Anisotropic transport in a two-dimensional electron gas in the presence of spin-orbit coupling

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In a two-dimensional electron gas as realized by a semiconductor quantum well, the presence of spin-orbit coupling of both the Rashba and Dresselhaus type leads to anisotropic dispersion relations and Fermi contours. We study the effect of this anisotropy on the electrical conductivity in the presence of fixed impurity scatterers. The conductivity also shows in general an anisotropy which can be tuned by varying the Rashba coefficient. This effect provides a method of detecting and investigating spin-orbit coupling by measuring spin-unpolarized electrical currents in the diffusive regime. Our approach is based on an exact solution of the two-dimensional Boltzmann equation and provides also a natural framework for investigating other transport effects including the anomalous Hall effect.

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I. INTRODUCTION

In the recent years the emerging field of spintronics[1,2] has generated an intense interest in effects of spin-orbit interaction in low-dimensional semiconductor heterostructures. For conduction-band electrons in zinc-blende semiconductors the dominant effects of spin-orbit interaction in low-dimensional geometry can be described in terms of two effective contributions to the Hamiltonian. On the one hand, there is the Rashba spin-orbit term[3] which is due to the inversion asymmetry of the confining potential and has the form

$$\mathcal{H}_R=\frac{\alpha}{\hbar}(p_x\sigma^y-p_y\sigma^x), \tag{1}$$

where $\vec{p}$ is the momentum of the electron confined in a two-dimensional geometry and $\vec{\sigma}$ the vector of Pauli matrices. The coefficient α is tunable in strength by the external gate perpendicular to the plane of the two-dimensional electron gas. The other contribution is the Dresselhaus spin-orbit term which is present in semiconductors lacking bulk inversion symmetry.[4] When restricted to a two-dimensional semiconductor nanostructure grown along the [001] direction this coupling is of the form[5,6]

$$\mathcal{H}_D=\frac{\beta}{\hbar}(p_y\sigma^y-p_x\sigma^x), \tag{2}$$

where the coefficient β is determined by the semiconductor material and the geometry of the sample.

The interplay of these two types of spin-orbit coupling has been investigated theoretically with respect to several physical phenomena including the spin splitting in zero magnetic field,[7–10] spin precession and relaxation,[5,6,10–17] contributions to magneto-oscillations,[18] quantum interference corrections to the conductivity,[19,20] mesoscopic transport through quantum dots,[21] and issues of gate operations between quantum dot spin qubits.[22] In the present work we point out another effect occurring in the presence of both Rashba and Dresselhaus spin-orbit coupling, namely, an anisotropy in the electrical conductivity for diffusive spin-unpolarized charge transport in the presence of fixed impurity scatterers.

One of the most important concepts in the discussion of spin-orbit coupling in semiconductors, and in the field of spintronics in general, is the spin field-effect transistor proposed by Datta and Das.[23] This proposal uses the Rashba spin-orbit coupling to perform controlled rotations of spins of electrons passing through a field effect transistor (FET) device operating in the ballistic transport regime. Ballistic transport is necessary to avoid randomization of the spin state by even spin-independent scatterers which would change the electron momentum and therefore also the effective field provided by the Rashba term in an uncontrolled way. The requirement of ballistic transport has been so far one of the major obstacles toward the practical realization of a spin FET. Recently, an alternative scenario for a spin FET was proposed which can also operate in the nonballistic regime.[14] This proposal exploits the fact that if the Rashba coefficient is tuned via external gates to be equal to the Dresselhaus coefficient, $\alpha=\beta$, a new conserved quantity arises which prohibits the randomization of the spin. As we shall see below, this particular point $\alpha=\beta$ in parameter space will also be of special interest in our present study.

Our description of diffusive two-dimensional transport in the presence of spin-orbit interaction and fixed impurities is based on an exact solution of the two-dimensional Boltzmann equation and provides also a natural framework for investigating other transport effects including the anomalous Hall effect. In particular, our scheme of generating exact solutions to the transport equation can deal with arbitrary dispersion relations and is not restricted to isotropic cases.

Another study on the possible influence of the Dresselhaus spin-orbit coupling on the operation of the spin FET was performed very recently by Lusakowski, Wrobel, and Dietl.[16] These investigations are restricted to the ballistic regime, but take into account, in addition to Hamiltonian (2), also contributions to the Dresselhaus term being trilinear in the momentum. In the present study we will mostly neglect these trilinear contributions, but discuss their possible influence briefly in Sec. IV C. In another very recent preprint

Winkler also studied spin transport in the presence of spin-orbit interaction stemming from both structure inversion asymmetry and bulk inversion asymmetry.[17] In particular, in Refs. 16 and 17 the possibilities arising from different growth directions for the two-dimensional electron system are explored. Yet another recent work dealing with transport in the presence of spin-orbit coupling was performed by Mishchenko and Halperin who derived the equations of motion for the single-electron-density matrix in Wigner representation in a two-dimensional free-electron gas.[24] The authors applied their results to the dynamic conductivity of the system taking into account, however, the Rashba term only such that no anisotropy in conductivity occurred. Finally we mention a very recent work by Ganichev *et al.*[25] who present an experimental method to distinguish the effects of Rashba and Dresselhaus spin-orbit coupling using the spin galvanic effect.

This paper is organized as follows. In Sec. II we review the dispersion relations and eigenstates of free electrons confined in two dimensions in the presence of spin-orbit coupling of both the Rashba and the Dresselhaus type, and we present results for the Fermi contours. In Sec. III we present a scheme to generate exact solutions to the two-dimensional Boltzmann equation that underlies our present study. This approximation-free solution to the semiclassical transport equation is then applied in Sec. IV to the case of free electrons being subject to spin-orbit interaction of the above type. We close with a summary and discussion of the results in Sec. V.

