Anomalous diffusion near the Fermi surface

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(Received 8 June 2003; published 11 November 2003)

We consider the three-dimensional dynamical evolution of localized hot spots in ultralow temperature Fermi liquids. Within a model calculation, it is found that such perturbations do not relax into a hydrodynamic profile at long times, as might be expected of thermal hot spots under less exotic conditions. Instead, the hot spot expands outward into a shell moving away from its center at a velocity comparable to the Fermi velocity v_F, which is consistent with the ''baked Alaska'' hypothesis proposed earlier by Leggett as a possible solution to the riddle of *B*-phase nucleation in ^{3}He.

DOI: 10.1103/PhysRevB.68.174516 PACS number(s): 67.55.Fa

I. INTRODUCTION

In a Fermi liquid at temperatures sufficiently low compared to the energies of excited quasiparticles, the decay of these quasiparticles occurs principally through the scattering of additional particle-hole pairs out of the ground state.[1] To leading order each collision produces only one extra pair, at a rate proportional to ϵ_k^2, where ϵ_k is the energy of the incident quasiparticle relative to the Fermi energy E_F. A cascade of pair creation events is thus initiated whereby one excitation multiplies itself over time into a large number (3^N) of lower energy ones. Assuming that, on average, the energy of these excitations divides evenly down the cascade, the mean free path increases by a factor ~ 9 at each level. On these intuitive grounds it was postulated[2,3] that the evolution of localized ''hot spots,'' associated with the action of ionizing radiation on the supercooled *A* phase of ^{3}He, should exhibit strongly nonhydrodynamic behavior. In particular, it was argued that the distribution function in real space should develop a shell structure moving at or somewhat below v_F away from the center of the hot spot. Some investigators[4,5] have questioned the plausibility of this ''baked Alaska hypothesis,'' favoring instead a more conventional interpretation[6] founded on local hydrodynamic concepts.

It is therefore of some interest whether hydrodynamic principles may be applied when the particle diffusion is governed by this kind of scattering. After all, in the various approaches to ordinary diffusion one usually makes the assumption that the diffusing particles lose all memory of their past trajectory after a finite number of collisions, resulting in the characteristically Gaussian distribution at long times. Here this is not the case, since after each collision the phase space available for all subsequent scattering has irreversibly contracted. There is thus a ''piling up'' and not an ''averaging out'' of this effect, and the distribution that emerges, even after a large number of collisions, must reflect this.

The above considerations apply when the quasiparticles in question are sufficiently dilute that collisions between them may be neglected. Within the baked Alaska picture, this condition almost certainly fails during the early-time regime immediately subsequent to the energy deposition in the liquid. The hot spot effective temperature is initially so high that the mean-free path of the interior quasiparticles is on the order of the dimension of the hot spot itself.[3,10] We are led therefore to divide the problem into two stages. The main body of this paper will be concerned with the later dilute stage which is dominated by the cascade effect. The remainder will be devoted to a study of the early-time dynamics of the hot spot, focussing in particular on an estimation of the crossover time to a nonmonotonic distribution.

Our approach to the long-time cascade problem consists of building up a single-particle Green's function designed to mimic the essential scattering physics in a tractable way. This is then reduced to an effective kernel valid in the long wavelength, low frequency limit appropriate to a consideration of local hydrodynamics. The basic parameter of our theory is the average multiplication factor ζ of the mean free path at each level of the cascade. In addition we shall find another parameter ν emerging in the long time limit which controls a scaling law $|\mathbf{r}-\mathbf{r}'|\sim(t-t')^\nu$ relating the root-mean-squared displacement of the particle with time. Our principal results are the following: for the $\zeta>1$ case of interest, $\nu=1$, and there is no transition to diffusive long-time behavior, this being inhibited by the phase-space memory. Further, the effective Green's function possesses a shell structure moving away from the origin at a renormalized Fermi velocity $\tilde{v}_F$. These properties are, of course, exactly those required by the baked Alaska hypothesis. Finally, it will be seen that despite the creation of additional particle-hole pairs by the cascade, the quasiparticle density is always decreasing. This ensures the self-consistency of the assumption that pair creation indeed dominates the evolution of the hot spot at long times.

We model the early-time regime as much as possible within the constraints of local hydrodynamics, in which the effects of collisions are systematically overrepresented. The inadequacy of this description is demonstrated by the divergence of a particular hydrodynamic parameter. This enables us to find an upper limit on the crossover time to nonmonotonic behavior which is somewhat less than R_c/v_F, where R_c is the critical radius[2,3] above which a *B*-phase bubble expands spontaneously in the presence of the bulk *A* phase.

