Approximate ground state of a confined Coulomb anyon gas in an external magnetic field

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We derive an analytic, albeit approximate, expression for the ground state energy of N Coulomb interacting anyons with fractional statistics ν, $0 \leqslant |\nu| \leqslant 1$, confined in a two-dimensional well (with characteristic frequency ω_0) and subjected to an external magnetic field (with cyclotron frequency ω_c). We apply a variational principle combined with a regularization procedure which consists of fitting a cutoff parameter to existing exact analytical results in the noninteracting case, and to numerical calculations for electrons in quantum dots in the interacting case. The resulting expression depends upon parameters of the system $|\nu|, N, \omega_0, r_0, a_B$ and ω_c, where r_0 represents a characteristic unit length and a_B the Bohr radius. Validity of the result is critically assessed by a comparison with exact, approximate, and numerical results from the literature.

I. INTRODUCTION

The exchange statistics of particles whose orbital motion is restricted to two space dimensions differs substantially from the three-dimensional (3D) case. The topology of their (multiply connected) configuration space allows for fractional statistics,[1] characterized by a continuous parameter ν which labels the possible one-dimensional representations of the braid group. For particles in two dimensional, ν may attain values between 0 (for bosons) and 1 (for fermions), thus two dimensional (2D) particles are called anyons.[2] The concept of anyons has been used to describe quasiparticle excitations in the fractional quantum Hall regime[3–5] and in high-T_c superconductors.[6]

A system of particular current interest is that of 2D electrons in a parabolic confinement potential, the so-called quantum dot or artificial atom.[7] These systems, realized in semiconductor nanostructures, are objects of fundamental studies of ground state properties of interacting N-particle systems and have also a potential for applications in quantum information and computation.[8] Exact closed-form solutions of the problem are reduced to a few simple cases due to its intrinsic mathematical complexity.[9,10] Typically, ground state calculations make use of numerical simulations for individual choices of the parameters of the system. It would be desirable to have an accurate, albeit approximate, analytical expression describing the ground state energy of such a system as a function of the parameters of the system (including an external magnetic field applied perpendicular to the plane of the dot). In order to derive such a formula we make use of the anyon concept including the effect of the Coulomb interaction.

Anyons in a parabolic confining potential (with and without an external magnetic field) has been the subject of several investigations in the past. For the $N=2$ case an exact solution to the spectral problem exists.[1–3,5] For the noninteracting case, its generalization to $N=3$ (without magnetic field) is considered in (Ref. 11). Numerical calculations have been performed for the lower part of the energy spectrum for

$N=3$ (Ref. 12) and 4.[13] Making use of the separation of the center-of-mass and relative motion one finds in the two-anyon case that the ground state energy for the relative motion is a linear function of ν for $0 \leqslant \nu \leqslant 1$. For $N=3$ and 4,[5,12,13] the ground state energy of the relative motion is a linear function of ν only near the bosonic limit $\nu \approx 0$. The ground state in the fermionic end ($\nu=1$) is continuously connected to an excited state of the bosonic spectrum ($\nu=0$) and consequently — when adding the lowest energy of the center-of-mass motion (which does not depend on ν) — one finds for $N=3$ and 4 (and likely for all higher N[14]) a nonlinear ν dependence of the ground state energy. In the case of interacting anyons in no external magnetic field the energy spectrum for two anyons at some fixed values of the Coulomb interaction parameter was found analytically in Ref. 15 and approximately, for two and three anyons, in Ref. 16.

In the presence of an external magnetic field one can study the interplay between the statistical and physical magnetic fluxes. This has been done analytically for the $N=2$ noninteracting anyon case with confinement,[17] while the cases with $N>2$ have been investigated preferentially without confinement.[18–20] The ground state for the $N=3$ case including confinement and magnetic field was calculated in Ref. 21. The case with the applied external magnetic field and Coulomb interaction for two anyons in a harmonic potential was considered in Ref. 22 and for two and three anyons in Ref. 23. Reference 5 provides a review of all these studies.

In our treatment of the N-anyon problem we make use of the bosonic representation of anyons that works with a gauge vector potential to account for the fractional exchange statistics but allows to use a product ansatz for the N-body wave function. We apply a variational principle by constructing this wave function from single-particle Gaussians of variable shape. It is well known from perturbative ground state calculations for anyons in an oscillator potential that the expression for the ground state energy has a logarithmic divergence connected with a cutoff parameter for the interparticle

distance.[14,24–28] We face the same problem in our variational treatment. Making use of the physical argument (see Ref. 4) that for $\nu \neq 0$ this distance has to have some finite value, we regularize the formula obtained for the ground state energy by an appropriate procedure that takes into account some existing exact analytical results, in the case without Coulomb interaction, and numerical results for electrons in quantum dots, in the case with Coulomb interaction. Our formula, which is an approximate closed-form expression depending upon ν, N, ω_0 (confinement parameter), and r_0/a_B (Coulomb interaction parameter), where $r_0 = [\hbar/(M\omega_0)]^{1/2}$ and $a_B = \hbar^2/(Me^2)$ (in the presence of a magnetic field also of the parameter ω_c/ω_0, and we need to replace ν by $|\nu|$), will be compared to exact, approximate, and numerical results for quantum dots reported in the literature.

The paper is organized as follows: In Sec. II we describe the system and motivate the ansatz for the variational treatment, in Sec. III we present the calculations without, and in Sec. IV with a homogeneous magnetic field for the case without Coulomb interaction. Calculations including Coulomb interactions are presented in Secs. V and VI. Finally, Sec. VII summarizes the main conclusions.

We would like to note the existence of two seemingly unrelated notions of anyonic statistics in the literature: one originally introduced in first quantization in the coordinate representation, and another derived within the framework of quantum field theory. In both cases the original motivation to introduce such particles was basically as an inherent possibility in the kinematics of $(2+1)$-dimensional quantum mechanics and clearly the concepts, if correctly implemented, should be equivalent whether one uses first quantization in the coordinate representation or second quantization.[29,30] Within the framework of quantum field theory fermions can be kinematically transformed into hard-core bosons (through statistical transmutation) but not into canonical ones, thus preserving the exclusion statistics properties of the particles. More generically, the Hamiltonian spectra of particles sharing the same exclusion statistics can be connected through a continuous mapping. The anyon notion used in the present paper is consistent with the one developed in the framework of the bosonic representation in first quantization. Had we used the fermionic representation we would have ended up in an excited bosonic state.