II. DISPERSION RELATIONS, EIGENSTATES, AND FERMI CONTOURS

We consider the single-particle Hamiltonian for a two-dimensional electron system

$$\mathcal{H}=\frac{\vec{p}^2}{2m}+\mathcal{H}_R+\mathcal{H}_D, \tag{3}$$

where m is an effective band mass. The eigenenergies are given by

$$\varepsilon_\pm(\vec{k})=\frac{\hbar^2k^2}{2m}\pm\sqrt{(\alpha k_y+\beta k_x)^2+(\alpha k_x+\beta k_y)^2}, \tag{4}$$

with eigenstates

$$\langle\vec{r}|\vec{k},\pm\rangle=\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{A}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm e^{i\chi(\vec{k})}\end{pmatrix}, \tag{5}$$

where A is the area of the system and

$$\chi(\vec{k})=\arg[-\alpha k_y-\beta k_x+i(\alpha k_x+\beta k_y)]. \tag{6}$$

The semiclassical particle velocities are given by

$$\vec{v}_\pm(\vec{k})=\frac{\partial\varepsilon_\pm(\vec{k})}{\hbar\,\partial\vec{k}}=\frac{\hbar\vec{k}}{m}\pm\frac{(\alpha^2+\beta^2)\vec{k}+2\alpha\beta(\tilde{\sigma}^x\vec{k})}{\hbar\sqrt{(\alpha^2+\beta^2)k^2+2\alpha\beta(\vec{k}^T\tilde{\sigma}^x\vec{k})}}, \tag{7}$$

where $\tilde{\sigma}^x$ is a usual Pauli matrix acting on the vector $\vec{k}$. As a consistency check let us consider the quantum-mechanical velocity operator

$$\dot{\vec{r}}=\frac{i}{\hbar}[\mathcal{H},\vec{r}]. \tag{8}$$

Using the above expressions for the eigenstates, it is straightforward to show that its matrix elements are given by

$$\langle\vec{k},\pm|\dot{\vec{r}}|\vec{k}',\pm\rangle=\delta_{\vec{k},\vec{k}'}\vec{v}_\pm(\vec{k}), \tag{9}$$

i.e., the semiclassical velocities $\vec{v}_\pm(\vec{k})$ are, as usual, the diagonal elements of the velocity operator.

Parametrizing wave vectors as $\vec{k}=k(\cos\varphi,\sin\varphi)$, one obtains for positive Fermi energy ε_f the following parametrization of the Fermi contours:

$$k_\pm^f(\varphi;\varepsilon_f)=\mp\sqrt{\left(\frac{m}{\hbar^2}\right)^2[\alpha^2+\beta^2+2\alpha\beta\sin(2\varphi)]}+\sqrt{\frac{2m}{\hbar^2}\varepsilon_f+\left(\frac{m}{\hbar^2}\right)^2[\alpha^2+\beta^2+2\alpha\beta\sin(2\varphi)]}. \tag{10}$$

Here the double sign corresponds to the above two dispersion branches, and the Fermi wave vector is given by

$$\vec{k}_\pm^f(\varphi;\varepsilon_f)=k_\pm^f(\varphi;\varepsilon_f)(\cos\varphi,\sin\varphi). \tag{11}$$

At negative Fermi energies the Fermi contours can become somewhat more complicated. This case corresponds to rather low electron densities and shall not be considered here further. In the following the Fermi energy is always assumed to be positive. From Eq. (10) one finds the electron density n as

$$n=\frac{1}{(2\pi)^2}\sum_{\mu=\pm}\int d\varphi\,\frac{1}{2}[k_\mu^f(\varphi;\varepsilon_f)]^2=\frac{1}{2\pi}\left[\frac{2m}{\hbar^2}\varepsilon_f+2\left(\frac{m}{\hbar^2}\right)^2(\alpha^2+\beta^2)\right]. \tag{12}$$

If $\alpha=0$ or $\beta=0$ the dispersions are isotropic and Fermi contours are concentric circles. For $\alpha\neq0\neq\beta$ the Fermi contours are anisotropic which, as we shall see below, leads in general to anisotropic transport properties. Note that the dispersion relations and Fermi contours are symmetric around the points $\varphi\in\{\pi/4,3\pi/4,5\pi/4,7\pi/4\}$, i.e., these quantities are invariant under reflections along the (1,1) and (1,−1) directions. These directions define the symmetry axes of the problem. In particular, for these directions the wave vectors and particle velocities are collinear.

The above findings for the Fermi contours are illustrated in Fig. 1 where we show data for typical values for the Fermi energy, Dresselhaus coefficient, and effective band mass[26–28] at various values for α.[29–34] If both α and β are nonzero the Fermi contours are anisotropic having the (1,1) and (1,−1) directions as symmetry axes. The case $\alpha=\pm\beta$ is

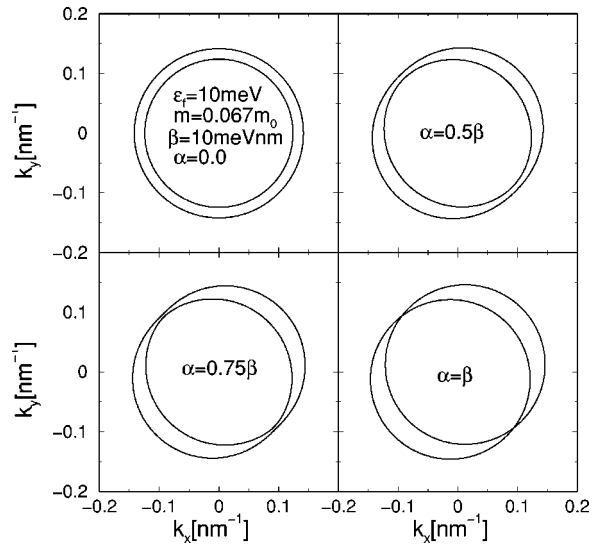

FIG. 1. Fermi contours for various values of the Rashba coefficient α at a Fermi energy of 10 meV, a Dresselhaus coefficient of 10 meV nm, and a band mass of 0.067 in units of the bare electron mass m_0. The upper left panel shows the isotropic Fermi contours at $\alpha=0$. In the upper right and lower left panel data at intermediate values of α are plotted, while the lower right panel shows data for $\alpha=\beta$. In this case the Fermi contours are two circles having the same radius and being displaced from the origin.

particular.[14] Here a new conserved quantity given by $\Sigma := (\sigma^x \mp \sigma^y)/\sqrt{2}$ arises,[35] and the spin state of the electrons becomes independent of the wave vector. For this situation the dispersion relations are more conveniently written as (choosing $\alpha=+\beta$)

$$\varepsilon_{(\pm)}(\vec{k}) = \frac{\hbar^2}{2m}|\vec{K}_{(\pm)}(\vec{k})|^2 - \frac{2m\beta^2}{\hbar^2}, \tag{13}$$

where

$$\vec{K}_{(\pm)}(\vec{k}) = \vec{k} \pm \frac{\sqrt{2}m\beta}{\hbar^2}(1,1) \tag{14}$$

is the distance vector between the centers of the circles and points on their circumference. The double sign labeling branches in Eq. (13) does *not* correspond to the one in Eq. (4) and is therefore put in parentheses. As seen in Fig. 1, right at $\alpha=\beta$ different parts of the dispersion branches for $\alpha\neq\beta$ merge to different circles inducing a relabeling of branches.