II. MODEL AND FORMALISM

In the above description of the problem it is clear that a large number of quasiparticles are produced by the cascade effect. It would only obscure the essential physics if we tried

from the beginning to keep track of all of them. Instead, we study the motion of a fictitious particle corresponding to a single line traced down through the cascade. The problem is thereby reduced to that of a single particle propagating in a continuous field of scatterers according to an infinite sequence of scattering rates $\{\Gamma_k\}$. The phase-space memory is incorporated by the parametrization $\Gamma_k=\zeta_k\Gamma\equiv\zeta^{-k}\Gamma$, where Γ is the starting rate at the ''top'' of the cascade. Naturally ζ itself is subject to fluctuation, since the energy cannot really be expected to distribute itself evenly down the cascade; we defer consideration of such fluctuations to the Appendix.

A mathematical description of this process may be written formally as a weighted sum over the various trajectories available to the particle

$$K(\mathbf{r}-\mathbf{r}',t-t')=\sum_N K_N(\mathbf{r}-\mathbf{r}',t-t')$$
$$=\sum_N \int\cdots\int \prod_k^N d^4\mathbf{r}_k W_N \times\delta\left(\mathbf{r}-\mathbf{r}'-\sum_k^N \mathbf{r}_k\right)\delta\left(t-t'-\sum_k^N \tau_k\right), \tag{1}$$

where $W_N=W_N[\{\mathbf{r}_k\},\{\tau_k\}]$ is the ''weight'' associated with a given configuration of collision coordinates $\{\mathbf{r}_k\},\{\tau_k\}$. Physically this represents the space-time probability density that a particle starting from $\mathbf{r}'$ propagates to $\mathbf{r}$ in a time $t-t'$, having undergone an indeterminate number of collisions in between. The (classical) trajectories intermediate to collisions are broken up into intervals $\tau_k=t_k-t_{k-1}$ which fluctuate according to the scattering statistics. For convenience we assume that the particle starts at the origin at $t=0$.

The structure of the N-point weight function W_N follows from simple physical arguments. First, collisions randomize the initial velocity at each step; for the sake of argument we assume isotropic scattering, affording us the greatest possible randomization at each collision and thereby isolating the physical effects of phase-space memory from those due to any anisotropies in the true scattering cross section. An additional important virtue of this assumption is a decoupling of the averages over ''in'' and ''out'' momenta at each collision.

In general, the classical kernel describing propagation from $\mathbf{r}_{k-1}$ to $\mathbf{r}_k$ in a time τ_k for a given initial velocity and a fixed collision rate Γ_k is a simple Green's function $G(\mathbf{r}_k-\mathbf{r}_{k-1},\tau_k)$ of the form

$$G(\mathbf{r}_k-\mathbf{r}_{k-1},\tau_k)=\delta\left(\mathbf{r}_k-\mathbf{r}_{k-1}-\int^{\tau_k} d\tau_k'\mathbf{v}(\tau_k')\right)\times\exp[-\Gamma_k\tau_k]$$

where $\mathbf{v}(\tau_k')$ is given by the particle's equations of motion. In the absence of external potentials, G is translationally invariant, so that $\mathbf{r}_k-\mathbf{r}_{k-1}\rightarrow\mathbf{r}_k$, and the quasiparticle trajectories reduce to straight-line paths. To obtain the appropriate probability density we average over all possible velocities subject to the constraint $|\mathbf{v}_k|\sim v_F$ coming from degenerate Fermi statistics

$$K_0(\mathbf{r}_k,\tau_k)=\frac{1}{4\pi r_k v_F\tau_k}\delta(r_k-v_F\tau_k)\exp[-\Gamma_k\tau_k]. \tag{2}$$

It should be noted that this propagator corresponds to the lowest order in perturbation theory, and is therefore the general solution at very early times. Thus the true propagator exhibits shell-like behavior in this limit regardless of the precise value of ζ.

The weight function is a probability density *vis a vis* the measure $\Pi_k d^4\mathbf{r}_k\equiv\Pi_k d^3\mathbf{r}_k d\tau_k$, which, in addition to the statistical independence of individual scattering events implies that W_N is just a product of propagators K_0 connecting N collision coordinates, multiplied by the probability per unit time for a collision at each point, i.e.,

$$W_N=K_0(\mathbf{r}_0,\tau_0)\Gamma_0 K_0(\mathbf{r}_1,\tau_1)\cdots\Gamma_{N-1}K_0(\mathbf{r}_N,\tau_N). \tag{3}$$

