this is the case the Green function satisfies the equation

$$\left[\hat{H}(\mathbf{r})-\varepsilon+\frac{\omega}{2}+\frac{\omega+i\delta}{2}\Lambda+iJ(\mathbf{r})\right]G(\mathbf{r},\mathbf{r}')=i\delta(\mathbf{r}-\mathbf{r}'). \quad (2.10)$$

Equation (2.10) was previously studied in the absence of the magnetic field in the quasiclassical approximation using a method of a quasiclassical Green function, Refs. 23, 27. This method is based on the assumption that the external fields and sources are smooth functions (i.e., slowly changing over the wavelength λ_F). Within this method the Green function $G(\mathbf{p},\mathbf{R})$ can be rewritten using the Wigner transformation:

$$G(\mathbf{r},\mathbf{r}')=\int \frac{d\mathbf{p}}{(2\pi)^2}e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}G(\mathbf{p},\mathbf{R}),\quad \mathbf{R}=(\mathbf{r}+\mathbf{r}')/2.$$

The function $G(\mathbf{p},\mathbf{R})$ has a sharp peak at the Fermi surface $\mathbf{p}=p_F\mathbf{n}$. This property is due to the fact that the long-range fields and sources weakly disturb the shape of the Fermi surface. Integrating the Green function $G(\mathbf{p},\mathbf{R})$ over the absolute value of the momentum $\mathbf{p}$ results in a new function $g_{\mathbf{n}}(\mathbf{r})$ that depends on the center-of-mass coordinate $\mathbf{R}$ and the unit vector $\mathbf{n}=\mathbf{p}/p$ determining the direction at the Fermi surface. The coordinate dependence of this function turns out to be smooth and therefore $g_{\mathbf{n}}(\mathbf{r})$ may be considered as the quasiclassical approximation of the exact Green function $G(\mathbf{r},\mathbf{r}')$. On the other hand, the partition function $Z_1[J]$, Eq. (2.8), can be expressed through $g_{\mathbf{n}}(\mathbf{r})$.

Before we start the calculation following this procedure let us make some remarks about differences between the RP and RMF models. First, the presence of the magnetic field breaks the time-reversal symmetry and, hence, excitations sensitive to the time reversal are suppressed. Therefore we consider only such correlation functions that can be obtained from the sources $\hat{a}(\mathbf{r})$ commuting with τ_3. The part of the Green function anticommuting with τ_3 is negligible and may be omitted from the further consideration.

The second remark is related to the physical aspects of the quasiclassical approximation in the presence of a magnetic field. It is known that systems placed in a magnetic field are invariant with respect to the magnetic translations $\hat{T}_{\mathbf{a}}=\exp[(\nabla_r-i(e/c)\hat{\tau}_3\mathbf{A})\mathbf{a}]$ instead of the ordinary ones.[31] The difference between these translations is relevant for an infinite system even if the magnetic field is weak. This means that electron states are to be characterized not by the ordinary momentum $\mathbf{p}_{kin}$ determining the kinetic energy but rather by the generalized momentum $\mathbf{p}=\mathbf{p}_{kin}+(e/c)\hat{\tau}_3\mathbf{A}(\mathbf{r})$. The generalized momentum $\mathbf{p}$ is a well-defined quantum number if the magnetic field is weak:

$$r_H\gg\lambda_F,\quad r_H=\frac{v_F}{\omega_H}, \quad (2.11)$$

where $\omega_H=eH/mc$ is the Larmor frequency and v_F Fermi velocity. Inequality (2.11) coincides with the condition of the applicability of the quasiclassical approximation. The Fermi surface is defined in the space of the generalized momentum $\mathbf{p}$ and, contrary to the case of zero magnetic field, has a rather complicated form. The value of the momentum $\mathbf{p}$ at the Fermi surface strongly depends on the direction $\mathbf{n}=\mathbf{p}/p$. Therefore we change the definition of the quasiclassical Green function by replacing the integration over the

absolute value of the generalized momentum $\mathbf{p}$ by that of the kinetic one, $\mathbf{p}_{kin}$ (see, e.g., Ref. 30):

$$g_{\mathbf{n}}(\mathbf{r})=\frac{1}{\pi}\int d\xi G\left(\mathbf{p}+\frac{e}{c}\hat{\tau}_3\mathbf{A}(\mathbf{r}),\mathbf{r}\right), \tag{2.12}$$

where the function $G(\mathbf{p},\mathbf{r})$ is the Green function taken in the Wigner representation and $\xi=\mathbf{p}^2/(2m)-\varepsilon_F$, $\mathbf{n}=\mathbf{p}/p$. The quasiclassical Green function $g_{\mathbf{n}}(\mathbf{r})$ defined by Eq. (2.12) is gauge invariant. The logarithmic derivative of the partition function $Z_1[J]$, Eq. (2.8), can be estimated as follows:

$$\frac{\delta \ln Z_1[J]}{\delta J(\mathbf{r})}=\frac{1}{2}G(\mathbf{r},\mathbf{r})\approx\frac{\pi\nu}{2}\int g_{\mathbf{n}}(\mathbf{r})d\mathbf{n}, \tag{2.13}$$

where ν is the density of states at the Fermi surface. Performing the Wigner transformation we subtract Eq. (2.10) from the conjugated one, then integrate the result over ξ as in Eq. (2.12) and obtain the following in the quasiclassical approximation:

$$\left(v_F\mathbf{n}\nabla_r+\frac{e}{mc}\hat{\tau}_3B(\mathbf{r})\partial_\varphi-p_F^{-1}\nabla_r u(\mathbf{r})\partial_{\mathbf{n}}\right)g_{\mathbf{n}}(\mathbf{r})$$
$$+\frac{i(\omega+i\delta)}{2}[\Lambda,g_{\mathbf{n}}]-[J(\mathbf{r}),g_{\mathbf{n}}]=0. \tag{2.14}$$