II. INTERACTING ANYONS IN A 2D PARABOLIC WELL IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD: GENERAL SETUP

The Hamiltonian of N spinless anyons of mass M and charge e confined to a 2D parabolic well, interacting through Coulomb repulsions, and in the presence of an external homogeneous magnetic field, $\vec{H} = H\vec{e}_z$ $[\omega_c = |eH|/(Mc)]$, is given by

$$\hat{H} = \frac{1}{2M} \sum_{k=1}^{N} (\{\vec{p}_k - [\vec{A}_\nu(\vec{r}_k) + e\vec{A}_{ext}(\vec{r}_k)/c]\}^2 + M^2\omega_0^2|\vec{r}_k|^2)$$

$$+ \frac{1}{2} \sum_{k,j \neq k}^{N} \frac{e^2}{|\vec{r}_{kj}|}. \tag{1}$$

Here $\vec{r}_k$ and $\vec{p}_k$ represent the position and momentum operators of the kth anyon in two space dimensions:

$$\vec{A}_\nu(\vec{r}_k) = \hbar\nu \sum_{j \neq k}^{N} \frac{\vec{e}_z \times \vec{r}_{kj}}{|\vec{r}_{kj}|^2} \tag{2}$$

is the anyon gauge vector potential,[11,31] $\vec{r}_{kj} = \vec{r}_k - \vec{r}_j$, and $\vec{e}_z$ is the unit vector normal to the 2D plane. The factor ν determines the fractional statistics (or spin) of the anyon: it varies between $\nu = 0$ (bosons) and $\nu = 1$ (fermions). The external magnetic field enters by minimally coupling the vector potential $\vec{A}_{ext}(\vec{r}_k) = \vec{H} \times \vec{r}_k/2$.

In order to find an analytic expression for the ground state energy as a function of ν, N, ω_0, r_0/a_B, and ω_c/ω_0 [in the presence of a magnetic field ν is replaced by $|\nu|$ (see Sec. IV below)] we employ a variational scheme by minimizing the expression for the total energy,

$$E = \frac{\displaystyle\int \Psi_T^*(\vec{R})\hat{H}\Psi_T(\vec{R})d\vec{R}}{\displaystyle\int \Psi_T^*(\vec{R})\Psi_T(\vec{R})d\vec{R}} \tag{3}$$

with a trial wave function $\Psi_T(\vec{R})$ depending on the configuration $\vec{R} = \{\vec{r}_1 \ldots \vec{r}_N\}$ of the N anyons. To motivate the choice of $\Psi_T(\vec{R})$ we invoke the mean-field approximation to the gauge vector field

$$\bar{\vec{A}}_\nu(\vec{r}) = \frac{1}{2}\vec{B}_\nu \times \vec{r} \tag{4}$$

introduced by Fetter, Hanna, and Laughlin.[32] This single-particle vector potential can be understood as that of a homogeneous "magnetic" field $\vec{B}_\nu = 2\pi\rho\hbar\nu\vec{e}_z$ connected with the carrier density ρ and the anyonic factor ν (note: $\vec{B}_\nu$ vanishes in the bosonic limit). By analogy to a physical magnetic field one can introduce a "magnetic" length $l_\nu = (\hbar/B_\nu)^{1/2}$. The other characteristic length of the system is the mean distance between particles $r_0 = 1/\sqrt{\pi\rho}$. Taking into account only this mean gauge vector field (and not the external parabolic confining potential) one obtains a Landau spectrum[33] and it is reasonable, in the bosonic representation of anyons when the many-body wave function takes the product form

$$\Psi_T(\vec{R}) = \prod_{k=1}^{N} \psi_T(\vec{r}_k), \tag{5}$$

to adopt the single-particle trial functions $\psi_T(\vec{r}_k)$ in the form

$$\psi_T(\vec{r}_k) = C\exp\left(-(\alpha' + \nu)\frac{(x_k^2 + y_k^2)}{2r_0^2}\right), \tag{6}$$

typical for the lowest Landau level. Here C is a normalization constant and α' a variational parameter. To include the external confining potential we identify r_0 with the characteristic length $(\hbar/M\omega_0)^{1/2}$ of this harmonic oscillator. When

energies are expressed in units of $\hbar\omega_0$ and lengths in units of r_0 the normalized trial wave function reads

$$\Psi_T(\vec{R}) = \left(\frac{\alpha}{\pi}\right)^{N/2} \prod_{k=1}^{N} \exp\left(-\alpha\frac{(x_k^2+y_k^2)}{2}\right), \qquad (7)$$

where $\alpha = \alpha' + \nu$.

In evaluating the expectation value E [Eq. (3)] it is convenient to consider the local energy $E_L(\vec{R}) = \Psi_T^{-1}(\vec{R})\hat{H}\Psi_T(\vec{R})$.[34] In general $E_L(\vec{R})$ is a complex function,

$$E_L(\vec{R}) = ReE_L(\vec{R}) + iImE_L(\vec{R}), \qquad (8)$$

with

$$ImE_L(\vec{R}) = -\alpha\sum_{k=1}^{N} [(\vec{A}_\nu(\vec{r}_k) + e\vec{A}_{ext}(\vec{r}_k)/c)\cdot\vec{r}_k]. \quad (9)$$

However, evaluation of the expectation value $E = \int \Psi_T(\vec{R})E_L(\vec{R})\Psi_T(\vec{R})d\vec{R}$ immediately yields

$$\int \Psi_T(\vec{R}) \; ImE_L(\vec{R})\Psi_T(\vec{R})d\vec{R} = 0, \qquad (10)$$

and, therefore, the only quantity to consider in the following is $ReE_L(\vec{R})$. Before proceeding, we would like to emphasize that the absolute ground state of the anyon system is a nonanalytic function of ν. Our calculations will simply provide a smooth interpolation.

III. NONINTERACTING CASE AND $H=0$

In the noninteracting case, in the absence of an external magnetic field, the local energy is

$$ReE_L(\vec{R}) = \sum_{k=1}^{N} \left[\alpha + \frac{x_k^2+y_k^2}{2}(1-\alpha^2) + \frac{\nu^2}{2}[\vec{A}_\nu(\vec{r}_k)]^2\right]. \qquad (11)$$