Upon transformation to $(\mathbf{q},\omega)$, our formal solution (1) factorizes term by term into products of basic kernels of the form $\tilde{K}_0(\mathbf{q},\omega)\equiv(1/qv_F)\tan^{-1}[qv_F/(\Gamma-i\omega)]$, allowing us to reorganize our expression for the true kernel $\tilde{K}$ as an expansion in powers of $\zeta\Gamma$:

$$\tilde{K}(\mathbf{q},\omega)=\tilde{K}_0(\mathbf{q},\omega)+\tilde{K}_0(\mathbf{q},\omega)\zeta\Gamma\tilde{K}_0(\zeta\mathbf{q},\zeta\omega)+\tilde{K}_0(\mathbf{q},\omega)\zeta\Gamma\tilde{K}_0(\zeta\mathbf{q},\zeta\omega)\zeta\Gamma\tilde{K}_0(\zeta^2\mathbf{q},\zeta^2\omega)+\cdots. \tag{4}$$

We have used a scaling property of $\tilde{K}_0$, namely,

$$\frac{\zeta^{-k}}{qv_F}\tan^{-1}\left(\frac{qv_F}{\Gamma_k-i\omega}\right)=\tilde{K}_0(\zeta^k\mathbf{q},\zeta^k\omega).$$

Inspection of the series (4) reveals that $\tilde{K}(\mathbf{q},\omega)$ inherits similar scaling properties:

$$\tilde{K}_\Gamma(\zeta^N\mathbf{q},\zeta^N\omega)=\zeta^{-N}\tilde{K}_{\Gamma_N}(\mathbf{q},\omega), \tag{5}$$

where the subscripts (usually suppressed) indicate scattering rates at the top of the associated cascade. This equation reflects a self-similarity property implicit to our model.

The series (4) may be written more compactly as a Dyson-like scattering equation

$$\tilde{K}(\mathbf{q},\omega)=\frac{1}{qv_F}\tan^{-1}\left(\frac{qv_F}{\Gamma-i\omega}\right)[1+\zeta\Gamma\tilde{K}(\zeta\mathbf{q},\zeta\omega)] \tag{6}$$

from which the series itself follows by iteration.

A final mathematical consideration essential to the consistency of this approach is the overall normalization of Eq. (1). This may be demonstrated by taking $\mathbf{q}\rightarrow 0$ (i.e., integrating over all space) in Eq. (6) and noting that, in order for the series to be normalized, $\tilde{K}(0,\omega)$ cannot depend on the parameter ζ, which must therefore drop out of the equation for

$\tilde{K}(0,\omega)$. This can only occur if $\tilde{K}(0,\zeta\omega)=\zeta^{-1}\tilde{K}(0,\omega)$; substitution of this condition yields the self-consistent solution $\tilde{K}(0,\omega)=i/\omega$.

III. ORDINARY DIFFUSION

In order to develop the physical principles necessary to the problem at hand, we consider first a more familiar problem but from a new perspective. In particular we explore the behavior of the kernel K in the $\zeta=1$ case describing continuous diffusion at constant velocity and with constant scattering rate (the discrete case corresponds to the usual random walk problem). This has of course been studied extensively in various contexts by a number of investigators; we make specific reference only to the work of Palmeri[7,8] who considered the baked Alaska problem in the simpler ''Lorentz gas'' system consisting of an aggregate of noninteracting fermions propagating in a background of fixed, isotropic scatterers. The simplification arises from the fixed character of the scatterers, rendering all collisions elastic (hence the $\zeta=1$ condition within the present formalism). He arrived at his solution by methods rather different from ours, preferring direct integration of a Boltzmann equation to the sum-over-trajectories approach developed here.

Setting $\zeta=1$, the Dyson-like equation reduces immediately to an algebraic equation in $\tilde{K}$ with the solution

$$\tilde{K}(\mathbf{q},\omega)=\frac{\dfrac{1}{qv_F}\tan^{-1}\left(\dfrac{qv_F}{\Gamma-i\omega}\right)}{1-\dfrac{\Gamma}{qv_F}\tan^{-1}\left(\dfrac{qv_F}{\Gamma-i\omega}\right)}$$

which is in exact correspondence with Palmeri's kernel. It should be noted that Palmeri was apparently the first to derive this analytic solution. An effective kernel valid in the limit of long wavelengths and low frequencies follows from this exact solution; as shown by Palmeri this is nothing more than the familiar Gaussian $[4\pi/(2\pi Dt)^{3/2}]\exp[-r^2/6Dt]$, indicating the applicability of local hydrodynamic concepts on long length scales for scattering of this type. Here $D=\frac{1}{3}v_F^2\Gamma^{-1}$ is the diffusion constant as it is ordinarily defined.

The extraction of this effective kernel by Palmeri is rather laborious and of course requires that one have an exact solution to begin with. We now discuss an alternative method based on direct consideration of the series (4), which is in general much easier to implement, and perhaps of some interest even beyond the context of the present discussion.