In Eq. (2.14), $B(\mathbf{r})=\partial_x A_y-\partial_y A_x$ is the magnetic field, $\partial_{\mathbf{n}}=\nabla_{\mathbf{n}}-\mathbf{n}$, $\nabla_{\mathbf{n}}=\mathbf{e}_\varphi\partial_\varphi$, $\mathbf{n}=(\cos\varphi,\sin\varphi)$, $\mathbf{e}_\varphi=(-\sin\varphi,\cos\varphi)$. In this approximation the solution of the Eq. (2.14) is to be sought with the usual constraint[27]

$$g_{\mathbf{n}}^2(\mathbf{r})=1 \tag{2.15}$$

and the boundary condition

$$g_{\mathbf{n}_\perp}(\mathbf{r})=g_{-\mathbf{n}_\perp}(\mathbf{r})|_{\mathbf{r}\in S}, \tag{2.16}$$

where $\mathbf{r}\in S$ stands for points on the surface of the sample and $\mathbf{n}_\perp$ means the component of the vector $\mathbf{n}$ perpendicular to the surface. Following Ref. 27 we write the solution of Eq. (2.14) in terms of a functional integral over supermatrices $Q_{\mathbf{n}}(\mathbf{r})$:

$$g_{\mathbf{n}}(\mathbf{r})=Z_2^{-1}[J]\int_{Q_{\mathbf{n}}^2=1}Q_{\mathbf{n}}(\mathbf{r})\exp\left(-\frac{\pi\nu}{2}\Phi_J[Q_{\mathbf{n}}]\right)DQ_{\mathbf{n}},$$

$$Z_2[J]=\int_{Q_{\mathbf{n}}^2=1}\exp\left(-\frac{\pi\nu}{2}\Phi_J[Q_{\mathbf{n}}]\right)DQ_{\mathbf{n}}, \tag{2.17}$$

$$\Phi_J[Q_{\mathbf{n}}]=\mathrm{Str}\int d\mathbf{r}d\mathbf{n}\Big[\Lambda\overline{T}_{\mathbf{n}}(\mathbf{r})\Big(v_F\mathbf{n}\nabla_r+\frac{eB(\mathbf{r})}{mc}\hat{\tau}_3\partial_\varphi$$
$$-p_F^{-1}\nabla_r u(\mathbf{r})\nabla_n\Big)T_{\mathbf{n}}(\mathbf{r})$$
$$+\left(\frac{i(\omega+i\delta)}{2}\Lambda-J(\mathbf{r})\right)Q_{\mathbf{n}}(\mathbf{r})\Big], \tag{2.18}$$

$$Q_{\mathbf{n}}(\mathbf{r})=T_{\mathbf{n}}(\mathbf{r})\Lambda\overline{T}_{\mathbf{n}}(\mathbf{r}),\quad \overline{T}_{\mathbf{n}}(\mathbf{r})T_{\mathbf{n}}(\mathbf{r})=1,$$

where ∂_φ stands for the derivative in the angle. The integration in Eq. (2.18) is performed over the self-conjugate supermatrices

$$\overline{Q}_{\mathbf{n}}(\mathbf{r})=Q_{\mathbf{n}}(\mathbf{r}),\quad \overline{Q}_{\mathbf{n}}(\mathbf{r})=CQ^T_{-\mathbf{n}}(\mathbf{r})C^T$$

with the constraint $Q_{\mathbf{n}}^2(\mathbf{r})=1$ and

$$Q_{\mathbf{n}_\perp}(\mathbf{r})|_S=Q_{-\mathbf{n}_\perp}(\mathbf{r})|_S \tag{2.19}$$

at the surface S of the sample. The structure of the supermatrix $Q_{\mathbf{n}}$ coincides with the structure of the supermatrix $M(\mathbf{r})$. We do not demonstrate here the equivalence of the matrices $g_{\mathbf{n}}(\mathbf{r})$, Eqs. (2.12) and (2.18), and refer to the proof given in Ref. 27. We mention here only that both the matrices are the logarithmic derivatives in the matrix $J(\mathbf{r})$ of the partition functions $Z_1[J]$, $Z_2[J]$, respectively. Hence, these functions are equal to each other up to some factor that is independent of $J(\mathbf{r})$. Due to the supersymmetry $Z_1[J]=Z_2[J]=1$ for $J(\mathbf{r})=0$, which means that the factor is unity and the partition functions are equal to each other,

$$Z_1[J]=Z_2[J]. \tag{2.20}$$

Below the magnetic field $B(\mathbf{r})$ is considered as a random function with a Gaussian distribution and the pair-correlation function of the form

$$\langle B(\mathbf{r})B(\mathbf{r}')\rangle=2\left(\frac{mc}{e}\right)^2\omega_c^2W_B(\mathbf{r}-\mathbf{r}'), \tag{2.21}$$

where ω_c is a coefficient that is the characteristic frequency of the cyclotron motion and the function $W_B(\mathbf{r}-\mathbf{r}')$ is assumed to fall off at distances $|\mathbf{r}-\mathbf{r}'|>b$ and to be normalized as $W_B(\mathbf{r}=0)=1$. The length b characterizes the decay of the correlations of the RMF $B(\mathbf{r})$. Substituting Eq. (2.20) into Eq. (2.7) and averaging the result over the magnetic field and long-ranged potential $u_l(\mathbf{r})$ we find for the partition function $Z[\hat{a}]$, Eq. (2.7),

$$Z[\hat{a}]=\int\exp(-F[Q_{\mathbf{n}}])DQ_{\mathbf{n}}, \tag{2.22}$$

where the free-energy functional $F[Q_{\mathbf{n}}]$ has the form

$$F[Q_{\mathbf{n}}]=F_{kin}[Q_{\mathbf{n}}]+F_{imp}[Q_{\mathbf{n}}]+F_{imp}^{(s)}[Q_{\mathbf{n}}]+F_m[Q_{\mathbf{n}}],$$

where

$$F_{kin}[Q_{\mathbf{n}}]=\frac{\pi\nu}{2}\mathrm{Str}\int d\mathbf{r}d\mathbf{n}\Big[\Lambda\overline{T}_{\mathbf{n}}(\mathbf{r})v_F\mathbf{n}\nabla_r T_{\mathbf{n}}(\mathbf{r})$$
$$+i\left(\frac{\omega+i\delta}{2}\Lambda-\hat{a}\right)Q_{\mathbf{n}}(\mathbf{r})\Big],$$