The expectation value of $ReE_L(\vec{R})$ can be easily calculated for the first two terms of Eq. (11). The last term contributes with integrals of the form

$$\int \Psi_T(\vec{R}) \frac{\vec{r}_{kj}\cdot\vec{r}_{kl}}{|\vec{r}_{kj}|^2|\vec{r}_{kl}|^2}\Psi_T(\vec{R})d\vec{R},$$

which fall into one class of $N(N-1)$ integrals with $j=l$ and a second class of $N(N-1)(N-2)$ integrals with $k\neq j, k\neq l$ and $j\neq l$. The first class of integrals can be evaluated using[35]

$$\int_0^\infty Ei(ax)e^{-\mu x}dx = -\frac{1}{\mu}\ln\left(\frac{\mu}{a}-1\right) \qquad (12)$$

with $a>0$, $Re\mu>0$, and $\mu>a$. [$Ei(y) = -\int_{-y}^\infty e^{-z}dz/z$ is the exponential integral with $y<0$.] The result is

$$\int \Psi_T(\vec{R})\frac{1}{|\vec{r}_{kj}|^2}\Psi_T(\vec{R})d\vec{R} \approx \alpha\ln\left(\frac{1}{2\delta}\right) \qquad (13)$$

which displays a logarithmic divergence with the cutoff parameter δ tending to zero. The integrals of the second class yield

$$\int \Psi_T(\vec{R}) \frac{\vec{r}_{kj}\cdot\vec{r}_{kl}}{|\vec{r}_{kj}|^2|\vec{r}_{kl}|^2}\Psi_T(\vec{R})d\vec{R} = -\alpha G \qquad (14)$$

where $G=3^{1/2}\ln(4/3)$. Putting together all these different contributions one obtains

$$E = \frac{N}{2}\left(\mathcal{N}\alpha + \frac{1}{\alpha}\right), \qquad (15)$$

with

$$\mathcal{N} = 1 + \nu^2(N-1)\left[\ln\left(\frac{1}{2\delta}\right) - G(N-2)\right], \qquad (16)$$

which attains a minimum ($dE/d\alpha=0$) for

$$\alpha_0 = \mathcal{N}^{-1/2}. \qquad (17)$$

Thus, the resulting expression for the ground state energy is

$$E_0 = N\;\mathcal{N}^{1/2}. \qquad (18)$$

The logarithmic divergence displayed in E_0 when $\delta\to 0$ has also been found in other approximate perturbative treatments of the problem and is widely discussed in the literature. To remedy this problem, various solutions were introduced: In Ref. 24 a hard-core centrifugal term and in Ref. 25 a pair correlation term were introduced in the trial wave function, while in Ref. 26 both modifications of $\psi_T(\vec{r})$ were used. An artificial repulsive deltalike potential was assumed in Refs. 14, 27 and 28 when the unperturbed ground state wave function is a product of single particle (gaussian) wave functions. Here we assume, as in Ref. 4, that the cutoff parameter δ cannot be zero for $\nu>0$, away from the bosonic limit, since it corresponds to the square of the nearest distance between the particles. Thus, for anyons in the parabolic confining potential δ is definitely smaller than 1 (in units of r_0^2). In the following we determine δ by fitting to appropriate results for special values of the parameters of the system.

Wu[11] has computed the ground state energy of N anyons in a 2D parabolic potential near the bosonic limit $\nu\simeq 0$ and obtained

$$E \approx [N + N(N-1)\nu/2]. \qquad (19)$$

To regularize the expression for E_0 we make use of this result by expanding E_0, [Eq. (18)] for $\nu\to 0$ and identify the leading term in ν^2 with the term linear in ν of Eq. 19, with the result

$$\delta = \frac{1}{2}\exp\left[-\frac{1+\nu G(N-2)}{\nu}\right]. \qquad (20)$$

With this value of the cutoff parameter the final analytic expression for the ground state energy is (also see Ref. 36)

$$E_0 = N[1 + \nu(N-1)]^{1/2}. \qquad (21)$$
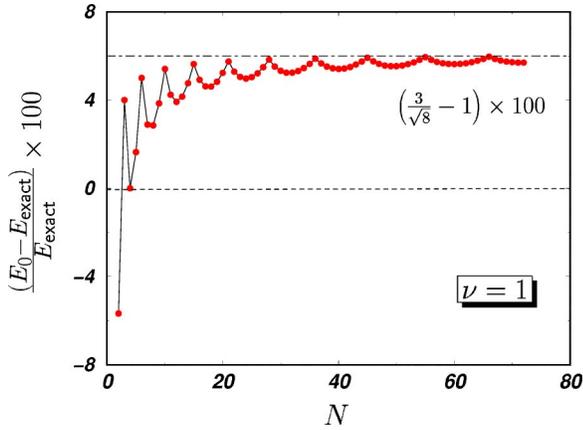
FIG. 1. Relative deviation (in percent) of the approximate ground state energy E_0 [Eq. (21)], from the exact ground state energy, E_{exact}, for up to $N=72$ noninteracting fermions ($\nu=1$) in a parabolic confining potential. The dash-dotted line indicates the asymptotic ($N\to\infty$) value.

By construction, it is evident that this formula reproduces the result of Wu[11] in the bosonic limit $\nu\to 0$. Less trivial, however, is the asymptotics in the fermionic end: For large N it is consistent (up to a numerical factor) with the approximate expression $E\approx\nu^{1/2}N^{3/2}$ of Chitra and Sen[14] calculated perturbatively from the bosonic end for $\nu>1/N$. These authors also studied the fermionic end $\nu\simeq 1$ and found for $N\gg 1$ the expression $E\approx(8N^3)^{1/2}/3$. This formula is asymptotic to the exact ground state energy $E=N_{cl}(1+8N_{cl})^{1/2}/3$ for N_{cl} fermions filling the first K closed shells[5], where $N_{cl}=K(K+1)/2$. Note that Eq. (21) provides a monotonically increasing function of ν while in the closed-shell case the exact fermionic end has lower energy (by a factor $8^{1/2}/3$) than the one calculated from the bosonic end.

In Fig. 1 we compare exact ground state energies (the sum of occupied harmonic oscillator states), for up to $N=72$ fermions ($\nu=1$), with the results obtained from Eq. (21). As it turns out, the relative deviation does not exceed 6%. Figure
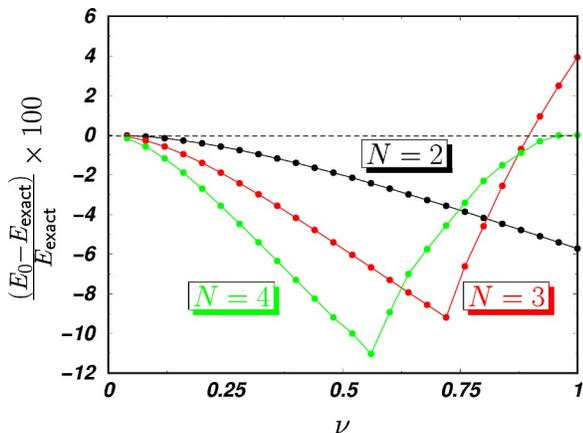
2 shows the relative deviation for two, three, and four anyons as a function of ν (the exact ground state energies here are taken from Refs. 1, 2, 12 and 13). In this figure we have considered all the cases for which the exact ground state energies are known. When the number of anyons N increases, the absolute ground state of the system is a nonanalytic function of ν because there is approximately $N^{1/2}$ number of level crossings[14]. Since our formula for E_0, [Eq. (21)] has the same ($N\to\infty$) asymptotics as the formula obtained by Chitra and Sen,[14] one expects that, for $0<\nu<1$, this relative deviation will be bounded as the number of particles is increased.