III. BOLTZMANN THEORY OF ANISOTROPIC TRANSPORT IN TWO DIMENSIONS

The Boltzmann equation for transport in the two-dimensional electron gas in the presence of fixed random impurities reads[36]

$$\frac{\partial f_\mu}{\partial t} + \dot{\vec{r}}\cdot\frac{\partial f_\mu}{\partial \vec{r}} + \dot{\vec{k}}\cdot\frac{\partial f_\mu}{\partial \vec{k}} = \left(\frac{\partial f_\mu}{\partial t}\right)_{\text{coll}}. \tag{15}$$

Here μ is a band index; in the context of the preceding section it corresponds to the double sign labeling the two dispersion branches. $f_\mu(\vec{r},\vec{k},t)$ is the usual semiclassical distribution function, and the collision term is given by

$$\left(\frac{\partial f_\mu}{\partial t}\right)_{\text{coll}} = \sum_{\mu'}\int \frac{d^2k'}{(2\pi)^2}[w(\vec{k},\mu;\vec{k}',\mu')f_{\mu'}(\vec{k}')\{1-f_\mu(\vec{k})\} - w(\vec{k}',\mu';\vec{k},\mu)f_\mu(\vec{k})\{1-f_{\mu'}(\vec{k}')\}], \tag{16}$$

where $w(\vec{k},\mu;\vec{k}',\mu')$ is a transition probability determined by the fixed impurities. The semiclassical equations of motion read

$$\dot{\vec{r}} = \vec{v}_\mu(\vec{k}) = \frac{\partial\varepsilon_\mu(\vec{k})}{\hbar\,\partial\vec{k}}, \quad \hbar\dot{\vec{k}} = -e\vec{E}, \tag{17}$$

where $\varepsilon_\mu(\vec{k})$ is the dispersion of the band μ, $(-e)=-|e|$ is the electron charge, and $\vec{E}$ is an external electric field in the plane of the two-dimensional gas. Assuming a homogeneous system in a stationary state, $f_\mu(\vec{r},\vec{k},t)=f_\mu(\vec{k})$, and elastic scattering fulfilling the microreversibility condition $w(\vec{k},\mu;\vec{k}',\mu')=w(\vec{k}',\mu';\vec{k},\mu)$, the Boltzmann equation[36] becomes in lowest order in $|\vec{E}|$

$$-e\vec{E}\cdot\vec{v}_\mu(\vec{k})\left(-\frac{\partial f^0}{\partial\varepsilon}\right) = S[f_\mu(\vec{k})], \tag{18}$$

with the scattering operator

$$S[f_\mu(\vec{k})] = \sum_{\mu'}\int\frac{d^2k'}{(2\pi)^2}[w(\vec{k},\mu;\vec{k}',\mu')\{f_\mu(\vec{k}) - f_{\mu'}(\vec{k}')\}]. \tag{19}$$

Here f^0 is the equilibrium Fermi distribution depending only on the energy ε, and the derivative in Eq. (18) has to be evaluated at $\varepsilon=\varepsilon_\mu(\vec{k})$.

Now let $\vartheta(\vec{a})$ be the angle a given vector $\vec{a}$ forms with the direction of $\vec{E}$, fulfilling the relations

$$\vec{E}\cdot\vec{a} = Ea\cos[\vartheta(\vec{a})], \tag{20}$$

$$(\vec{e}_z\times\vec{E})\cdot\vec{a} = Ea\sin[\vartheta(\vec{a})], \tag{21}$$

where $\vec{e}_z$ is the direction perpendicular to the two-dimensional (xy) plane. The form of the transport equation (18) suggests to study the action of the scattering operator (19) on the functions $f_\mu(\vec{k})=|v_\mu(\vec{k})|\cos[\vartheta(v_\mu(\vec{k}))]$ and $f_\mu(\vec{k})=|v_\mu(\vec{k})|\sin[\vartheta(v_\mu(\vec{k}))]$. In fact, by inserting these functions into Eq. (19) and expressing the angle $\vartheta(v_{\mu'}(\vec{k}'))$ in terms of $\vartheta(v_\mu(\vec{k}))$ and $[\vartheta(v_\mu(\vec{k}))-\vartheta(v_{\mu'}(\vec{k}'))]$ via elementary trigonometric relations, it is easy to show that

$$S\left[\begin{pmatrix} |\vec{v}_\mu(\vec{k})|\cos[\vartheta(\vec{v}_\mu(\vec{k}))] \\ |\vec{v}_\mu(\vec{k})|\sin[\vartheta(\vec{v}_\mu(\vec{k}))] \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{\tau_\mu^\parallel(\vec{k})} & -\frac{1}{\tau_\mu^\perp(\vec{k})} \\ \frac{1}{\tau_\mu^\perp(\vec{k})} & \frac{1}{\tau_\mu^\parallel(\vec{k})} \end{pmatrix} \begin{pmatrix} |\vec{v}_\mu(\vec{k})|\cos[\vartheta(\vec{v}_\mu(\vec{k}))] \\ |\vec{v}_\mu(\vec{k})|\sin[\vartheta(\vec{v}_\mu(\vec{k}))] \end{pmatrix}, \quad (22)$$

with

$$\frac{1}{\tau_\mu^\parallel(\vec{k})} = \sum_{\mu'} \int \frac{d^2k'}{(2\pi)^2}\left[w(\vec{k},\mu;\vec{k}',\mu')\cdot\left(1-\frac{|\vec{v}_{\mu'}(\vec{k}')|}{|\vec{v}_\mu(\vec{k})|}\times\cos[\vartheta(\vec{v}_\mu(\vec{k}))-\vartheta(\vec{v}_{\mu'}(\vec{k}'))]\right)\right], \quad (23)$$

$$\frac{1}{\tau_\mu^\perp(\vec{k})} = \sum_{\mu'} \int \frac{d^2k'}{(2\pi)^2}\left[w(\vec{k},\mu;\vec{k}',\mu')\cdot\frac{|\vec{v}_{\mu'}(\vec{k}')|}{|\vec{v}_\mu(\vec{k})|}\times\sin[\vartheta(\vec{v}_\mu(\vec{k}))-\vartheta(\vec{v}_{\mu'}(\vec{k}'))]\right]. \quad (24)$$

Note that $\tau_\mu^\parallel(\vec{k})$ and $\tau_\mu^\perp(\vec{k})$ are independent of the common direction with respect to the angles $\vartheta(\vec{v}_\mu(\vec{k}))$ and $\vartheta(\vec{v}_{\mu'}(\vec{k}'))$ in the above integrals are defined, since only differences of those angles occur. Therefore, $\tau_\mu^\parallel(\vec{k})$ and $\tau_\mu^\perp(\vec{k})$ are independent of the direction of the electric field $\vec{E}$.