$$F_{imp}[Q_{\mathbf{n}}]=-\frac{1}{8}\left(\frac{\pi\nu}{p_F}\right)^2\int d\mathbf{r}d\mathbf{n}d\mathbf{r}'d\mathbf{n}'\nabla_{\mathbf{r}}^i\nabla_{\mathbf{r}'}^j W(\mathbf{r}-\mathbf{r}')$$
$$\times\mathrm{Str}[\Lambda\overline{T}_{\mathbf{n}}(\mathbf{r})\nabla_{\mathbf{n}}^i T_{\mathbf{n}}(\mathbf{r})]$$
$$\times\mathrm{Str}[\Lambda\overline{T}_{\mathbf{n}'}(\mathbf{r}')\nabla_{\mathbf{n}'}^j T_{\mathbf{n}'}(\mathbf{r}')], \tag{2.23}$$

$$F_{imp}^{(s)}[Q_{\mathbf{n}}]=-\frac{\pi\nu}{8\tau_s}\int \mathrm{Str}\left(\int Q_{\mathbf{n}}(\mathbf{r})d\mathbf{n}\right)^2 d\mathbf{r},$$

$$F_m[Q_{\mathbf{n}}]=\left(\frac{\pi\nu}{2}\omega_c\right)^2\int d\mathbf{r}d\mathbf{n}d\mathbf{r}'d\mathbf{n}'W_B(\mathbf{r}-\mathbf{r}')\times\mathrm{Str}[\Lambda\hat{\tau}_3\overline{T}_{\mathbf{n}}(\mathbf{r})i\partial_\varphi T_{\mathbf{n}}(\mathbf{r})]\times\mathrm{Str}[\Lambda\hat{\tau}_3\overline{T}_{\mathbf{n}'}(\mathbf{r}')i\partial_{\varphi'}T_{\mathbf{n}'}(\mathbf{r}')].$$

The first term $F_{kin}[Q_{\mathbf{n}}]$ describes the free motion and is what remains when external fields and impurities are absent. The second and the third terms $F_{imp}[Q_{\mathbf{n}}]$, $F_{imp}^{(s)}[Q_{\mathbf{n}}]$ are responsible for the scattering on the long- and short-ranged potentials, respectively. The last term $F_m[Q_{\mathbf{n}}]$ is due to the presence of the random magnetic field. Correlation functions of interest can be obtained by calculating derivatives in the source $\hat{a}(\mathbf{r})$ of the partition function $Z[\hat{a}]$, Eq. (2.22).

It is important to emphasize that the structure of the terms $F_{imp}[Q_{\mathbf{n}}]$ and $F_m[Q_{\mathbf{n}}]$ describing the electron scattering on the random potential and on the random magnetic field, respectively, is clearly different. The term $F_{imp}[Q_{\mathbf{n}}]$ contains the components of the gradients parallel to the plane, whereas the term $F_m[Q_{\mathbf{n}}]$ contains the perpendicular one.

Nevertheless, at longer distances the RP and RMF models are very similar and we show this in the following section, carrying out a coarse-graining procedure suggested in Ref. 27. The latter means integrating out degrees of freedom at distances inside the Lyapunov region.

For simplicity of the presentation we will consider in the following sections only effects related to the random magnetic field and disregard the scattering on the random potentials, omitting $F_{imp}[Q_{\mathbf{n}}]$, $F_{imp}^{(s)}[Q_{\mathbf{n}}]$ in the free energy, Eq. (2.23). Accordingly, we will consider the symmetry of the supermatrices Q corresponding to the unitary ensemble. We will study the behavior of the σ model, Eq. (2.23), on different length scales and discuss the connection of this model with the models previously obtained in Refs. 19–22.

III. REDUCED σ MODEL

The σ model obtained in Eq. (2.23) is valid for the length scales down to the wavelength λ_F and has the form which differs from the σ model found in Refs. 20, 22. The latter model has been derived for the spatially uncorrelated magnetic field and is applicable at the length scale restricted from below by the single-particle relaxation length l but not by the wavelength λ_F. The length l could not be consistently estimated within the consideration of Refs. 20, 22 and remained without a clear physical interpretation. At the same time, the analysis of Refs. 27, 28, 32, leads to the conclusion that the role of this length is played by the Lyapunov length $l_L=v_F\tau_L$. Here τ_L is the inverse Lyapunov exponent and is the time during which two close trajectories increase the distance between them by a factor of the order of unity. On the other hand, according to Ref. 28, τ_L is the time which is required for two scattered particles to diverge over the distance of the order of the range of the potential (or the correlation length). In the Appendix we discuss the problem of the particle motion in a RMF and estimate the Lyapunov length l_L for weak fields as

$$l_L\sim l_{tr}\left(\frac{b}{l_{tr}}\right)^{2/3}. \tag{3.1}$$

This result shows that the Lyapunov length l_L is between the correlation b and transport l_{tr} lengths: $b\ll l_L\ll l_{tr}$.

The Lyapunov length l_L divides the length scales into two regions. At small distances, two particles propagate in the same magnetic field and correlations between them are relevant. Following the terminology of Ref. 28 we call these distances the Lyapunov region. In the second region, when the scales of interest are larger than the Lyapunov length, the motion of the particles is not correlated and they are scattered by the RMF independently. This can be called the collision region because the corresponding classical motion at such distances is described by the conventional Boltzmann equation with a collision term corresponding to the scattering on the RMF. The electron motion at these long distances should be described by a reduced σ model and one can expect that this reduced σ-model is just the σ model of Refs. 20, 22. In order to obtain the reduced σ model one should integrate out in Eqs. (2.22) and (2.23) the degrees of freedom related to the Lyapunov region. This coarse-graining procedure has been worked out in Ref. 27 for the RP model and we will repeat it now for the RMF model.