It should be noted, that due to our regularization procedure the ground state energy obtained, Eq. (21), is not an upper bound to the ground state as one would expect from a variational principle. In fact, for $N=2$ and $\nu=1$ Eq. (21) yields a value below the exact ground state (see Fig. 1). This is a consequence of the fitting of δ to the result of Wu,[11] which for $N=2$ leads to a square root dependence in ν, while the exact result for this case gives a linear dependence. On the other hand, Eq. (21) applies for the whole range of parameters of the system N,ν, and ω_0.

IV. NONINTERACTING CASE AND $H\neq 0$

In this section we include an external homogeneous magnetic field. In the presence of an external magnetic field $\vec{H}$ the statistical factor ν may change sign because $\nu=e\phi/2\pi\hbar$ is a fraction of the flux quantum carried by each anyon, $\phi_0=2\pi\hbar c/|e|$, and this flux can be antiparallel to the magnetic field.[4,19] The Hamiltonian is invariant under the transformation $(x_k,y_k,\nu,\beta)\to(x_k,-y_k,-\nu,-\beta)$, where $\beta=eH/|eH|$, and thus the energy spectrum is invariant under $(\nu,\beta)\to(-\nu,-\beta)$ (see Ref. 19). The spectrum only depends on $|\nu|$, $\nu\beta$ and the cyclotron frequency ω_c (apart from N and ω_0).

The real part of the local energy is given by

$$Re E_L(\vec{R})=\sum_{k=1}^{N}\left[\alpha+\frac{x_k^2+y_k^2}{2}\left(1-\alpha^2+\frac{\omega_c^2}{4\omega_0^2}\right)\right.$$
$$\left.+\frac{|\nu|^2}{2}[\vec{A}_\nu(\vec{r}_k)]^2+\frac{\nu\beta\omega_c}{2\omega_0}\sum_{j\neq k}^{N}\frac{\vec{r}_{kj}\cdot\vec{r}_k}{|\vec{r}_{kj}|^2}\right]. \tag{22}$$

We need to compute the contribution coming from the last term in $Re E_L$. To this end, we have to solve the integral

$$\int\Psi_T(\vec{R})\frac{\vec{r}_{kj}\cdot\vec{r}_k}{|\vec{r}_{kj}|^2}\Psi_T(\vec{R})d\vec{R}$$

($N(N-1)$ integrals of this form contribute to the energy). With the help of the integral[37]

$$\int_0^\infty e^{-a^2x^2}I_\nu(bx)dx=\frac{\pi^{1/2}}{2a}e^{b^2/8a^2}I_{\frac{\nu}{2}}\left(\frac{b^2}{8a^2}\right), \tag{23}$$

FIG. 2. Relative deviation (in percent) of the approximate ground state energy E_0, [Eq. (21)], from the exact ground state energy for two, three, and four anyons in a parabolic confinement potential.

where $\mathrm{Re}\,\nu > -1$, $\mathrm{Re}\,a^2 > 0$ [$I_\nu(x)$ is the modified Bessel function], and[38]

$$\int_0^\infty e^{-ax} \sin h(bx)\,dx = \frac{b}{a^2 - b^2}, \qquad (24)$$

with $a > b \geq 0$, one gets

$$\int \Psi_T(\vec{R}) \sum_{k,j \neq k}^N \frac{\vec{r}_{kj} \cdot \vec{r}_k}{|\vec{r}_{kj}|^2} \Psi_T(\vec{R})\,d\vec{R} = \frac{-N(N-1)}{2}. \quad (25)$$

The averaged real part of the local energy is

$$E = \frac{N}{2}\left[\mathcal{N}\left(\alpha + \frac{1}{\alpha}\left(1 + \frac{\omega_c^2}{4\omega_0^2}\right)\right) - \frac{\nu\beta\omega_c}{2\omega_0}\frac{N(N-1)}{2}\right], \qquad (26)$$

and takes its minimum value for

$$\alpha_0 = \left(1 + \frac{\omega_c^2}{4\omega_0^2}\right)^{1/2} \mathcal{N}^{-1/2}. \qquad (27)$$

The resulting energy minimum is given by (for now we return to standard units of energy and length)

$$E_0 = N\hbar\left(\omega_0^2 + \frac{\omega_c^2}{4}\right)^{1/2}, \quad \mathcal{N}^{1/2} - \frac{\nu\beta\hbar\omega_c}{4}N(N-1). \qquad (28)$$

As in Sec. III this expression diverges logarithmically in the limit of a vanishing cutoff parameter δ. Following the line of arguments of the Sec. III, the cutoff parameter — representing the squared minimum particle distance — should not be zero except for the bosonic limit $\nu = 0$. Having this in mind we determine δ by fitting to known exact results for the ground state energy.

To establish these results we calculate the fermion ground state energy from the single-particle spectrum of the 2D harmonic oscillator in an external magnetic field perpendicular to the 2D plane (Fock-Darwin spectrum[39]) (also see Ref. 33):

$$E_{n,m} = \hbar\sqrt{\omega_0^2 + \frac{\omega_c^2}{4}}\,(2n + |m| + 1) + m\frac{\hbar\omega_c}{2}. \quad (29)$$

In Eq. (29) n and m are the radial and angular momentum quantum numbers, respectively. The ground state energy of N spinless fermionic particles is the sum of the N lowest single-particle energies (Pauli exclusion principle). Following Ref. 5 we introduce a parameter $z = R/P$, where $R = \hbar\omega_c/2$ and $P = \hbar(\omega_0^2 + \omega_c^2/4)^{1/2}$ and express the ground state energy in units of P. The parameter z changes between 0 and 1 when the external magnetic field is changed between 0 and infinity. The Fock-Darwin spectrum is characterized by level crossings. These crossings, occurring at $z = z_b, z_2, z_3, \ldots z_l$, have to be considered in evaluating the ground state energy, their number therefore depending upon N. Every interval between level crossings is characterized by its own expression for the ground state energy. However, only for the intervals $0 \leq z \leq z_b$ and $z_l \leq z \leq 1$ can one write down the expressions for the energy as a function of the

number of particles N. For the sake of clarity let us consider some special values of N. The cases with one and two fermions are not affected by crossings. For $N = 3$ we have one crossing at $z_b = z_l = 1/3$ and two expressions for the ground state energy: $E/P = 5$ and $E/P = 6 - 3z$. This crossing point coincides with the one considered in Refs. 5 and 21 for three anyons. The case with $N = 5$ has two crossings $z_b = 1/3$ and $z_l = 3/5$ and three expressions for energy: $E/P = 11 - 2z$, $E/P = 12 - 5z$ and $E/P = 15 - 10z$ in the intervals $0 \leq z \leq z_b$, $z_b \leq z \leq z_l$ and $z_l \leq z \leq 1$, respectively. On the basis of these special cases one can make the following generalizations.