Now consider the deviation $g_\mu(\vec{k})$ of the distribution function $f_\mu(\vec{k})$ from equilibrium,

$$g_\mu(\vec{k}) = f_\mu(\vec{k}) - f^0. \quad (25)$$

Making the ansatz

$$g_\mu(\vec{k}) = \left(-\frac{\partial f^0}{\partial\varepsilon}\right)|\vec{v}_\mu(\vec{k})|\{A_\mu(\vec{k})\cos[\vartheta(\vec{v}_\mu(\vec{k}))] + B_\mu(\vec{k})\sin[\vartheta(\vec{v}_\mu(\vec{k}))]\}, \quad (26)$$

with two parameters $A_\mu(\vec{k})$, $B_\mu(\vec{k})$, one finds from the above equations

$$A_\mu(\vec{k}) = \frac{-eE\tau_\mu^\parallel(\vec{k})}{1+\left(\frac{\tau_\mu^\parallel(\vec{k})}{\tau_\mu^\perp(\vec{k})}\right)^2}, \quad (27)$$

$$B_\mu(\vec{k}) = \frac{-eE\tau_\mu^\perp(\vec{k})}{1+\left(\frac{\tau_\mu^\perp(\vec{k})}{\tau_\mu^\parallel(\vec{k})}\right)^2}, \quad (28)$$

or

$$g_\mu(\vec{k}) = g_\mu^\parallel(\vec{k}) + g_\mu^\perp(\vec{k}), \quad (29)$$

with

$$g_\mu^\parallel(\vec{k}) = -e\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\frac{\tau_\mu^\parallel(\vec{k})}{1+\left(\frac{\tau_\mu^\parallel(\vec{k})}{\tau_\mu^\perp(\vec{k})}\right)^2}\vec{E}\cdot\vec{v}_\mu(\vec{k}), \quad (30)$$

$$g_\mu^\perp(\vec{k}) = -e\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{\tau_\mu^\perp(\vec{k})}{1+\left(\frac{\tau_\mu^\perp(\vec{k})}{\tau_\mu^\parallel(\vec{k})}\right)^2}\right]\cdot(\vec{e}_z\times\vec{E})\cdot\vec{v}_\mu(\vec{k}). \quad (31)$$

From this distribution function the electrical current of particles in band μ can be obtained as

$$\vec{j}_\mu = -e\int \frac{d^2k}{(2\pi)^2}\vec{v}_\mu(\vec{k})g_\mu(\vec{k}), \quad (32)$$

and the total electrical current is given by

$$\vec{j} = \sum_\mu \vec{j}_\mu. \quad (33)$$

From these relations one obtains the following conductivity tensor:

$$\sigma = \begin{pmatrix} \sigma_{xx}^\parallel + \sigma_{xy}^\perp & \sigma_{xy}^\parallel - \sigma_{xx}^\perp \\ \sigma_{xy}^\parallel + \sigma_{yy}^\perp & \sigma_{yy}^\parallel - \sigma_{xy}^\perp \end{pmatrix}, \quad (34)$$

where we have introduced the definitions

$$\sigma_{ij}^\parallel = e^2\sum_\mu \int \frac{d^2k}{(2\pi)^2}\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\cdot\left[\frac{\tau_\mu^\parallel(\vec{k})}{1+\left(\frac{\tau_\mu^\parallel(\vec{k})}{\tau_\mu^\perp(\vec{k})}\right)^2}\right] \times[\vec{v}_\mu(\vec{k})]_i[\vec{v}_\mu(\vec{k})]_j, \quad (35)$$

$$\sigma_{ij}^\perp = e^2\sum_\mu \int \frac{d^2k}{(2\pi)^2}\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\cdot\left[\frac{\tau_\mu^\perp(\vec{k})}{1+\left(\frac{\tau_\mu^\perp(\vec{k})}{\tau_\mu^\parallel(\vec{k})}\right)^2}\right] \times[\vec{v}_\mu(\vec{k})]_i[\vec{v}_\mu(\vec{k})]_j. \quad (36)$$

Several remarks are in order.

(i) For an isotropic dispersion and scattering potentials isotropic in real space, only $\sigma_{xx}^\parallel = \sigma_{yy}^\parallel$ are different from zero, and the conductivity tensor is proportional to the unit matrix. If additionally only one dispersion branch is there, the parameter $\tau^\parallel$ becomes

$$\frac{1}{\tau^\parallel(\vec{k})} = \int \frac{d^2k'}{(2\pi)^2}[w(\vec{k};\vec{k}')\{1-\cos[\vartheta(\vec{k})-\vartheta(\vec{k}')]\}]. \quad (37)$$

This is just the usual expression for the relaxation time in the isotropic standard case[36] and is independent of the wave vec-

tor $\vec{k}$, $\tau^{\parallel}(\vec{k})=:\tau_0$. In fact, the above considerations can be seen as a generalization of the standard isotropic case to general anisotropic dispersions in two dimensions. Note that, although the parameters $\tau_\mu^{\parallel}$ and $\tau_\mu^{\perp}$ have dimension of time, this does not mean that any relaxation-time *approximation* has been used to treat the case of anisotropic dispersions. In fact Eqs. (29)–(31) constitute an *exact solution* of the Boltzmann equation (18).

(ii) If in addition to $\sigma_{xx}^{\parallel}=\sigma_{yy}^{\parallel}\neq 0$, the contributions $\sigma_{xy}^{\parallel}=\sigma_{yx}^{\parallel}$ are nonzero, the degeneracy of the conductivity eigenvalues is lifted. In the case $\sigma_{xx}^{\parallel}=\sigma_{yy}^{\parallel}$ these eigenvalues are then given by $\sigma_{xx}^{\parallel}\pm\sigma_{xy}^{\parallel}$ with the eigendirections $(1,\pm 1)$.

(iii) Provided that $\sigma_{xx}^{\perp}=\sigma_{yy}^{\perp}\neq 0$, this contribution to the conductivity tensor corresponds to the anomalous or extraordinary Hall effect. This is an antisymmetric contribution to the conductivity tensor which does not stem from an external magnetic field but entirely from scattering processes. For such a contribution to be present, time-reversal symmetry has to be broken.

For the case of anisotropic dispersions induced by spin-orbit coupling as discussed in detail in the following section, we will see that there is no anomalous Hall effect (since time-reversal symmetry is intact), but there is a symmetric off-diagonal contribution to the conductivity tensor which stems from both $\sigma_{xy}^{\parallel}=\sigma_{yx}^{\parallel}\neq 0$ and $\sigma_{xx}^{\perp}=-\sigma_{yy}^{\perp}\neq 0$.