First, one should explicitly decouple the original mode $T_{\mathbf{n}}(\mathbf{r})$ into the ''slow'' and ''fast'' parts. We make this separation in the way preserving the rotational invariance of the initial model Eq. (2.23):

$$T_{\mathbf{n}}(\mathbf{r})=\tilde{T}_{\mathbf{n}}(\mathbf{r})V_{\mathbf{n}}(\mathbf{r}). \tag{3.2}$$

Here $\tilde{T}_{\mathbf{n}}(\mathbf{r})$, $V_{\mathbf{n}}(\mathbf{r})$ are slow and fast modes describing the fluctuations in the collision and Lyapunov regions, respectively. As soon as the mode separation is made one should substitute Eq. (3.2) into the free energy $F[Q_{\mathbf{n}}]$, Eq. (2.23), and then average it over the fast fluctuations $V_{\mathbf{n}}(\mathbf{r})$:

$$Z[\hat{a}]=\int_{\tilde{Q}_{\mathbf{n}}^2=1} e^{-F_{eff}[\tilde{Q}_{\mathbf{n}}]}D\tilde{Q}_{\mathbf{n}}, \tag{3.3}$$

where

$$e^{-F_{eff}[\tilde{Q}_{\mathbf{n}}]}=\int \exp(-F[Q_{\mathbf{n}}^{(0)}]-F_{int}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}])DV_{\mathbf{n}}, \tag{3.4}$$

$$Q_{\mathbf{n}}^{(0)}(\mathbf{r})=V_{\mathbf{n}}(\mathbf{r})\Lambda\overline{V}_{\mathbf{n}}(\mathbf{r}),\quad \tilde{Q}_{\mathbf{n}}(\mathbf{r})=\tilde{T}_{\mathbf{n}}(\mathbf{r})\Lambda\overline{\tilde{T}}_{\mathbf{n}}(\mathbf{r}).$$

The functional $F[Q_{\mathbf{n}}^{(0)}]$ in Eq. (3.4) coincides with the free energy, Eq. (2.23), provided the source is omitted in the latter

expression. The functional $F_{int}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]$ determines the interaction between the fast and slow modes $Q_{\mathbf{n}}^{(0)}$, $\tilde{Q}_{\mathbf{n}}$ and has the form

$$F_{int}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]=F'_{kin}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]+F'_{m}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}],$$

where

$$\begin{aligned}F'_{kin}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]=\frac{\pi\nu}{2}\mathrm{Str}\int d\mathbf{r}d\mathbf{n}Q_{\mathbf{n}}^{(0)}(\mathbf{r})\Big[\overline{\tilde{T}}_{\mathbf{n}}(\mathbf{r})v_F\mathbf{n}\nabla_{\mathbf{r}}\tilde{T}_{\mathbf{n}}(\mathbf{r})\\ +i\frac{\omega+i\delta}{2}(\overline{\tilde{T}}_{\mathbf{n}}(\mathbf{r})\Lambda\tilde{T}_{\mathbf{n}}(\mathbf{r})-\Lambda\\ -i\overline{\tilde{T}}_{\mathbf{n}}(\mathbf{r})\hat{a}(\mathbf{r})\tilde{T}_{\mathbf{n}}(\mathbf{r}),\end{aligned}$$

$$\begin{aligned}F'_{m}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]=\left(\frac{\pi\nu}{2}\omega_c\right)^2\int d\mathbf{r}d\mathbf{n}d\mathbf{r}'d\mathbf{n}'W_B(\mathbf{r}-\mathbf{r}')\\ \times\mathrm{Str}[\hat{\tau}_3Q_{\mathbf{n}}^{(0)}(\mathbf{r})\Phi_{\mathbf{n}}(\mathbf{r})]\\ \times\mathrm{Str}[\hat{\tau}_3Q_{\mathbf{n}'}^{(0)}(\mathbf{r}')\Phi_{\mathbf{n}'}(\mathbf{r}')]\\ +2\left(\frac{\pi\nu}{2}\omega_c\right)^2\int d\mathbf{r}d\mathbf{n}d\mathbf{r}'d\mathbf{n}'W_B(\mathbf{r}-\mathbf{r}')\\ \times\mathrm{Str}[\hat{\tau}_3Q_{\mathbf{n}}^{(0)}(\mathbf{r})\Phi_{\mathbf{n}}(\mathbf{r})]\\ \times\mathrm{Str}[\Lambda\hat{\tau}_3\overline{V}_{\mathbf{n}'}(\mathbf{r}')i\partial_{\varphi'}V_{\mathbf{n}'}(\mathbf{r}')],\end{aligned}\tag{3.5}$$

$$\Phi_{\mathbf{n}}(\mathbf{r})=\overline{\tilde{T}}_{\mathbf{n}}(\mathbf{r})i\partial_{\varphi}\tilde{T}_{\mathbf{n}}(\mathbf{r}).$$

Before the averaging over the fast fluctuations $Q_{\mathbf{n}}^{(0)}$ we make the following essential remark.

The separation into the fast and slow modes, Eq. (3.2), requires a more accurate definition. The point is that the excitations in the model, Eq. (2.23), reveal a strong anisotropy in the phase space $(\mathbf{r},\mathbf{n})$ due to the specific form of the free-energy functional, Eq. (2.23). Since only the first-order derivatives in $\mathbf{r}$ and $\mathbf{n}$ enter the free energy, Eq. (2.23), the dependence of the excitations on the coordinates $(\mathbf{r},\mathbf{n})$ will resemble a propagation along a classical trajectory. Such an anisotropy demands a care and should be performed in an invariant way. As in Ref. 27, the scale separation can be performed introducing an additional term into the functional $F[Q_{\mathbf{n}}^{(0)}]$, Eq. (3.4),

$$F_L[Q_{\mathbf{n}}^{(0)}]=-\frac{\pi\nu}{4}\lambda_L\mathrm{Str}\int d\mathbf{r}d\mathbf{n}\Lambda Q_{\mathbf{n}}^{(0)}(\mathbf{r}).\tag{3.6}$$

Then, we extend the region of the integration over $Q_{\mathbf{n}}^{(0)}(\mathbf{r})$ to all possible matrices with the constraints Eq. (2.19). The parameter λ_L is just the Lyapunov exponent τ_L^{-1} and the term $F_L[Q_{\mathbf{n}}^{(0)}]$, Eq. (3.6), serves to suppress fluctuations of the matrices $Q_{\mathbf{n}}^{(0)}$ outside the Lyapunov region.