(i) There are $N_{cl} - K$ crossing points $z = z_b, z_2, z_3, \ldots z_l$ for N_{cl} fermions in K closed shells. Therefore, there are $N_{cl} - K + 1$ expressions for the ground state energy of fermions and ground state spectra of anyons for this number of particles.

(ii) One can write the expression $z_b = 1/(2K - 1)$ for K closed shells. The last crossing point does not depend on K, it is a function of N, $z_l = (N-2)/N$, and, thus, it is applicable for any number N.

(iii) One can write the expression for the ground state energy in the interval $0 \leq z \leq z_b$ (we choose the smallest z_b for all particles filling the given shell) or $0 \leq \omega_c \leq \omega_0/[K(K-1)]^{1/2}$ for $K \geq 2$.

$$E/P \approx N^{3/2} + zS. \qquad (30)$$

This expression is approximate [to within 6% accuracy (see Sec. III)] for $N \neq N_{cl}$, where $S = \Sigma_{j=0}^{N - N_l - 1}(-N_s + 2j)$, N_s is the integer part of $[-1 + (1 + 8N)^{1/2}]/2$ and $N_l = N_s(N_s + 1)/2$, and becomes exact in the form $E/P = N_{cl}\sqrt{(1 + 8N_{cl})}/3$ for closed shells, i.e., $N = N_{cl}$.

(iv) The expression for the ground state energy in the interval $z_l \leq z \leq 1$ or $\omega_c \geq \omega_0(N-2)/(N-1)^{1/2}$, determined by the lowest levels from each shell in agreement with the Fock-Darwin formula [Eq. (29)], is

$$E/P = N[(N+1) - z(N-1)]/2. \qquad (31)$$

Having discussed the case of fermions ($|\nu| = 1$) we now determine the approximate expressions of the ground state energy of N noninteracting anyons for the two interesting ranges of weak $0 \leq \omega_c \leq \omega_0/[K(K-1)]^{1/2}$ and strong $\omega_c \geq \omega_0(N-2)/(N-1)^{1/2}$ magnetic fields. In the weak magnetic field regime it is

$$E_0 = PN_{cl}[1 + |\nu|(N_{cl} - 1)]^{1/2} \qquad (32)$$

for closed shells and

$$E_0 = PN[1 + |\nu|(N-1)]^{1/2} + \nu\beta RS \qquad (33)$$

otherwise. Note that these expressions coincide with Eq. (21)

in the absence of an external magnetic field, i.e., $\omega_c = 0$. We find the cutoff parameters by equating these expressions to Eq. (28). The result for closed shells is

$$\delta = \frac{1}{2} \exp\left[-\frac{1}{|\nu|} \left(1 + |\nu|[(N_{cl}-1)(G+z^2/4)-G] \right.\right.$$
$$\left.\left. + \frac{\nu \beta z}{|\nu|}[1+|\nu|(N_{cl}-1)]^{1/2} \right) \right] r_0^2, \quad (34)$$

while for open shells it is

$$\delta = \frac{1}{2} \exp\left[-\frac{1}{|\nu|} \left(1 + |\nu|G(N-2) \right.\right.$$
$$\left.\left. + \frac{2\nu \beta z T}{|\nu|(N-1)}[1+|\nu|(N-1)]^{1/2} + \frac{|\nu|z^2T^2}{(N-1)} \right) \right] r_0^2, \quad (35)$$

where $N \geq 2$ and $T = (N-1)/2 + S/N$.

In the high magnetic field regime (or for weak confinement) the Fock-Darwin single-particle energies tend toward Landau levels, the lowest energy state having the quantum numbers $n=0$ and $m \leq 0$ and energy R. In this limit the ground state energy of N particles is NR, independent of $|\nu|$.[13,20,5] This exact result can be reproduced with

$$\mathcal{N}^{1/2} = 1 + \nu \beta (N-1)/2, \quad (36)$$

which gives the cutoff parameter

$$\delta = \frac{1}{2} \exp\left[-\frac{1}{4|\nu|} \{ 4\nu \beta / |\nu| + |\nu|[N(4G+1) \right.$$
$$\left. -(8G+1)] \} \right] r_0^2. \quad (37)$$

Using this choice in the general formula [Eq. (28)] we arrive at the closed analytic expression for the approximate ground state energy

$$E_0 = PN[1 + \nu \beta (N-1)/2] - \nu \beta R N (N-1)/2. \quad (38)$$

Besides the high magnetic field (or weak confinement) limit used here to fix the cut-off parameter δ, the expression obtained for E_0 reproduces the exact ground state energy of noninteracting fermions (the sum of the N lowest Fock-Darwin energies) in the whole magnetic field range beyond the last crossing of the Nth level with $n=0, m=-(N-1)$, which defines the so-called *maximum density droplet*.[7] Finally, approximate expressions Eqs. (32),(33) and (38) give the ground state energy $E_0 = PN$ of N bosons ($\nu=0$) in a magnetic field and harmonic confining potential. The separate discussion provided here for small and large magnetic fields is in correspondence with the treatment of the $N=3$ case discussed in Refs. 21 and 5.

V. COULOMB-INTERACTING CASE AND $H=0$

We now include the effect of the Coulomb repulsions between anyons

$$\frac{r_0}{2a_B} \sum_{k,j \neq k}^{N} \frac{1}{|\vec{r}_{kj}|}$$

in the expression for the real part of local energy $\text{Re}E_L(\vec{R})$, [Eq. 11], but in a vanishing external magnetic field. Here, as in Sec. III, we assume that $\nu \equiv |\nu|$.

The Coulomb interaction part contributes with $N(N-1)$ integrals of the form $\int \Psi_T(\vec{R})(1/|\vec{r}_{kj}|)\Psi_T(\vec{R})d\vec{R}$. These integrals can be evaluated using Eq. 23 and[35]

$$\int_0^\infty e^{-ax}I_\nu(bx)dx = \frac{b^\nu}{\sqrt{a^2-b^2}(a+\sqrt{a^2-b^2})^\nu}, \quad (39)$$

where $\text{Re}\nu > -1$ and $\text{Re}a > |\text{Re}b|$. The result is

$$\int \Psi_T(\vec{R}) \frac{1}{|\vec{r}_{ij}|} \Psi_T(\vec{R})d\vec{R} = \left(\frac{\pi \alpha}{2}\right)^{1/2}. \quad (40)$$

The averaged (real part of the) local energy is

$$E = \frac{N}{2}\left(\mathcal{N}\alpha + \frac{1}{\alpha} + 2\mathcal{M}\ \alpha^{1/2} \right), \quad (41)$$

with

$$\mathcal{M} = \left(\frac{\pi}{2}\right)^{1/2} \frac{N-1}{2} \frac{r_0}{a_B}. \quad (42)$$