IV. CONDUCTIVITY IN THE PRESENCE OF SPIN-ORBIT COUPLING

We now proceed with calculating transport properties for Fermi-liquid electrons in two dimensions in the presence of spin-orbit coupling, using the formalism of the preceding section. To be specific, we will evaluate the transition probabilities in the scattering operator (19) by Fermi's golden rule,

$$w(\vec{k},\mu;\vec{k}',\mu')=\frac{2\pi}{\hbar}\frac{\nu}{A}|(\vec{k},\mu|V|\vec{k}',\mu')|^2\cdot\delta(\varepsilon_\mu(\vec{k})-\varepsilon_{\mu'}(\vec{k}')), \tag{38}$$

where V is the operator of a single scatterer and ν is the density of scatterers. The momentum eigenstates involved above are normalized as

$$(\vec{k},\mu|\vec{k}',\mu')=A\,\delta_{\vec{k},\vec{k}'}\,\delta_{\mu,\mu'}, \tag{39}$$

with A being the area of the system. As a further simplification we will consider fixed impurities with δ-function shaped scattering potentials (s-wave approximation),

$$V(\vec{r})=\kappa\delta(\vec{r}), \tag{40}$$

where κ parametrizes the strength of the potential. The square moduli of the matrix elements read

$$|(\vec{k},\mu|V|\vec{k}',\mu')|^2=\frac{\kappa^2}{2}\{1+\mu\mu'\cos[\chi(\vec{k})-\chi(\vec{k}')]\}. \tag{41}$$

Moreover, we will concentrate on the case of zero temperature, where the derivative of the Fermi function as it arises in the integral expressions for transport parameters is equal to the negative of a δ function peaked at the Fermi energy. Thus the integrations over momentum space in Eqs. (35) and (36) reduce to integration over the Fermi contour.

However, even with these simplifications, the integrations involved are in general nonelementary. In order to make analytical progress we concentrate on the case of finite Dresselhaus coupling and small Rashba coupling ($|\alpha|\ll\beta$, Sec. IV A) and the particular case $\alpha=\beta$ (Sec. IV B).

A. The case $\alpha\ll\beta$

It is straightforward to expand the quantities entering the transport parameters and conductivities discussed in Sec. III for $\alpha\ll\beta$ in lowest order in α. However, since the calculations are somewhat lengthy, details are given in the Appendix. The full result for the elements of the conductivity tensor (34) up to linear order in α but general values for Dresselhaus coefficient β and positive Fermi energy ε_f is stated in Eqs. (A17) and (A18). These expressions are still somewhat complicated but simplify significantly if one additionally assumes that the "Dresselhaus energy" $\varepsilon_D:=m\beta^2/\hbar^2$ is small compared to ε_f as it is usually the case for realistic situations. In other words, defining the "Rashba energy" as $\varepsilon_R:=m\alpha^2/\hbar^2$, we consider the situation

$$\varepsilon_R\ll\varepsilon_D\ll\varepsilon_f, \tag{42}$$

where the Fermi energy is related to the electron density n via [cf. Eq. (12)]

$$n=\frac{1}{2\pi}\left[\frac{2m}{\hbar^2}\varepsilon_f+2\left(\frac{m}{\hbar^2}\right)^2\beta^2\right]+\mathcal{O}(\alpha^2). \tag{43}$$

Then one has

$$\sigma_{xx}=\sigma_{yy}=\sigma_0+\mathcal{O}\left(\frac{\varepsilon_R}{\varepsilon_f},\frac{\varepsilon_D}{\varepsilon_f}\right), \tag{44}$$

$$\sigma_{xy}=\sigma_{yx}=\sigma_0(-\mathrm{sign}(\alpha))\frac{7}{8}\sqrt{\frac{\varepsilon_R\varepsilon_D}{\varepsilon_f}}+\mathcal{O}\left(\frac{\varepsilon_R}{\varepsilon_f},\sqrt{\frac{\varepsilon_R}{\varepsilon_f}}\frac{\varepsilon_D}{\varepsilon_f}\right), \tag{45}$$

where σ_0 is the usual Drude conductivity,

$$\sigma_0=\frac{e^2\tau_0 n_0}{m}, \tag{46}$$

and

$$\tau_0=\frac{\hbar^3}{\nu\kappa^2 m}, \tag{47}$$

$$n_0=\frac{k_f^2}{2\pi} \tag{48}$$

are the momentum relaxation time and particle density, respectively, in the absence of spin-orbit coupling. The eigenvalues of the conductivity tensor are

$$\sigma^+ = \sigma_{xx} + \sigma_{xy}, \quad \sigma^- = \sigma_{xx} - \sigma_{xy}, \tag{49}$$

with corresponding eigendirections (1,1) and (1,−1), respectively. These directions are the symmetry axes of the underlying dispersion relations; the same eigendirections are found from Eqs. (A17) and (A18) where the Dresselhaus energy has not been assumed to be small compared to the Fermi energy. From Eqs. (44) and (45), the conductivity anisotropy $\Delta\sigma$ is given by

$$\Delta\sigma := \frac{|\sigma^+ - \sigma^-|}{\sigma^+ + \sigma^-} = \frac{7}{8}\frac{\sqrt{\varepsilon_R \varepsilon_D}}{\varepsilon_f} + \mathcal{O}\left(\frac{\varepsilon_R}{\varepsilon_f}, \frac{\varepsilon_D}{\varepsilon_f}\right). \tag{50}$$

We note that changing the sign of α (by reversing the potential gradient across the quantum well) results in a shift by $\pi/2$ in the wave-vector dependence of dispersion relations and eigenstates. Such a shift leads to a sign change in $(\sigma^+ - \sigma^-)$. Therefore, this quantity contains only odd powers of α.