As soon as the mode separation is properly defined one can carry out the integration in Eq. (3.4) and evaluate the effective energy $F_{eff}[\tilde{Q}_{\mathbf{n}}]$. We perform this computation using the cumulant expansion in F_{int}, Eq. (3.4), and approximation of the weak magnetic field. In the same way as it was done in Ref. 27 for the model of the long-ranged disorder one can show that this is an expansion in powers of the operator $l_L\nabla_r$ which is small outside the Lyapunov region. Considering only the first order we find

$$F_{eff}[\tilde{Q}_{\mathbf{n}}]=\langle F_{int}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]\rangle_0,\tag{3.7}$$

where the brackets $\langle\cdots\rangle_0$ stand for integration over $Q_{\mathbf{n}}^{(0)}$. Due to the supersymmetry $\langle Q_{\mathbf{n}}^{(0)}(\mathbf{r})\rangle_0=\Lambda$, which gives

$$\langle F'_{kin}[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]\rangle_0=F_{kin}[\tilde{Q}_{\mathbf{n}}]\tag{3.8}$$

with the same functional $F_{kin}[\tilde{Q}_{\mathbf{n}}]$ as in Eq. (2.23). The second term in the functional $F'_m[Q_{\mathbf{n}}^{(0)},\tilde{Q}_{\mathbf{n}}]$, Eq. (3.5), vanishes after the averaging due to the symmetry as well. The contribution coming from the first term can be divided into two parts: the first one comes from the reducible average and coincides with the magnetic energy $F_m[\tilde{Q}_{\mathbf{n}}]$ of the initial functional, Eq. (2.23), whereas the other is given by the irreducible average $\langle\langle Q_{\mathbf{n}}^{(0)}Q_{\mathbf{n}'}^{(0)}\rangle\rangle_0=\langle Q_{\mathbf{n}}^{(0)}Q_{\mathbf{n}'}^{(0)}\rangle_0-\langle Q_{\mathbf{n}}^{(0)}\rangle_0\langle Q_{\mathbf{n}'}^{(0)}\rangle_0$ of the supermatrices $Q_{\mathbf{n}}^{(0)}$.

In order to find the contribution coming from the irreducible average we consider the matrix

$$\tilde{g}_{\mathbf{n}_1}(\mathbf{r}_1;\alpha)=\frac{\left\langle Q_{\mathbf{n}_1}^{(0)}(\mathbf{r}_1)\exp\left[\frac{\pi\nu}{2}\mathrm{Str}\int d\mathbf{r}d\mathbf{n}\hat{a}_{\mathbf{n}}(\mathbf{r})Q_{\mathbf{n}}^{(0)}(\mathbf{r})\right]\right\rangle_0}{\left\langle\exp\left[\frac{\pi\nu}{2}\mathrm{Str}\int d\mathbf{r}d\mathbf{n}\hat{a}_{\mathbf{n}}(\mathbf{r})Q_{\mathbf{n}}^{(0)}(\mathbf{r})\right]\right\rangle_0},\tag{3.9}$$

where the new source $\hat{a}_{\mathbf{n}}(\mathbf{r})$ is

$$\hat{a}_{\mathbf{n}}(\mathbf{r})=\alpha(\mathbf{r})\hat{\tau}_3\Phi_{\mathbf{n}}(\mathbf{r}),$$

$\alpha(\mathbf{r})$ is some function. Due to the supersymmetry $\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha=0)=\Lambda$. The first derivative in the function $\alpha(\mathbf{r})$ gives

$$\begin{aligned}\left.\frac{\delta\tilde{g}_{\mathbf{n}_1}(\mathbf{r}_1;\alpha)}{\delta\alpha(\mathbf{r}_2)}\right|_{\alpha(\mathbf{r})=0}\\ =\frac{\pi\nu}{2}\left\langle\left\langle Q_{\mathbf{n}_1}^{(0)}(\mathbf{r}_1)\mathrm{Str}\int d\mathbf{n}'Q_{\mathbf{n}'}^{(0)}(\mathbf{r}_2)\hat{\tau}_3\Phi_{\mathbf{n}'}(\mathbf{r}_2)\right\rangle\right\rangle_0.\end{aligned}\tag{3.10}$$

On the other hand, the matrix $\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha)$ satisfies the equation

$$\begin{aligned}v_F\mathbf{n}\nabla_r\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha)+i\frac{\omega+i\lambda_L}{2}[\Lambda,\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha)]\\ =\alpha(\mathbf{r})[\hat{\tau}_3\Phi_{\mathbf{n}}(\mathbf{r}),\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha)]\end{aligned}\tag{3.11}$$

and condition $\tilde{g}_{\mathbf{n}}^2(\mathbf{r};\alpha)=1$. Differentiating in $\alpha(\mathbf{r})$ both sides of this condition and then putting $\alpha(\mathbf{r})=0$ we find that the matrix $\delta\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha)/\delta\alpha(\mathbf{r}')|_{\alpha=0}$ in Eq. (3.10) is off diagonal. Equation (3.11) can be considered for the off-diagonal part of the matrix $\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha)$ and rewritten in the integral form

$$\tilde{g}^{\perp}_{\mathbf{n}}(\mathbf{r};\alpha)=\int d\mathbf{r}'\mathcal{G}_{\mathbf{n}}(\mathbf{r}-\mathbf{r}')\alpha(\mathbf{r}')[\hat{\tau}_3\Phi_{\mathbf{n}}(\mathbf{r}),\tilde{g}_{\mathbf{n}}(\mathbf{r};\alpha)]^{\perp}, \tag{3.12}$$

where the superscript $\perp$ stands for the part of the supermatrices anticommuting with Λ. The kernel $\mathcal{G}_{\mathbf{n}}(\mathbf{r}-\mathbf{r}')$ is the solution of the equation

$$[v_F\mathbf{n}\nabla_r+i(\omega+i\lambda_L)\Lambda]\mathcal{G}_{\mathbf{n}}(\mathbf{r}-\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \tag{3.13}$$

Differentiating in $\alpha(\mathbf{r})$ both sides of Eq. (3.12) and putting $\alpha(\mathbf{r})=0$ we obtain

$$\frac{\pi\nu}{2}\left\langle\left\langle Q^{(0)}_{\mathbf{n}_1}(\mathbf{r}_1)\mathrm{Str}\int d\mathbf{n}'Q^{(0)}_{\mathbf{n}'}(\mathbf{r}_2)\hat{\tau}_3\Phi_{\mathbf{n}'}(\mathbf{r}_2)\right\rangle\right\rangle_0 =\mathcal{G}_{\mathbf{n}_1}(\mathbf{r}_1-\mathbf{r}_2)[\hat{\tau}_3\Phi_{\mathbf{n}_1}(\mathbf{r}_2),\Lambda]. \tag{3.14}$$