The extremum condition $dE/d\alpha = 0$ leads to the equation

$$X^4 - \mathcal{M}X - \mathcal{N} = 0 \quad (43)$$

for $X = 1/\alpha^{1/2}$. Two complex and two real solutions of this equation can be found by the Descartes-Euler method.[40] The minimum energy is given by the expression

$$E_0 = \frac{N}{2}\left[\frac{\mathcal{N}}{X_0^2} + X_0^2 + \frac{2\mathcal{M}}{X_0} \right], \quad (44)$$

and it is achieved at the point

$$X_0 = (A+B)^{1/2} + [-(A+B)+2(A^2-AB+B^2)^{1/2}]^{1/2}, \quad (45)$$

where

$$A = \{\mathcal{M}^2/128 + [(\mathcal{N}/12)^3 + (\mathcal{M}^2/128)^2]^{1/2}\}^{1/3},$$

$$B = \{\mathcal{M}^2/128 - [(\mathcal{N}/12)^3 + (\mathcal{M}^2/128)^2]^{1/2}\}^{1/3}. \quad (46)$$

Again, the ground state energy E_0 [Eq. (44)] has a logarithmic divergence in the limit $\delta \to 0$. Assuming that $\mathcal{N}$ can be regularized, one can recognize two limits of interest: one corresponding to weak correlations, $r_0/a_B \ll 1$,
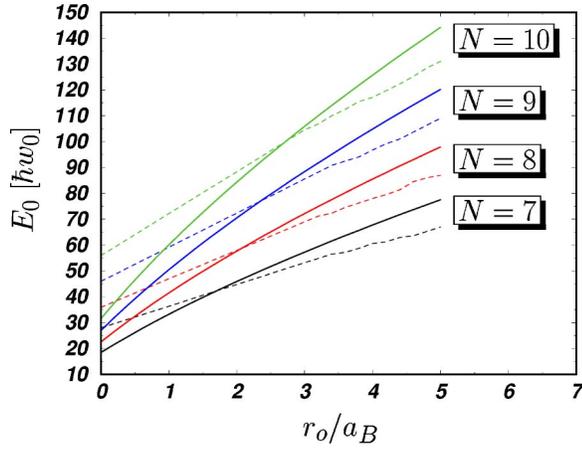
FIG. 3. Coulomb interaction parameter r_0/a_B dependence of the ground state energy for 7 – 10 electrons calculated by variational (Ref. 41) and fixed-node quantum Monte Carlo methods (Ref. 42) (dashed curves) (results of both calculations are indistinguishable in these curves) and by formula (44) (solid curves).

$$E_0 \approx N \left(\mathcal{N}^{1/2} + \frac{\mathcal{M}}{\mathcal{N}^{1/4}} \right) \tag{47}$$

and another for strong correlations, $r_0/a_B \gg 1$,

$$E_0 \approx \frac{3N}{2} \left(\mathcal{M}^{2/3} + \frac{\mathcal{N}}{3 \mathcal{M}^{2/3}} \right). \tag{48}$$

In order to determine the cutoff parameter δ, and due to the lack of analytic results, we need to fit to known numerical results for the ground state energy at special values of the parameter r_0/a_B.

In Fig. 3 we compare the ground state energies calculated for 7–10 electrons using Eq. (44), with the *noninteracting* $\mathcal{N}=1+\nu(N-1)$, to variational[41] and fixed-node quantum Monte Carlo calculations[42] (also see Ref. 43).

From Eqs. (47) and (48) follows that the contribution of the statistics (dependence upon ν) in the ground state energy is important for weak, and negligible for strong, Coulomb correlations r_0/a_B. For large values of r_0/a_B one can compare the dependence of the ground state energy, [Eq. (48)], with the estimate given in Ref. 44. The asymptotic behavior of the ground state energy with N derived from our expression is $E_0 \sim N^{5/3}$ (as in Ref. 44).

VI. COULOMB-INTERACTING CASE AND $H \neq 0$

Finally, we consider the case of a confined Coulomb anyon gas in an external magnetic field. The resulting expression for the averaged local energy is

$$E = \frac{N}{2} \left[\mathcal{N} \, \alpha + \frac{1}{\alpha} \left(1 + \frac{\omega_c^2}{4\omega_0^2} \right) + 2\mathcal{M} \, \alpha^{1/2} \right]$$
$$- \frac{\nu\beta\omega_c}{2\omega_0} \frac{N(N-1)}{2}. \tag{49}$$

Following steps similar to previous sections we obtain the ground state energy (in standard units)

$$E_0 = \frac{N}{2} \hbar \omega_0 \left[\frac{\mathcal{N}}{\bar{X}_0^2} + \bar{X}_0^2 \left(1 + \frac{\omega_c^2}{4\omega_0^2} \right) - \frac{\nu\beta\omega_c}{2\omega_0}(N-1) + \frac{2\mathcal{M}}{\bar{X}_0} \right], \tag{50}$$

where $\bar{X}_0$ is formally the same expression as Eq. (45) after replacing $\mathcal{N} \to \mathcal{N}_1 = \mathcal{N}/[1 + \omega_c^2/(4\omega_0^2)]$ and $\mathcal{M} \to \mathcal{M}_1 = \mathcal{M}/[1 + \omega_c^2/(4\omega_0^2)]$ in Eq. (46). The asymptotic expressions are

$$E_0 \approx N\hbar\omega_0 \left[\left(\mathcal{N}_1^{1/2} + \frac{\mathcal{M}_1}{\mathcal{N}_1^{1/4}} \right) \left(1 + \frac{\omega_c^2}{4\omega_0^2} \right) - \frac{\nu\beta\omega_c}{2\omega_0} \frac{(N-1)}{2} \right] \tag{51}$$

and

$$E_0 \approx \frac{3N\hbar\omega_0}{2} \left[\left(\mathcal{M}_1^{2/3} + \frac{\mathcal{N}_1}{3\mathcal{M}_1^{2/3}} \right) \left(1 + \frac{\omega_c^2}{4\omega_0^2} \right) \right.$$
$$\left. - \frac{\nu\beta\omega_c}{3\omega_0} \frac{(N-1)}{2} \right] \tag{52}$$

for very small and very large values of r_0/a_B, respectively. We regularize the logarithmic divergence by fitting δ to known numerical results for the ground state energy of quantum dots in external magnetic fields.