From a physical standpoint, the derivation of the effective kernel ought not to require the summation of each and every term in perturbation theory. The basic idea of our method is to delay summation of perturbation theory until one is well into the asymptotic regime $\Gamma t\gg 1$, at which point the structure of the series simplifies enormously. In fact, at long times only a very few terms in the series should be relevant: terms that deviate significantly from the expected number of collisions at a given time will be exponentially small. To show this, let us consider first only the Nth term $K_N(\mathbf{r},t)$ ($N\gg 1$) of the series for long wavelengths and low frequencies. In this limit K_N factorizes into separate space and time dependent parts $K_N\rightarrow\Omega_N(t)Q_N(\mathbf{r})$ with

$$\Omega_N(t)=\Gamma^{N-1}\int d\omega\frac{e^{-i\omega t}}{(\Gamma-i\omega)^N}$$

and

$$Q_N(\mathbf{r})=\int d^3\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{r}}\left(1-\frac{1}{3}v_F^2\Gamma^{-2}q^2\right)^N$$

$$\rightarrow\int d^3\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{r}-(N/3)\Gamma^{-2}(qv_F)^2}.$$

It should be noted that $\Omega_N(t)=\int d\omega e^{-i\omega t}\lim_{q\rightarrow 0}\tilde{K}_N(\mathbf{q},\omega)$, so that the integrand is actually valid for all ω and the limits of integration may be taken to be $(-\infty,\infty)$. The same does not hold true for the integrand in the second factor, which is strictly valid only for $qv_F,\omega\ll\Gamma$. However the swift convergence of the Gaussian integral renders the introduction of a cutoff superfluous.

The first factor may be evaluated by the method of contours in the standard way by enclosing the Nth order pole at $\omega=-i\Gamma$; this gives us $\Omega_N(t)=[(\Gamma t)^{N-1}/(N-1)!]e^{-\Gamma t}$, which will be recognized as the Poisson distribution. We note that this could have been anticipated on purely physical grounds by considering first the distribution in the number of collisions with respect to time.

This makes evident an important feature of the long-time limit which essentially fixes a relation between N and t and gives us a means of performing the required sum over N. For N large enough, the Poisson formula reduces via Stirling's approximation to $\Omega_N(t)\sim\exp[(N-1)(1+\ln\Gamma t)-(N-1)\ln(N-1)-\Gamma t]$. The argument is stationary under variations in the number of collisions for $N=\Gamma t$, which is the mean of the Poisson distribution. Fluctuations $\delta N=N-\Gamma t\ll N$ are therefore distributed according to $\exp[-(\delta N)^2/\Gamma t]$; hence in the sum over N only relatively few terms make a contribution. The fractional width falls off as $(\Gamma t)^{-1/2}$, and $K(\mathbf{r},t)$ may be treated as a weighted sum over the very narrow distribution Ω_N. We may therefore, to a very good approximation, replace this sum by that $Q_N(\mathbf{r})$ corresponding to its mean value. This yields for the effective kernel an expression of the form

$$K(\mathbf{r},t)\sim\int d^3\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{r}-(1/3)v_F^2\Gamma^{-1}tq^2}$$

$$=\frac{4\pi}{(2\pi Dt)^{3/2}}\exp\left[-\frac{r^2}{6Dt}\right] \tag{7}$$

which is of course the same result as that obtained from the exact solution.

IV. ANOMALOUS DIFFUSION

We proceed conceptually along lines parallel to those of the preceding section. We will deviate in certain mathematical particulars, however, in order to accommodate the scattering statistics peculiar to this case. For example, the ana-

logue of $\Omega_N(t)$ is here somewhat complicated and less obviously useful for finding that term in the series which is most probable at long times. Instead we locate the N closest to the point where the ratio $R=\Omega_{N+1}(t)/\Omega_N(t)$ crosses unity for a given interval t.

Keeping only the slowest decaying terms, R becomes

$$R\sim\exp[-(\zeta^{-N-1}-\zeta^{-N})\Gamma t-N\ln\zeta]. \tag{8}$$

Setting this equal to 1 is the same as finding the stationary point of an argument of the form $-\zeta^{-N}\Gamma t-\frac{1}{2}N^2\ln\zeta$, which is precisely what we should have expected on the grounds that, roughly speaking, the most probable N results from a competition between factors such as $\Pi_k^N\zeta^{-k}\sim\zeta^{-(1/2)N^2}$ and the ever-slower decay of the diffusing particle. This yields, to leading order in N, the scaling relation $N\sim(\ln\Gamma t/\ln\zeta)$. This is very close to the mean value given by $\Gamma t\sim(1-\zeta^{N+1})/(1-\zeta)$, the difference arising from the skewness in the distribution of small fluctuations $P_N(\delta N)$, which goes as

$$P_N(\delta N)\sim\begin{cases}\zeta^{-|\delta N|} & \text{for } \delta N>0,\\ \exp[-\zeta^{|\delta N|}] & \text{for } \delta N<0.\end{cases}$$