Substitution of Eq. (3.14) into Eq. (3.5) gives

$$\langle F'_m[\tilde{Q}_{\mathbf{n}},Q^{(0)}_{\mathbf{n}}]\rangle_0=F_m[\tilde{Q}_{\mathbf{n}}]-\pi\nu\omega_c^2\int d\mathbf{r}d\mathbf{r}'d\mathbf{n}W_B(\mathbf{r}-\mathbf{r}') \times\mathrm{Str}[\Phi^{\perp}_{\mathbf{n}}(\mathbf{r})\mathcal{G}_{\mathbf{n}}(\mathbf{r}-\mathbf{r}')\Lambda\Phi^{\perp}_{\mathbf{n}}(\mathbf{r}')]. \tag{3.15}$$

Characteristic values of the difference $\mathbf{r}-\mathbf{r}'$ in $\mathcal{G}_{\mathbf{n}}(\mathbf{r}-\mathbf{r}')$ are in the Lyapunov region, whereas $\Phi_{\mathbf{n}}(\mathbf{r})$ is a smooth function. This allows us to make the replacement $\mathbf{r}'\to\mathbf{r}$ in one of the $\Phi_{\mathbf{n}}$ in Eq. (3.15). The integral over the difference $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}'$ is calculated as follows. First, we rewrite this integral using integration in the momentum space instead of the coordinate one,

$$\int \mathcal{G}_{\mathbf{n}}(\boldsymbol{\rho})W_B(\boldsymbol{\rho})d\boldsymbol{\rho}=\int\frac{d\mathbf{q}}{(2\pi)^2}W_B(\mathbf{q})\frac{i}{v_F\mathbf{n}\mathbf{q}-(\omega+i\lambda_L)\Lambda}. \tag{3.16}$$

The momentum $\mathbf{q}$ may be considered as the transfer momentum $\mathbf{q}=\mathbf{p}'-\mathbf{p}$, where $\mathbf{p}'=p_F\mathbf{n}'$, $\mathbf{p}=p_F\mathbf{n}$ are momenta of a particle after and before the scattering. Since for a weak scattering the characteristic length b of the distribution $W_B(\mathbf{r}-\mathbf{r}')$ is much smaller than the Lyapunov length, $b\ll l_L$, Eq. (3.1), the fraction in Eq. (3.16) can be replaced by the δ function:

$$\int\frac{d\mathbf{q}}{(2\pi)^2}W_B(\mathbf{q})\frac{i}{v_F\mathbf{n}\mathbf{q}-(\omega+i\lambda_L)\Lambda} \approx-\pi\Lambda\int\frac{d\mathbf{q}}{(2\pi)^2}W_B(\mathbf{q})\delta(v_F\mathbf{n}\mathbf{q}). \tag{3.17}$$

The δ function fixes the value of the final momentum $\mathbf{p}'$ on the Fermi surface: $\delta(v_F\mathbf{n}\mathbf{q})=\delta[v_F\mathbf{n}(\mathbf{p}'-\mathbf{p})]=\delta[(\partial\varepsilon/\partial\mathbf{p})\times(\mathbf{p}'-\mathbf{p})]=\delta[\varepsilon(\mathbf{p}')-\varepsilon(\mathbf{p})]$. Integrating over the energy $\varepsilon'\equiv\varepsilon(\mathbf{p}')$ we find for the integral, Eq. (3.16), the following expression:

$$-\pi\nu\Lambda\int d\mathbf{n}'W_B[p_F(\mathbf{n}-\mathbf{n}')]. \tag{3.18}$$

Taking together Eqs. (3.15), (3.16), and (3.18) we obtain the free energy $F_{eff}[Q_{\mathbf{n}}]$ of the reduced σ model:

$$F_{eff}[Q_{\mathbf{n}}]=F[Q_{\mathbf{n}}]+F'[Q_{\mathbf{n}}], \tag{3.19}$$

where

$$F[Q_{\mathbf{n}}]=\frac{\pi\nu}{2}\int d\mathbf{r}d\mathbf{n}\mathrm{Str}\left[\overline{T}_{\mathbf{n}}(\mathbf{r})v_F\mathbf{n}\nabla_r T_{\mathbf{n}}(\mathbf{r}) +i\left(\frac{\omega+i\delta}{2}\Lambda-\hat{a}(\mathbf{r})\right)Q_{\mathbf{n}}(\mathbf{r})+\frac{1}{4\tau_{tr}}(\partial_\varphi Q_{\mathbf{n}})^2\right] \tag{3.20}$$

$$F'[Q_{\mathbf{n}}]=-\left(\frac{\pi\nu}{2}\omega_c\right)^2\int d\mathbf{r}d\mathbf{n}d\mathbf{r}'d\mathbf{n}'W_B(\mathbf{r}-\mathbf{r}') \times\mathrm{Str}[\Lambda\hat{\tau}_3\overline{T}_{\mathbf{n}}(\mathbf{r})\partial_\varphi T_{\mathbf{n}}(\mathbf{r})] \times\mathrm{Str}[\Lambda\hat{\tau}_3\overline{T}_{\mathbf{n}'}(\mathbf{r}')\partial_{\varphi'}T_{\mathbf{n}'}(\mathbf{r}')]. \tag{3.21}$$

The collision term in the free-energy functional is expressed through the transport time τ_{tr},

$$(2\pi\nu\tau_{tr})^{-1}=\int d\mathbf{n}'\omega_c^2W_B[p_F(\mathbf{n}'-\mathbf{n})], \tag{3.22}$$

and agrees with the results of Refs. 20, 22, 27, where the RMF and long-range disorder models, respectively, were considered in the limit of small scattering angles. The second term $F'[Q_{\mathbf{n}}]$ in Eq. (3.19) is small and can be neglected. This can be easily understood using the fact that the Fourier transform of the function W_B in Eq. (3.22) contains momenta of the order of p_F, which corresponds to short distances of the order of λ_F. In contrast, the main contribution to the integral over the coordinates in Eq. (3.21) comes at weak RMF from larger distances of order l_L where the function W_B is small. Therefore, everywhere below we will imply that the reduced ballistic σ model is described by the free-energy functional $F[Q_{\mathbf{n}}]$ from Eq. (3.20).