We compare our results with the ground state calculations of Ref. 45 for GaAs dots in an external magnetic field. The results of these calculations are very close to the results of Refs. 46 and 47 computed by exact diagonalization and quantum Monte Carlo methods, respectively. For GaAs quantum dots $M^* = 0.067M$, and the dielectric constant is $\epsilon = 12.4$. Therefore, the effective Bohr radius $a_B^* = \hbar^2 \epsilon/M^* e^2$ is $a_B^* \simeq 97.90$ Å and the unit of energy – the effective Hartree $[H^* = M^* e^4/(\epsilon^2 \hbar^2)]$ — is $H^* \simeq 11.86$ meV. The cyclotron frequency is $\omega_c = eH/(M^*c)$ [for simplicity, in this part of our work we assume $\nu \equiv |\nu|$, $e \equiv |e|$, and $H \equiv |H|$ while a correct combination of the signs of these quantities is given in Eqs. (50)– (52)]. Thus, the energy quanta for this frequency is $\hbar\omega_c = 1.7269 \cdot H \cdot$ (meV/T). Here we took into account that $\hbar\omega_c = \hbar eH/(M^*c) = 2M\mu_B H/M^*$, the Bohr magneton is $\mu_B = e\hbar/(2Mc) = 0.05785$ meV/T, and the magnetic field H is measured in Tesla (T) magnetic units. The Coulomb interaction parameter r_0/a_B in our case is equal to $r_0/a_B = \sqrt{H^*/(\hbar\omega_0)}$.

To compare with the results of Ref. 45 for spin polarized electrons, we calculated the ground state energy using Eq. (50) with the expression $\mathcal{N}=N$ (i.e., *noninteracting* $\mathcal{N}$ with $\nu=1$), for three and four particles as a function of the magnetic field strength H. As in Ref. 45 we considered $\hbar\omega_0 = 3.37$ meV. Comparison of these results is displayed in
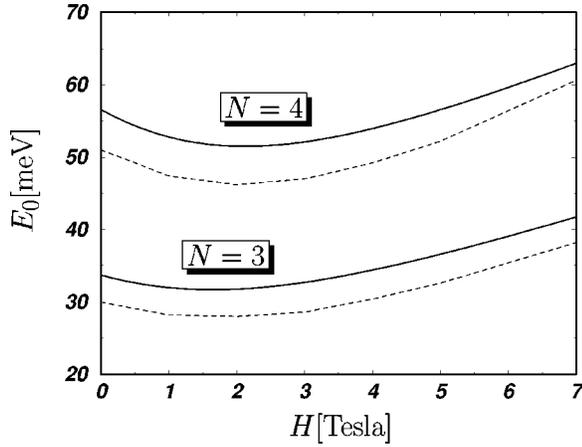
FIG. 4. Magnetic field H dependence of the ground state energy for $N=3$ and $N=4$ spin-polarized electrons in a harmonic potential calculated in Ref. 45 (the dashed curves), and using Eq. (50) (the solid curves). As in Ref. 45 we used $\hbar\omega_0=3.37$ meV ($r_0/a_B = \sqrt{H^*/(\hbar\omega_0)}$, where the effective Hartree H^* is equal to $H^* \simeq 11.86$ meV).

Fig. 4. The deviation of our results with respect to the ones given in Ref. 45 is no more than 10%.

It turns out that this expression for $\mathcal{N}$ with $\nu=1$ is appropriate for the description of a small number of electrons ($N = 3,4,5,6$) and not large magnetic fields. For large number of particles and a wide range of magnetic fields one can write the approximate expression for $\mathcal{N}$ (here we return to the original signs of ν, e, and H),

$$\mathcal{N}=F\Bigg[[1+|\nu|(N-1)]^{1/2}$$
$$+\frac{\nu\beta\omega_c(N-1)}{4\omega_0}-\left(\frac{\nu\beta\omega_cN^{1/2}}{\omega_0}\right)^{1/2}\Bigg]^2, \qquad (53)$$

and thus for δ,

$$\delta=\frac{1}{2}\exp\Bigg[-\frac{F}{|\nu|}\Bigg(1+|\nu|G(N-2)$$
$$+\frac{|\nu|\omega_c^2D}{4\omega_0^2}+\frac{\nu\beta\omega_cC}{2|\nu|\omega_0}-Q\Bigg)r_0^2\Bigg], \qquad (54)$$

with

$$F=\frac{1}{1+\dfrac{|\nu|^2\omega_c^2}{4\omega_0^2}}, \qquad (55)$$

$$D=\frac{N^2-2N-3}{4(N-1)}+|\nu|^2G(N-2), \qquad (56)$$

$$C=\frac{2N^{1/2}}{N-1}+[1+|\nu|(N-1)]^{1/2}-\left(\frac{\nu\beta\omega_cN^{1/2}}{\omega_0}\right)^{1/2}, \qquad (57)$$

and
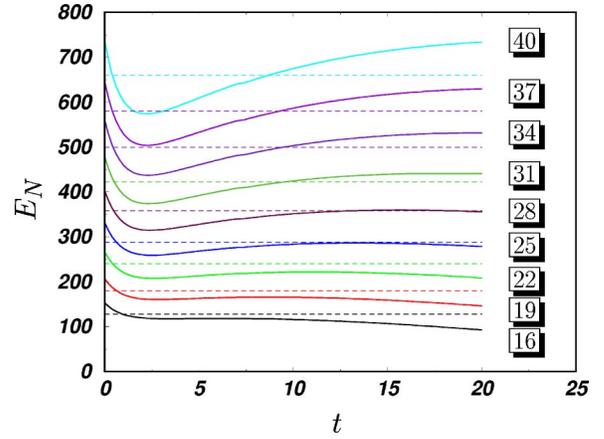
FIG. 5. Ground state energy $E_N=(E_0-N\hbar\omega)/(\hbar\omega_0)$ for $16-40$ electrons calculated using the expression Eq. (50) for $r_0/a_B = 1.911$, applying the expression for $\mathcal{N}$ Eq. (53) with $|\nu|=1$ (solid curves), and energy for classical electrons (Ref. 48) (dashed lines). Here $\omega=(\omega_0^2+\omega_c^2/4)^{1/2}$ and $t=\omega_c/\omega_0$.

$$Q=2\Bigg[\frac{\nu\beta\omega_cN^{1/2}}{|\nu|^2\omega_0(N-1)^2}[1+|\nu|(N-1)]\Bigg]^{1/2}, \qquad (58)$$

for $N\geqslant2$. Here we took into account that $\mathcal{N}$ depends weakly on the Coulomb parameter r_0/a_B (the results indicated in Fig. 3 have been calculated with $\mathcal{N}$ not having this parameter dependence).