Hence the fractional width of the distribution at long times is quite narrow, and we may restrict our attention to the single term $\tilde{K}_N$; however, because $\zeta>1$ the structure of such a term is somewhat different from what we encountered in Sec. III in that a large number of length and time scales appear in the various factors. Most of these may be safely expanded for small (q,ω) without affecting the long-wavelength, long-time properties of the resulting integral, but the factor representing variations on the largest such scales will be kept intact. This is to ensure that we do not 'wash away' details on the these scales by looking too coarsely at the distribution. This gives the expression

$$K_N(\mathbf{r},t)\sim\frac{\Gamma}{(2\pi)^4}\int_0^\infty\int_{-\infty}^\infty\int_{-1}^1 d^3\mathbf{q}\,d\omega\,\frac{du}{2}\times\frac{ie^{-i\mathbf{q}\cdot\mathbf{r}-i\omega(t-\Gamma^{-1}\beta_1)-1/3(qv_F)^2\Gamma^{-2}\beta_2}}{\omega+qv_Fu+i\zeta^{-N}\Gamma}, \tag{9}$$

where the parameters $\beta_k=\Gamma^{-1}(1-\zeta^{kN})/(1-\zeta^k)$ have been introduced for compactness, and the last arctangent factor has been rewritten as an integral over the parameter u. Because we have kept the last factor there is no need to impose cutoffs, and the integrations are therefore relatively benign:

$$K(\mathbf{r},t)\sim\frac{\eta e^{-\Gamma_N(t-\beta_1)}}{rv_F(t-\beta_1)\beta_2^{1/2}}\left(\exp\left[-\frac{[r-v_F(t-\beta_1)]^2}{\frac{4}{3}v_F^2\Gamma^{-1}\beta_2}\right]-\exp\left[-\frac{[r+v_F(t-\beta_1)]^2}{\frac{4}{3}v_F^2\Gamma^{-1}\beta_2}\right]\right). \tag{10}$$

Here η is a dimensionless numerical constant fixed by normalization. The decay factor may be ignored, being in fact irrelevant since the sharpness of the distribution in N leads us to identify N with its mean value as given by the relation $t\sim\Gamma^{-1}(1-\zeta^{N+1})/(1-\zeta)$. This further implies that the parameters $\beta_1\to(1-\zeta^{-1})t$ and $\beta_2\to(\Gamma/\zeta^2)(\zeta-1)/(\zeta+1)t^2$ for N,t large. These substitutions produce finally the effective kernel $K(\mathbf{r},t)$:

$$\frac{\eta}{r(\tilde{v}_Ft)^2}\left(\exp\left[-\frac{(r-\tilde{v}_Ft)^2}{\frac{4}{3}\frac{1}{\zeta^2-1}(\tilde{v}_Ft)^2}\right]-\exp\left[-\frac{(r+\tilde{v}_Ft)^2}{\frac{4}{3}\frac{1}{\zeta^2-1}(\tilde{v}_Ft)^2}\right]\right). \tag{11}$$

Here $\tilde{v}_F=(1-\zeta^{-1})v_F$ is a renormalized Fermi velocity.

We note at this point that the propagator (11) is independent of Γ, the starting rate for our quasiparticle. At long times, therefore, the initial distribution of quasiparticle energies becomes irrelevant to the spatial profile. To estimate the effects of the growing number of particles in the shell, we multiply this kernel by $3^{N(t)}$, where $N(t)$ is the mean number of collisions that have occurred up until time t. Because N scales only logarithmically with time, however, $3^N\sim t^{1+\alpha}$ where $\alpha<1$, so the density is everywhere strictly decreasing as a function of time. Thus we do not return to the density dominated regime following the crossover to cascade scattering.

V. DISCUSSION

By the nature of the scaling laws emerging in the long-time limit we see that the most natural definition of the exponent ν characterizing these laws is of the form

$$\nu=\lim_{N\to\infty}\frac{\ln\left(\sum_k^N\zeta^{2k}\right)^{1/2}}{\ln\sum_k^N\zeta^k} \tag{12}$$

which is a nonanalytic function of ζ with the following values:

$$\nu=\begin{cases}0 & \text{for } 0<\zeta<1,\\ \frac{1}{2} & \text{for } \zeta=1,\\ 1 & \text{for } \zeta>1.\end{cases} \tag{13}$$

These values correspond to the three possible effective kernels emerging from Eq. (1) under coarse graining. We now describe the properties of each in turn.