Thus, we have demonstrated that, although the ballistic σ model for the RMF is different from the one for the RP (the terms $F_{imp}[Q_{\mathbf{n}}]$ and $F_m[Q_{\mathbf{n}}]$ in Eq. (2.23) are different), the reduced σ models describing the electron motion exceeding the Lyapunov length l_L have the same form as Eq. (3.20). The similarity of the RMF and RP models has been emphasized in Ref. 21 and the final conclusion of Ref. 22 was the same. However, the methods used in these works were based on writing first the self-consistent Born approximation for one-particle Green functions (saddle-point equation in the σ model formulation) and on a subsequent expansion in slow modes, which could not be justified at short distances. Now we see that the equivalence of the RMF and RP models can hold at distances exceeding the Lyapunov length. This naturally leads to the equivalence of the diffusive σ models that can be written in the standard form

$$F[Q]=\frac{\pi\nu}{8}\mathrm{Str}\int[D(\nabla Q)^2+2i(\omega+i\delta)\Lambda Q]d\mathbf{r}, \tag{3.23}$$

where $D=v_F^2\tau_{tr}/2$. For the RMF problem the transport time τ_{tr} is given by Eq. (3.22).

Equation (3.23) is valid unless the correlations of the magnetic field are very long ranged. Only if

$$\langle B_q B_{-q}\rangle \sim q^{-2}, \tag{3.24}$$

an additional term can appear.[22] The symmetry of the diffusive σ model, Eq. (3.23), corresponds to the unitary ensemble and one comes to the standard conclusion about the localization.

Of course, the coarse-graining procedure leading to the ballistic σ model, Eq. (3.20), is possible only if the ground state of the initial σ model, Eq. (2.23), is achieved at $Q=\Lambda$. One can imagine such functions $W_B(\mathbf{r}-\mathbf{r}')$ that this ground state is no longer stable. However, this could be possible only if the Fourier transform $W_B(\mathbf{q})$ was negative for certain $\mathbf{q}$, which is excluded in the case of real magnetic fields. Therefore, beyond the Lyapunov region, the ballistic σ model, Eq. (3.20), and, correspondingly, the diffusive σ model, Eq. (3.23), seem to be unavoidable.

IV. DISCUSSION

In the present paper we considered the problem of the two-dimensional electron gas in a random magnetic field (RMF) using the nonlinear supermatrix σ model approach. We derived a ballistic σ model avoiding the standard scheme based on finding a saddle point in the integral over supervectors and expanding in slow modes near this point. Such a scheme explicitly relies on the assumption of a sufficiently short correlation length of a random potential (see, e.g., in Ref. 18) and its validity for a long-range disorder is not clear. As the vector potential entering the RMF model has a large correlation length even when the magnetic field is δ correlated in space, the procedure of singling out slow modes used in the standard derivation is not well justified at least at not very large distances. Besides, the saddle-point approximation is hardly allowed in this case as well.

Instead of following the standard scheme we used the method based on writing quasiclassical equations for Green functions and the exact representation of their solutions in terms of integrals over supermatrices $Q_{\mathbf{n}}$ with the constraint $Q_{\mathbf{n}}^2=1$. This method needs neither singling out the fast and slow parts from the interaction nor the saddle-point approximation. Conditions of the applicability of the method coincide with those of the quasiclassical approximation. Therefore, the σ model obtained should be applicable over the distances down to the Fermi wavelength, which makes it more general in comparison with the σmodels derived earlier on the basis of the standard scheme, Refs. 19,20,22,26. The latter models are justified at distances exceeding the single-particle mean free path l as in the Refs. 20,22 or the transport length l_{tr} as in Ref. 19.

We have demonstrated that similar to the problem of long-range random potential, there is a characteristic or Lyapunov length l_L dividing the length scale into the Lyapunov and collision regions. The first region corresponds to the small distances over which the particle motion is strongly correlated. Correlations disappear over the larger lengths where the particle interaction can be considered in terms of collisions. In the Appendix we estimate the Lyapunov length for RMF problem, restricting our consideration by the limit of a weak field. The estimated length is expressed through the transport length l_{tr} and the correlation length b of the RMF by a formula similar to the one obtained previously in Ref. 28 in the model of a long-ranged potential.

The reduced σ model obtained after integrating over the fluctuations in the Lyapunov region coincides with the model of Ref. 22 provided the latter is considered in the limit of a small-angle weak scattering. The reduced σ model obtained in this way is equivalent to the model found in the problem of a long-range potential disorder, Ref. 27. At the same time, it is relevant to emphasize that at short distances inside the Lyapunov region the RMF and RP models correspond to different σ models.

At distances exceeding the transport length $l_{tr}=v_F\tau_{tr}$ one comes to the standard diffusion σ model, Eq. (3.23), unless the correlation of the magnetic fields obeys Eq. (3.24). Calculations for the σ model, Eq. (3.23), within the renormalization-group scheme leads to the standard conclusion about the localization. This conclusion is in contradiction with the numerical results of Ref. 25 where the existence of hidden degrees of freedom was proposed, which could lead to the existence of extended states. We did not find any indication for such degrees of freedom. Of course, our consideration was performed in the quasiclassical limit, such that we did not take into account a possibility of a quantization of the energy levels. However, it is not easy to understand how taking into account distances shorter than the wavelength λ_F could lead to a destruction of the localization.

APPENDIX: LYAPUNOV EXPONENT IN RMF PROBLEM

Here we study the classical scattering of two particles in a random magnetic field (RMF). The presence of the RMF leads to an effective interaction between the particles. The radius of this interaction is equal to the correlation length of the field. The scattering process lasts a finite time after which the particles diverge over the distance exceeding the correlation length and begin to move without any interaction. The aim of the calculation presented below is to estimate this time. It is clear that for larger times the particle scattering may be considered in terms of collisions. We restrict our calculation by the case of a weak magnetic field.