B. The case $\alpha = \beta$

As discussed already in detail in Ref. 14 and in Sec. II, the case $\alpha = \beta$ is special under several aspects. Here the transport quantities are readily obtained using form (13) for the dispersion relations. As a result, the conductivity tensor is isotropic with

$$\sigma_{xx} = \sigma_{yy} = \sigma_0 = \frac{e^2 \tau_0 n_0}{m}, \tag{51}$$

where $\tau_0 = \hbar^3/\nu\kappa^2 m$ as in Eq. (47), and

$$n_0 = \frac{|\vec{K}_f|^2}{2\pi} = \frac{1}{2\pi}\frac{2m}{\hbar^2}\left(\varepsilon_f + \frac{2m\beta^2}{\hbar^2}\right) \tag{52}$$

is the density of electrons. At small deviations from the point $\alpha = \beta$ one should expect the conductivity tensor to develop again an anisotropy. However, this cannot be analyzed in the same way as the case $|\alpha| \ll \beta$ since the particle velocities and other quantities entering the integrands in Eqs. (23) and (24) do not allow for an expansion in $|\alpha - \beta|$ around $\alpha = \beta$ for wave vectors with $k_x = k_y$. At these points the dispersion branches (4) continuously merge into two new circles when approaching $\alpha = \beta$, cf. the lower right panel of Fig. 1. Therefore, in order to evaluate the conductivity tensor around $\alpha = \beta$, one should use other methods rather than expanding the dispersion relations. For our purposes here, we shall be content with the statement that the conductivity tensor is of course continuous around $\alpha = \beta$, and is isotropic exactly at that point.

C. The influence of trilinear contributions to the Dresselhaus term

Hamiltonian (2) is derived from the bulk Dresselhaus spin-orbit coupling being trilinear in the momentum operators,[4]

$$\mathcal{H}_D^{bulk} = \frac{\gamma}{\hbar}[\sigma^x p_x(p_y^2 - p_z^2) + \sigma^y p_y(p_z^2 - p_x^2) + \sigma^z p_z(p_x^2 - p_y^2)], \tag{53}$$

with a coupling parameter γ. In a sufficiently narrow quantum well grown along the [001] direction one can approximate the operators p_z and p_z^2 by their expectation values $\langle p_z \rangle$, $\langle p_z^2 \rangle$. This leads to the following two contributions to spin-orbit coupling resulting from bulk inversion asymmetry: The Dresselhaus term (2) linear in the momenta with $\beta = \gamma\langle p_z^2 \rangle$, and the trilinear term

$$\mathcal{H}_D^{(3)} = \frac{\gamma}{\hbar}(\sigma^x p_x p_y^2 - \sigma^y p_y p_x^2). \tag{54}$$

Clearly the typical magnitude of $\mathcal{H}_D^{(3)}$ compared to the linear term $\mathcal{H}_D$ is given by the ratio of the Fermi energy ε_f of the in-plane motion to the kinetic energy of the quantized degree of freedom in the growth direction. For typical values of ε_f of about 10 meV and not too broad quantum wells this ratio is small, and we have therefore neglected the Dresselhaus term trilinear in the momentum components. If desired, it is straightforward to include this term in the calculations of transport quantities, although the procedure becomes considerably more involved and will require numerical calculations. However, we do not expect, for the following reasons, that including the trilinear Dresselhaus term but not the Rashba term will lead to anisotropic charge transport: The Hamiltonian

$$\mathcal{H} = \frac{\vec{p}^2}{2m} + \mathcal{H}_D + \mathcal{H}_D^{(3)} \tag{55}$$

gives the following dispersions for wave vectors $\vec{k} = k(\cos\varphi, \sin\varphi)$:

$$\varepsilon_\pm(k,\varphi) = \frac{\hbar^2 k^2}{2m} \pm \sqrt{\beta^2 k^2 - [1 - \cos(4\varphi)]\left(\frac{\hbar^2}{2}\beta\gamma k^4 - \frac{\hbar^4}{8}\gamma^2 k^6\right)}. \tag{56}$$

The angular variable φ enters only terms of $\cos(4\varphi)$ which leads to Fermi contours with fourfold symmetry, differently from the just twofold symmetry in the case of Rashba and linear Dresselhaus term. In particular, for Hamiltonian (55) the dispersions are symmetric with respect to both the axes pairs (1,0), (0,1) and (1,1), (1,−1), and these axes pairs are the possible candidates for eigendirections of the conductivity tensor. However, since directions in the above pairs are equivalent due to the existence of the other pair of symmetry axes, we do not expect an anisotropy in transport quantities. Moreover, as seen above, such anisotropies arise from the interplay of the Rashba and the Dresselhaus term and are tunable by external gates. Concerning gating the Rashba coefficient by an electric field across the quantum well, one should keep in mind that such an operation might effectively also alter the Dresselhaus coefficient $\beta = \gamma\langle p_z^2 \rangle$ by changing

the expectation value $\langle p_z^2 \rangle$. However, this additional effect cannot change our principle findings concerning anisotropic transport.

V. SUMMARY AND DISCUSSION

We have presented a theory of anisotropic transport in a two-dimensional electron gas. The anisotropy in the electrical conductivity is induced by the interplay between Rashba and Dresselhaus spin-orbit coupling in the semiconductor quantum well confining the electron gas. The principle axes for anisotropic diffusive charge transport are given by the symmetry axes of the single-particle dispersion relations, which are anisotropic if both Rashba and Dresselhaus spin-orbit interactions are present. We have evaluated the conductivity tensor at zero temperature for scattering on fixed random impurities whose potentials are modeled by δ functions. However, because of the anisotropic properties of the underlying dispersions, we do not expect our results to change qualitatively if other impurity potentials are considered. In particular, the differential cross section for δ-function potentials is isotropic, which makes obvious that our result is due to the spin-orbit induced effects and not due to special properties of the scatterers.

To enable analytical progress in the evaluation of transport properties, we have concentrated on the case of a finite Dresselhaus term and a small Rashba term ($|\alpha| \ll \beta$), and on the case where the Rashba and Dresselhaus coefficients are equal ($\alpha = \beta$). For $|\alpha| \ll \beta$ we have found the anisotropic corrections to the conductivity tensor due to the presence of the Rashba term. These findings demonstrate the principle result that diffusive charge transport becomes anisotropic if both Rashba and Dresselhaus spin-orbit coupling are present. This anisotropy can be tuned by external gates which provides the possibility of detecting and investigating spin-orbit interaction by measuring spin-unpolarized diffusive electrical currents. Apart from possible device applications of this effect, the experimental observation of such a tunable anisotropy in spin-unpolarized diffusive transport would certainly significantly confirm and deepen our understanding of spin-orbit coupling in semiconductors.

In our calculations we have concentrated on the Dresselhaus contributions being linear in the momentum components, as it is appropriate for not too wide quantum wells. The possible influence of the trilinear Dresselhaus term is discussed in Sec. IV C.

The case $\alpha = \beta$ is special under several aspects due to the additional conserved quantity that arises at this point.[14] Here the conductivity tensor is found to be isotropic.