First, the $\nu=0$ propagator, which is not derived in this paper, is basically a δ function situated at $\mathbf{r}$; the geometric growth of the scattering rate in this case prevents the diffusing particle from traveling very far from its starting point, leading in the limit to a time-independent, highly localized distribution. The kernel for $\nu=\frac{1}{2}$ is the usual type of propagation one expects when collisions are uncorrelated. Although we know from Secs. II and III that at early times this

kernel exhibits a shell-like structure, this is apparently completely washed out by collisions; no artifact of this early, ''organized'' behavior remains at long times.

The last, and most important, propagator, at least from the point of view of the present paper, is the one corresponding to $\nu=1$. First, we note that the overall shape and scaling of this kernel are completely consistent with the baked Alaska idea. Second, this kernel possesses a rather striking property in that we are able, merely by varying parameters appearing in the kernel itself, to recover the kernel valid at very early times $0<t\ll\Gamma^{-1}$. In particular, taking the parameter $\zeta\to\infty$ in Eq. (11) reproduces this early time behavior, essentially by pushing the intervals between collisions to an arbitrarily large value. The same can be achieved by treating N as a continuous parameter in Eq. (10) and taking the limit $N\to 0$. This reversibility is not evident in either of the other two propagators, from which nothing of the corresponding early-time behavior may be inferred.

This effect is a consequence of the fact that the number of collisions rises only logarithmically with time, which is simply too slow to drive a transition to the hydrodynamic or Gaussian profile. Further, in contrast to the $\nu=0,\frac{1}{2}$ cases where the ζ parameter drops out completely, here it remains, like a fossil buried in the long-time distribution; even at long times the propagator retains a memory of the initial δ-function pulse moving out from the origin (which is precisely the lowest order in perturbation theory). Thus we can see that the propagator is distinctly nonmonotonic at *all* times.

VI. CONNECTION WITH THE REAL PROBLEM IN ^{3}He

If particle-hole creation were in fact the only scattering mode relevant to the real problem of *B*-phase nucleation in ^{3}He, then the whole baked Alaska scenario would follow quite naturally from the discussion in Sec. IV. However, for some period of time immediately following the energy deposition process which produces the hot spot, its dynamics will be characterized instead by strong interquasiparticle scattering.

This stage is clearly unstable to the physics of Sec. IV in that ultimately the density of the hot spot must decrease to the point where collisions among the constituent quasiparticles may be neglected at all later times. This is because the scattering mode in question conserves both the total number and energy of the associated quasiparticles, and must spread out as a function of time.

However, it is conceivable that the presence of this interquasiparticle scattering can delay the crossover to nonmonotonic behavior for a time $\tau_{\rm cross}$ which is long on the scale of R_c/v_F. This would mean that, although baked Alaskas might themselves correspond to real phenomena, they would not be relevant to the *B*-phase nucleation problem referred to in Sec. I.

In what follows we work as much as possible within the framework of local hydrodynamics, in order to demonstrate that even within this picture (1) diffusive propagation is simply an inadequate description, (2) if we violate this picture in a particular physically reasonable way we immediately obtain baked Alaska-type distributions, and (3) the time scale for the appearance of these distributions is set, as it should be, by R_c/v_F.

A. Breakdown of the diffusion picture

If we are to take the concept of local hydrodynamics seriously, it ought to be possible to define an effective temperature $T(\mathbf{r},t)$ and energy density $\mathcal{E}(\mathbf{r},t)$ which are governed by reasonable differential equations. In the case of a highly degenerate Fermi liquid these two quantities are not independent: the energy density $\mathcal{E}\sim$ (number of excited quasiparticles)$\times$(mean energy of excited quasiparticles)$\propto T^2$. We begin by defining[9] the energy current $\mathbf{j}_\epsilon(\mathbf{r},t)=-\kappa_T\nabla T(\mathbf{r},t)$ where $\kappa_T\equiv\kappa T^{-1}$ is the thermal conductivity and the constant $\kappa=\frac{1}{3}C_N(T)Tv_Fl(T)$, with $C_N(T)(\propto T)$ the normal state heat capacity per unit volume and $l(T)(\propto T^{-2})$ the mean free path. We have explicitly neglected any convective currents in the liquid; i.e., we have assumed the dynamics is purely diffusive. Because of the relationship between $\mathcal{E}$ and T we may rewrite the energy current entirely in terms of $\mathcal{E}$ itself, so that $\mathbf{j}_\epsilon(\mathbf{r},t)=(\kappa/2)\mathcal{E}(\mathbf{r},t)^{-1}\nabla\mathcal{E}(\mathbf{r},t)$.