Let us consider two particles on a plane with the coordinates $\mathbf{r}_1$, $\mathbf{r}_2$ and momenta $\mathbf{p}_1$, $\mathbf{p}_2$ moving in a perpendicular magnetic field. The equations of the motion for each particle are

$$\dot{\mathbf{r}}_i=\frac{\mathbf{p}_i}{m}, \quad \dot{\mathbf{p}}_i=\frac{eB(\mathbf{r}_i)}{mc}[\mathbf{p}_i\times\hat{e}_z], \tag{A1}$$

where $\hat{e}_z$ is the unit vector perpendicular to the plane of the motion. Let $\boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{p}=\mathbf{p}_1-\mathbf{p}_2$ be coordinate and momentum of the relative motion. We assume that the particles start their motion close to each other and have parallel momenta $\mathbf{p}_1=\mathbf{p}_2$ so that $\mathbf{p}=0$ and $\boldsymbol{\rho}=\boldsymbol{\rho}_0$ in the beginning; $\boldsymbol{\rho}_0$ is

assumed to be perpendicular to the direction **n** of the motion of the center of mass. Since the energy does not change in the magnetic field, the absolute value of the momenta $\mathbf{p}_1, \mathbf{p}_2$ will remain constant and equal to each other, $|\mathbf{p}_1| = |\mathbf{p}_2|$. Therefore, the direction of the relative motion will always be perpendicular to the direction of the motion of the mass center **n**: $(\mathbf{pn}) = 0$. This allows us to write $\boldsymbol{\rho} = \rho[\mathbf{n} \times \hat{e}_z]$. Using Eq. (A1) we find

$$\dot{\rho} = \frac{p}{m}, \quad \dot{p} = e\frac{v_F}{c}(B_1 - B_2), \tag{A2}$$

where $B_i \equiv B(\mathbf{r}_i)$, and $p = |\mathbf{p}|$ is the absolute value of the momentum of the relative motion. At the beginning of the motion ρ is rather small and the difference $B_1 - B_2$ can be approximately written as $B_1 - B_2 \approx (\partial B/\partial R_\perp)\rho$, where $R_\perp$ is the coordinate of the mass center in the direction perpendicular to **n**. Equation (A2) considered in this approximation reduces to a linear system of first-order differential equations. Hence, the distance ρ will grow exponentially as a function of time. The mean rate of the divergency or the Lyapunov exponent determines the scattering time involved.

To study statistics of the relative motion we introduce a distribution function $W(t, \rho, p)$. By definition, it is the probability for the relative distance and momentum to be ρ and p at the time t, respectively, provided they have been initially ρ_0, $p = 0$. Let $W(t_0, \rho, p)$ be the distribution at the time t_0. Then, it can be written at the time $t_0 + \Delta t$ as

$$W(t_0 + \Delta t, \rho, p) = \int P(t_0 + \Delta t, \rho, p; t_0, \rho', p') W(t_0, \rho', p') d\rho' dp', \tag{A3}$$

where $P(t, \rho, p; t', \rho', p')$ is the transition probability. This probability is determined by the equation of motion, Eq. (A2), and is introduced as

$$P(t, \rho, p; t', \rho', p') = \delta\left(\rho - \rho' - \int_{t'}^{t} \frac{p(\tau)}{m} d\tau\right) \delta\left(p - p' - e\frac{v_F}{c}\int_{t'}^{t} [B_1(\tau) - B_2(\tau)] d\tau\right), \tag{A4}$$

where $B_i(\tau) = B[\mathbf{r}_i(\tau)]$, $\mathbf{r}_i(\tau) = \mathbf{R}(\tau) \pm \boldsymbol{\rho}(\tau)/2$ and $\rho(\tau)$, $p(\tau)$ are the solution of the classical motion equation (A2). Substitution of Eq. (A4) into Eq. (A3) gives a relation between the distributions W at times t_0 and $t_0 + \Delta t$. Assuming that Δt is smaller than the inverse Lyapunov exponent we expand this relation in Δt and then average over the magnetic field $B(\mathbf{r})$. Since the magnetic field is assumed to be weak, we neglect the influence of the field on the trajectory of the mass center and obtain

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial \rho} - \frac{2}{\tau_{tr}}\varepsilon(\rho) p_F^2 \frac{\partial^2 W}{\partial p^2} = 0, \tag{A5}$$

where τ_{tr} is the transport time, Eq. (3.22), that can also be written as

$$\frac{1}{\tau_{tr}} = \omega_c^2 \int_{-\infty}^{+\infty} W_B(v_F \mathbf{n}\tau) d\tau. \tag{A6}$$

The function $\varepsilon(\rho)$ is, by definition,

$$\varepsilon(\rho) = 1 - \frac{\int_{-\infty}^{+\infty} W_B(v_F \mathbf{n}\tau + \rho[\mathbf{n} \times \hat{e}_z]) d\tau}{\int_{-\infty}^{+\infty} W_B(v_F \mathbf{n}\tau) d\tau}. \tag{A7}$$

The distance between the particles in the Lyapunov region is smaller than the correlation length of the magnetic field b. Hence, one may expand the function $\varepsilon(\rho)$ in ρ, which gives $\varepsilon(\rho) \approx \rho^2/2b^2$. This relation is to be considered as a definition of the length b. Substituting this expansion into Eq. (A5) we come to the same equation as the one derived in Ref. 28 where electron scattering in a long-ranged potential disorder was considered,

$$\left[\frac{\partial}{\partial t} - v_F \phi \frac{\partial}{\partial \rho} - \frac{\rho^2}{\tau_{tr} b^2}\frac{\partial^2}{\partial \phi^2}\right] W = 0. \tag{A8}$$

Using the result of that paper we find that the function $W(t, \rho)$ determining the distribution of the distance ρ (the momentum of the relative motion, **p**, is implied to be averaged in this function) satisfies the equation

$$\left[\tau_L \frac{\partial}{\partial t} - \beta \frac{\partial}{\partial z}\right] W = 0, \tag{A9}$$

where β is a numerical coefficient equal to $\beta \approx 0.365$ and $z = \ln(b/\rho)$. It follows from Eq. (A9) that the coefficient τ_L is in fact a characteristic time of the divergency of the trajectories of the particles calculated from the classical motion equation (A2). According to Ref. 28 this time is equal to

$$\tau_L = \tau_{tr}\left(\frac{b}{l_{tr}}\right)^{2/3} \tag{A10}$$

and this is at the same time the inverse Lyapunov exponent. As mentioned above, the quantity τ_L has the meaning of a characteristic time that two scattered particles spend moving together until the distance between them starts exceeding the correlation length b.

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