Our approach to anisotropic transport in two dimensions is based on an exact solution of the Boltzmann equation where the drift term is linearized in the in-plane electric field driving the current. This formalism can also deal with the case of anisotropic single-particle dispersions and should be seen as a generalization of the usual isotropic case. We expect this approach to be also useful in the study of other transport effects such as thermal conductivity, magnetothermal effects, and the anomalous Hall effect.

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APPENDIX: CALCULATION OF TRANSPORT PROPERTIES AT $|\alpha| \ll \beta$

In this appendix we present details of the calculation of transport properties at $|\alpha| \ll \beta$ using Fermi's golden rule (38) in the case of vanishing temperature.

1. Dispersion relations, eigenstates, and Fermi contours at $|\alpha| \ll \beta$

For the single-particle energies and the phases $\chi(\vec{k})$ entering eigenvectors (5), one finds the following expansions:

$$\varepsilon_{\pm}(\vec{k}) = \frac{\hbar^2 k^2}{2m} \pm \beta k \sqrt{\left(1 + \frac{\alpha^2}{\beta^2}\right) + 2\frac{\alpha}{\beta}\frac{(\vec{k}^T \tilde{\sigma}^x \vec{k})}{k^2}}$$
$$= \frac{\hbar^2 k^2}{2m} \pm \beta k \left[1 + \frac{\alpha}{\beta}\frac{(\vec{k}^T \tilde{\sigma}^x \vec{k})}{k^2} + \mathcal{O}\left(\frac{\alpha^2}{\beta^2}\right)\right], \quad \text{(A1)}$$

$$\chi(\vec{k}) = \arg\left[\left(-k_x - \frac{\alpha}{\beta}k_y\right) + i\left(k_y + \frac{\alpha}{\beta}k_x\right)\right]$$
$$= \arg\left[(-k_x + ik_y)\left(1 - \frac{\alpha}{\beta}\frac{i}{k^2}(k_x + ik_y)^2\right)\right]$$
$$= \arg\left\{(-k_x + ik_y)\cdot\left[\exp\left(-\frac{\alpha}{\beta}\frac{i}{k^2}(k_x + ik_y)^2\right) + \mathcal{O}\left(\frac{\alpha^2}{\beta^2}\right)\right]\right\}$$
$$= \arg(-k_x + ik_y) - \frac{\alpha}{\beta}\frac{(\vec{k}^T \tilde{\sigma}^z \vec{k})}{k^2} + \mathcal{O}\left(\frac{\alpha^2}{\beta^2}\right). \quad \text{(A2)}$$

Here $\tilde{\sigma}^x$ and $\tilde{\sigma}^z$ are again usual Pauli matrices acting on the two-component vectors $\vec{k}$. Note that $\chi(\vec{k})$ and therefore the eigenstates remain unchanged if $\vec{k}$ points along the directions (1,1) or (1,−1). This can also be seen directly from Eq. (6). The expansion of the particle velocities reads

$$\vec{v}_{\pm}(\vec{k}) = \frac{\hbar\vec{k}}{m} \pm \frac{\beta}{\hbar}\left[\frac{\vec{k}}{k} - \frac{\alpha}{\beta}\left(\frac{(\vec{k}^T \tilde{\sigma}^x \vec{k})}{k^2} - 2\tilde{\sigma}^x\right)\frac{\vec{k}}{k} + \mathcal{O}\left(\frac{\alpha^2}{\beta^2}\right)\right], \quad \text{(A3)}$$

$$|\vec{v}_{\pm}(\vec{k})| = \frac{\hbar k}{m} \pm \frac{\beta}{\hbar} \pm \frac{\alpha}{\hbar}\frac{(\vec{k}^T \tilde{\sigma}^x \vec{k})}{k^2} + \mathcal{O}(\alpha^2). \quad \text{(A4)}$$

In order to evaluate the transport parameters $\tau_{\pm}^{\parallel}$ and $\tau_{\pm}^{\perp}$ according to Eqs. (23) and (24) one needs angles of the type $\vartheta(\vec{v}_{\mu}(\vec{k}))$. As already remarked in Sec. III, $\tau_{\pm}^{\parallel}$ and $\tau_{\pm}^{\perp}$ are independent of the direction with respect to which these angles are defined. It is convenient to choose this direction along the x axis. Then one finds analogously to Eq. (A2)

$$\vartheta(\vec{v}_{\pm}(\vec{k})) = \arg\{[\vec{v}_{\pm}(\vec{k})]_x + i[\vec{v}_{\pm}(\vec{k})]_y\}$$
$$= \vartheta(\vec{k}) \pm \frac{2\alpha}{\frac{\hbar^2 k}{m} \pm \beta} \frac{(\vec{k}^T \tilde{\sigma}^z \vec{k})}{k^2} + \mathcal{O}(\alpha^2). \quad \text{(A5)}$$

For the Fermi contour as given in Eq. (10) for positive $\varepsilon_f \geqslant 0$ one has the expansion

$$k^f_{\pm}(\varphi;\varepsilon_f) = \sqrt{\frac{2m}{\hbar^2}\varepsilon_f + \left(\frac{m}{\hbar^2}\right)^2 \beta^2} \mp \frac{m}{\hbar^2}\beta + \frac{m\alpha}{\hbar^2}\left(\frac{1}{\sqrt{1+2\frac{\hbar^2}{m\beta^2}\varepsilon_f}} \mp 1\right)\sin(2\varphi) + \mathcal{O}(\alpha^2). \quad \text{(A6)}$$

Note that, at a given electron density n, the Fermi energy ε_f is according to Eq. (43) unchanged in first order in α. Moreover, when inserting the Fermi momentum in zeroth order in α,

$$(k^f_{\pm})_0 = \sqrt{\frac{2m}{\hbar^2}\varepsilon_f + \left(\frac{m}{\hbar^2}\right)^2 \beta^2} \mp \frac{m}{\hbar^2}\beta, \quad \text{(A7)}$$

in Eq. (A4) one obtains

$$[|\vec{v}_{\pm}(\vec{k}^f_{\pm})|]_0 = \sqrt{\frac{2\varepsilon_f}{m} + \left(\frac{\beta}{\hbar}\right)^2}, \quad \text{(A8)}$$

i.e., the Fermi velocity is in zeroth order in α independent of the band index $\mu \in \{+,-\}$. Using Eq. (A8), expansion (A6) for the Fermi momentum can be rewritten up to linear order in α as

$$k^f_{\pm}(\varphi;\varepsilon_f) = \frac{m}{\hbar}\left([|\vec{v}_{\pm}(\vec{k}^f_{\pm})|]_0 \mp \frac{\beta}{\hbar}\right) \times \left(1 \mp \frac{\alpha}{\hbar[|\vec{v}_{\pm}(\vec{k}^f_{\pm})|]_0}\sin(2\varphi)\right) + \mathcal{O}(\alpha^2). \quad \text{(A9)}$$