In the next step we impose the constraint of local energy conservation, or

$$\partial_t\mathcal{E}(\mathbf{r},t)+\nabla\cdot\mathbf{j}_\epsilon(\mathbf{r},t)=0.$$

This gives us the equation

$$\partial_t\mathcal{E}(\mathbf{r},t)=\frac{\kappa}{2}\nabla^2\ln\mathcal{E}(\mathbf{r},t). \tag{14}$$

The highly singular nature of this equation makes it very difficult to characterize its solutions; in order to give the local hydrodynamics picture the benefit of the doubt we ''soften'' the singularity by removing the explicit position-dependence of the diffusion coefficient, replacing it instead with a self-consistent value corresponding to the maximum energy density at that time. In this way we systematically *overestimate* the influence of collisions on the energy density by assuming a diffusion coefficient which is spatially uniform at all times. We rewrite the above diffusion equation in the somewhat more conventional form

$$[\partial_t-D(t)\nabla^2]\mathcal{E}(\mathbf{r},t)=0, \tag{15}$$

where $D(t)\equiv\kappa/2\mathcal{E}[r_{\rm max}(t),t]$. In three dimensions we can write down the spherically symmetric solutions formally as $\mathcal{E}(\mathbf{r},t)=E/[\pi^{1/2}R(t)]^3\exp\{-[r/R(t)]^2\}$ for a Gaussian source at $r=0$ of dimension R_0 and total energy E. Here $R(t)^2\equiv\int^t dt'D(t')$, and the form of the solution indicates that the radius of maximum density $r_{\rm max}=0$ at all times. This implies the self-consistency relation

$$\partial_tR^2(t)=\frac{\pi^{3/2}\kappa}{2E}R(t)^3, \tag{16}$$

which in turn yields a width function $R(t)=R_0[1-(\pi^{3/2}\kappa R_0/4E)t]^{-1}$. This diverges after a time $\tau_{\rm div}=4\pi^{-3/2}(E/\kappa R_0)$, which we can rewrite in somewhat more familiar terms by noting that, according to the hydrodynamic

definitions we have adopted $\frac{1}{2}C_N(T_0)T_0R_0^3 \sim E$, where T_0 is the effective temperature of the initial hot spot. Taking $l(T_0) \equiv l_0$, this gives an estimate $\tau_{\rm div} \sim (R_0/l_0)R_0/v_F$.

B. Estimation of $\tau_{\rm cross}$

We can interpret the breakdown of local hydrodynamics in this instance by looking again at the results of Sec. IV. There the effective propagator (11) consists of a hybrid of quasi-diffusive behavior in the relative coordinate $r-\tilde{v}_F t$ and a rapid ''drift'' of the peak of the distribution away from the origin at a velocity $\tilde{v}_F$. Physically, the diverging mean-free path, combined with expansion from a localized source, leads to strong radial correlations in the quasiparticle velocities, which manifests in the propagator as a drift effect. This is absent in the usual local hydrodynamics picture, in which the average velocity for a diffusing particle is zero for any sufficiently coarse-grained subvolume in the system.

It is likely that in the present case the divergence of $R(t)$, which expresses a clear tendency for the distribution to spread very rapidly away from the origin, results from our neglect of a corresponding drift effect in Eq. (15). We have, after all, the same basic ingredients: a diverging mean-free path combined with radial expansion. Under this interpretation, the neglect of such a drift term so constrains the possible solutions (by forcing them into a monotonic ''strait-jacket''), that the rapid spreading tendency can manifest itself only in the unphysical divergence of a parameter in the solution, in this case $R(t)$.

Consider now the fuller version of Eq. (15) with a drift term added:

$$[\partial_t + \mathbf{v}_d(t)\cdot\nabla - D(t)\nabla^2]\mathcal{E}(\mathbf{r},t) = 0. \tag{17}$$

Although we have not provided any physical constraints on the detailed time dependence of $\mathbf{v}_d(t)$, we may nevertheless write down formally the spherically symmetric solutions of the above equation. These are

$$\frac{\eta E}{r r_d(t) R(t)}\left(\exp\left[-\frac{[r-r_d(t)]^2}{R(t)^2}\right] - \exp\left[-\frac{[r+r_d(t)]^2}{R(t)^2}\right]\right), \tag{18}$$

where $r_d(t) = \int^t dt' v_d(t')$, and η is a dimensionless numerical constant. This bears a striking resemblance to the effective kernel (11), though in this case the form of the solution is supposed to be valid for all times $t \geqslant 0$. Expanding this solution for small r

$$\mathcal{E}(\mathbf{r},t) = \frac{4\eta E}{R(t)^3}\exp\left[-\left(\frac{r_d(t)}{R(t)}\right)^2\right] \times\left\{1+\left(\frac{2}{3}\cdot\frac{r_d(t)^2}{R(t)^4} - \frac{1}{R(t)^2}\right)r^2 + \mathcal{O}(r^4)\right\}$$

reveals two qualitative regimes for the evolution of $\mathcal{E}(\mathbf{r},t)$ depending on the sign of the coefficient of r^2; i.e., monotonic at early times, then crossing over to nonmonotonic behavior when $r_d(t) \geqslant \sqrt{\frac{3}{2}}R(t)$.

The self-consistency relation during this monotonic phase is very simple: $\partial_t R^2(t) = (\pi^{3/2}\kappa/2E)R(t)^3\exp\{[r_d(t)/R(t)]^2\}$. This is quite similar to the corresponding relation with drift neglected, except for the exponential factor which varies between 1 and $e^{3/2}$. Thus $R(t)$ will diverge also in this case at a time $\tau'_{\rm div} < \tau_{\rm div}$. The divergence is avoided, however, if the crossover to the nonmonotonic solution occurs sometime before this. Based on the analogy to the situation discussed in Sec. IV, there are strong physical reasons for supposing this to be true. In this sense the precise time dependence of $v_d(t)$ is irrelevant so long as it compensates the unphysical divergence that occurs when it is strictly zero. This means in particular that $\tau_{\rm cross} < \tau'_{\rm div}$, or $\tau_{\rm cross} < \tau_{\rm div}$. We therefore take $\tau_{\rm div}$ as an absolute upper limit on the crossover time to a shell-like energy density, for any *physical* definition of the drift velocity $v_d(t)$.

Based on the estimates of R_0/l_0 appearing in Ref. 10, together with the fact that $R_c \sim 10R_0$ or greater, we see that $\tau_{\rm cross} \sim R_c/v_F$ at most. Of course, this estimate is based on calculations which strongly overemphasize the influence of collisions at each step, and place no restrictions on the magnitude of $v_d(t)$ (which, strictly speaking, should be $\leqslant v_F$), so that we may reasonably expect $\tau_{\rm cross}$ to be somewhat less than this, probably significantly less.

VII. CONCLUSIONS

We have argued that the baked Alaska scenario, as first described in Ref. 2 and discussed in somewhat greater detail in Ref. 3, is not only qualitatively plausible but even quantitatively accurate as a model of energy transport following intense local heating in a normal Fermi liquid. First, a hydrodynamic description of the usual sort breaks down when considering either interquasiparticle scattering or particle-hole creation processes within their respective regions of applicability. In the latter case it is even possible to find, within a simple model calculation, the effective propagator for single quasiparticles which shows explicitly a nonmonotonic shell structure moving out from the origin at a velocity $\sim v_F$. The time scale for the onset of baked Alaska behavior, which is set by the interquasiparticle scattering that dominates at early times, is found to be of the right order for the range of values of R_0/l_0 appropriate to liquid ^{3}He at these (effective) temperatures.

ACKNOWLEDGMENTS

One of us (G.L.W.) wishes to thank William McMahon for numerous discussions. The authors gratefully acknowledge support from NSF Grant No. DMR99-86199.

APPENDIX

In this Appendix we discuss briefly the nature of the fluctuations in ζ and their effect on the $\nu = 1$ propagator. For energetic (i.e., $\epsilon \gg kT$) quasiparticles the energy dependence of the Fermi functions appearing in the scattering rate is extremely weak, its variations being confined to a width $\sim kT$ about the Fermi surface. Therefore every configuration,

in energy space, of the final-state quasiparticles subsequent to a collision event are equally probable, and the probability of one of the final-state particles of energy ϵ to scatter with a fraction $f=\epsilon/\epsilon_i$ of the energy ϵ_i of the incident quasiparticle is determined entirely by the number of configurations available to the remaining two (labeled f',f''), subject to the constraint of energy conservation. This gives

$$P(f)=2\int_0^1\int_0^1 df'df''\delta(1-f-f'-f'')=2(1-f), \tag{A1}$$

the factor of 2 arising from the possible interchange of the quasiparticles. This is a triangular distribution weighted toward $f=0$. The mean $\langle f\rangle=\frac{1}{3}$ and the root-mean-squared deviation $\langle(f-\frac{1}{3})^2\rangle^{1/2}=\frac{1}{6}$, indicating that, as far as ζ is concerned, fluctuations from step to step in the diffusion process will tend to favor larger rather than smaller ζs.

In terms of the behavior of the kernel derived in Sec. IV, its qualitative features should be relatively insensitive to these fluctuations, which can be seen in the following manner. Imagine that we cut off the distribution in f at a point $f_c<1$ somewhere close to 1, since the probability of a large number of fs inside this region will be vanishingly small in the large N limit. Now choose an $f\equiv f_>$ such that $f_c<f_><1$, and hence a $\zeta'=1/f_>^2>1$. Because $\zeta'>1$ the effective kernel will be of the form shown in Sec. IV. Thus we do not expect fluctuations in ζ to change the behavior of the effective kernel in a qualitative way.

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