2. Transport quantities

Using the expansions given in the preceding section it is a little tedious but straightforward to obtain expressions for the transport quantities discussed in Sec. III up to linear order in the Rashba coefficient α. For the parameter $1/\tau^{\parallel}_{\mu}(\vec{k})$, $\mu \in \{+,-\}$, one finds in zeroth order in α,

$$\left(\frac{1}{\tau^{\parallel}_{\mu}(\vec{k})}\right)_0 = \nu\kappa^2 \frac{m}{\hbar^3}\left(1 + \frac{\mu}{2}\frac{\beta}{\hbar[|\vec{v}_{\mu}(\vec{k})|]_0}\right), \quad \text{(A10)}$$

and the first order is given by

$$\left(\frac{1}{\tau^{\parallel}_{\mu}(\vec{k})}\right)_1 = -\nu\kappa^2 \frac{m}{\hbar^3}\frac{1}{2}\frac{\alpha\beta}{\{\hbar[|\vec{v}_{\mu}(\vec{k})|]_0\}^2}\frac{(\vec{k}^T\tilde{\sigma}^x\vec{k})}{k^2}. \quad \text{(A11)}$$

For $1/\tau^{\perp}_{\mu}(\vec{k})$, the zeroth order in α vanishes while the first order reads

$$\left(\frac{1}{\tau^{\perp}_{\mu}(\vec{k})}\right)_1 = \nu\kappa^2 \frac{m}{\hbar^3}\frac{\alpha}{\hbar[|\vec{v}_{\mu}(\vec{k})|]_0} \times \left(\frac{1}{2}\mu - \frac{\beta}{\hbar[|\vec{v}_{\mu}(\vec{k})|]_0}\right)\frac{(\vec{k}^T\tilde{\sigma}^z\vec{k})}{k^2}. \quad \text{(A12)}$$

We now turn to the parameters entering the conductivity tensor. For the diagonal elements of $\sigma^{\parallel}$ we find in zeroth order

$$(\sigma^{\parallel}_{xx})_0 = (\sigma^{\parallel}_{yy})_0 = \frac{e^2}{h}\frac{\hbar^3}{\nu\kappa^2 m}\frac{m}{\hbar}\left[\frac{2\varepsilon_f}{m} + \left(\frac{\beta}{\hbar}\right)^2\right]\frac{\frac{2\varepsilon_f}{m} + \frac{3}{2}\left(\frac{\beta}{\hbar}\right)^2}{\frac{2\varepsilon_f}{m} + \frac{3}{4}\left(\frac{\beta}{\hbar}\right)^2}, \quad \text{(A13)}$$

while the contributions in first order in α vanish. The off-diagonal elements of $\sigma^{\parallel}$ are zero at $\alpha = 0$, and the first order reads

$$(\sigma^{\parallel}_{xy})_1 = (\sigma^{\parallel}_{yx})_1 = \frac{e^2}{h}\frac{\hbar^3}{\nu\kappa^2 m}\frac{m}{\hbar}\frac{\alpha}{\hbar} \cdot \frac{\beta}{\hbar}\left(\frac{2\varepsilon_f}{m} + \left(\frac{\beta}{\hbar}\right)^2\right)\frac{\frac{-11}{4}\frac{2\varepsilon_f}{m} - \frac{27}{16}\left(\frac{\beta}{\hbar}\right)^2}{\left(\frac{2\varepsilon_f}{m} + \frac{3}{4}\left(\frac{\beta}{\hbar}\right)^2\right)^2}. \quad \text{(A14)}$$

Finally, the off-diagonal elements of $\sigma^{\perp}$ vanish up to linear order in α,

$$\sigma^{\perp}_{xy} = \sigma^{\perp}_{yx} = 0 + \mathcal{O}(\alpha^2). \quad \text{(A15)}$$

The diagonal elements are also zero at vanishing α, and the contribution in first order in α is

$$(\sigma^{\perp}_{xx})_1 = -(\sigma^{\perp}_{yy})_1 = -\frac{e^2}{h}\frac{\hbar^3}{\nu\kappa^2 m}\frac{m}{\hbar}\frac{\alpha}{\hbar}\frac{\beta}{\hbar}\left[\frac{2\varepsilon_f}{m} + \left(\frac{\beta}{\hbar}\right)^2\right]\cdot\frac{\frac{2\varepsilon_f}{m} + \frac{27}{16}\left(\frac{\beta}{\hbar}\right)^2}{\left[\frac{2\varepsilon_f}{m} + \frac{3}{4}\left(\frac{\beta}{\hbar}\right)^2\right]^2}. \quad \text{(A16)}$$

From this one finds the elements of the conductivity tensor as

$$\sigma_{xx} = \sigma_{yy} = \sigma^{\parallel}_{xx} = \frac{e^2}{h}\frac{\hbar^3}{\nu\kappa^2 m}\frac{m}{\hbar}\left[\frac{2\varepsilon_f}{m} + \left(\frac{\beta}{\hbar}\right)^2\right]\frac{\frac{2\varepsilon_f}{m} + \frac{3}{2}\left(\frac{\beta}{\hbar}\right)^2}{\frac{2\varepsilon_f}{m} + \frac{3}{4}\left(\frac{\beta}{\hbar}\right)^2} + \mathcal{O}(\alpha^2), \quad \text{(A17)}$$

$$\sigma_{xy}=\sigma_{yx}=\sigma_{xy}^{\parallel}-\sigma_{xx}^{\perp}=\frac{e^2}{h}\frac{\hbar^3}{\nu\kappa^2 m}\frac{m}{\hbar}\frac{\alpha}{\hbar}$$

$$\cdot\left(-\frac{7}{4}\right)\frac{\beta}{\hbar}\left(\frac{2\varepsilon_f}{m}+\left(\frac{\beta}{\hbar}\right)^2\right)\frac{\frac{2\varepsilon_f}{m}}{\left(\frac{2\varepsilon_f}{m}+\frac{3}{4}\left(\frac{\beta}{\hbar}\right)^2\right)^2}$$

$$+\mathcal{O}(\alpha^2). \tag{A18}$$

Therefore, the eigenvalues of the conductivity tensor are

$$\sigma^{+}=\sigma_{xx}+\sigma_{xy}, \quad \sigma^{-}=\sigma_{xx}-\sigma_{xy}, \tag{A19}$$

with corresponding eigendirections (1,1) and (1,−1), respectively. These directions are the symmetry axes of the underlying dispersion relations.

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