In Fig. 5 we compare the ground state energy calculated with the expression Eq. (50), [using Eq. (53) for $\mathcal{N}$ with $|\nu|=1$] for $16\leqslant N\leqslant40$ and $0\leqslant\omega_c/\omega_0\leqslant20$ ($r_0/a_B = 1.911$), with the calculations for a classical system of electrons of Ref. 48. The deviation is maximal (no more than 15%) in the range $1/[K(K-1)]^{1/2}\leqslant\omega_c/\omega_0\leqslant(N-2)/(N-1)^{1/2}$, where K is the number of closed shells. This range of magnetic fields corresponds to the crossings of Fock-Darwin levels (see Ref. 39 and Sec. IV) and, therefore, the single particle ground state energy of electrons changes many times as ω_c increases. One can suppose that the Coulomb interaction shifts the levels but the qualitative structure of the many particle ground state is still complex. Thus, in this range of parameter ω_c/ω_0 the expression for $\mathcal{N}$ is not uniquely defined. We could not find a more appropriate expression for $\mathcal{N}$ than Eq. (53), for the magnetic fields indicated in Fig. 5.

VII. CONCLUSION

We have used the anyon concept combined with a variational calculation to obtain an analytic closed-form expression for the approximate ground state energy of N noninteracting and Coulomb-interacting particles in a 2D harmonic confining potential, with and without an external magnetic field. The crucial point of this approach is the appearance of a logarithmic divergence connected with a cut-off parameter, when evaluating the contribution of the gauge field vector

potential. Following arguments from the literature, according to which the cutoff parameter cannot be zero (except for the bosonic limit), we used it to fit our results to exact and numerical ground state energies known for special values of the system parameters. In doing so we provided closed analytic expressions for the approximate ground state energy depending upon $|\nu|, N, \omega_0, r_0/a_B$ and ω_c/ω_0.

ACKNOWLEDGMENTS

[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento Soc. Ital. Fis., B **37**, 1 (1977).

[2] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).

[3] S. Forte, Rev. Mod. Phys. **64**, 193 (1992); R. Iengo and K. Lechner, Phys. Rep. **213**, 179 (1992); *Quantum Hall Effect*, edited by M. Stone (World Scientific, Singapore, 1992).

[4] A. Lerda, *Anyons* (Springer-Verlag, Berlin, 1992).

[5] A. Khare, *Fractional Statistics and Quantum Theory* (World Scientific, Singapore, 1997).

[6] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).

[7] L. Jacak, P. Hawrylak, and A. Wojs, *Quantum Dots* (Springer-Verlag, Berlin, 1997).

[8] G. Burkard, D. Loss, and D. P. Di Vincenzo, Phys. Rev. B **59**, 2070 (1999).

[9] M. G. Taut, cond-mat/0002067.

[10] N. F. Johnson and L. Quiroga, Phys. Rev. Lett. **74**, 4277 (1995).

[11] Y. -S. Wu, Phys. Rev. Lett. **53**, 111 (1984), *ibid.* **53**, 1028(E) (1984).

[12] M. Sporre, J. J. M. Verbaarschot, and I. Zahed, Phys. Rev. Lett. **67**, 1813 (1991); M. V. N. Murthy, J. Law, M. Brack, and R. K. Bhaduri, *ibid.* **67**, 1817 (1991).

[13] M. Sporre, J. J. M. Verbaarschot, and I. Zahed, Phys. Rev. B **46**, 5738 (1992).

[14] R. Chitra and D. Sen, Phys. Rev. B **46**, 10 923 (1992).

[15] A. Vercin, Phys. Lett. B **260**, 120 (1991).

[16] J. Myrheim, E. Halrorsen, and A. Vercin, Phys. Lett. B **278**, 171 (1992); A. Gonsales, J. Phys.: Condens. Matter **9**, 4643 (1997); A. Gonsales, R. Perez, and P. Fileviez, *ibid.* **9**, 8465 (1997).

[17] A. Comtet, Y. Georglin, and S. Ouvry, J. Phys. A **22**, 3917 (1989); M. D. Johnson and C. S. Canright, Phys. Rev. B **41**, 6870 (1990).

[18] A. P. Polychronakos, Phys. Lett. B **264**, 362 (1991); J. Grunberg, T. H. Hansson, A. Karlhede, and E. Westerberg, Phys. Rev. B **44**, 8373 (1991); G. Dunne, A. Lerda, S. Sciuto, and C. A. Trugenberger, Nucl. Phys. B **370**, 601 (1992); K. H. Cho and C. Rim, Ann. Phys. (N.Y.) **213**, 295 (1992).

[19] A. Dasnieres de Veigy and S. Ouvry, Phys. Rev. Lett. **72**, 600 (1994).

[20] F. Iluminati, F. Ravndal, and J. Aa. Ruud, Phys. Lett. A **161**, 323 (1992); J. Aa. Ruud and F. Ravndal, Phys. Lett. B **291**, 137 (1992).

[21] A. Khare, J. McCabe, and S. Ouvry, Phys. Rev. D **46**, 2714 (1992).

[22] B. Roy, P. Roy, and Y. P. Varshni, Mod. Phys. Lett. B **8**, 159 (1994).

[23] R. Perez and A. Gonsales, Phys. Rev. B **58**, 7412 (1998).

[24] C. Chou, Phys. Rev. D **44**, 2533 (1991); *ibid.* **45**, 1433 (1992); C. Chou, L. Hua, and G. Amelino-Camelia, Phys. Lett. B **286**, 329 (1992); G. Amelino-Camelia, *ibid.* **299**, 83 (1992).

[25] A. Comtet, J. McCabe, and S. Ouvry, Phys. Lett. B **260**, 372 (1991); J. McCabe and S. Ouvry, *ibid.* **260**, 113 (1991); A. Dasnieres de Veigy and S. Ouvry, *ibid.* **291**, 130 (1992); Nucl. Phys. B **388**, 715 (1992).

[26] G. Amelino-Camelia, Phys. Lett. B **286**, 97 (1992).

[27] D. Sen, Nucl. Phys. B **360**, 397 (1991); D. Sen and R. Chitra, Phys. Rev. B **45**, 881 (1991).

[28] G. Amelino-Camelia, Phys. Lett. B **326**, 282 (1994).

[29] C. D. Batista and G. Ortiz, cond-mat/0207106 (unpublished).

[30] B. Abdullaev, C. D. Batista, and G. Ortiz (unpublished).

[31] R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988).

[32] A. L. Fetter, C. B. Hanna, and R. B. Laughlin, Phys. Rev. B **39**, 9679 (1989).

[33] V. M. Galicki, B. M. Karnakov and V. I. Kogan, *Problems in Quantum Mechanics* (in Russian) (Nauka, Moscow, 1981)

[34] D. M. Ceperley and M. H. Kalos, in *Monte Carlo Methods in Statistical Physics*, edited by K. Binder (Springer-Verlag, Berlin, 1979).

[35] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, London, 1980).

[36] B. Abdullaev, M. Musakhanov, and A. Nakamura, cond-mat/0012423 (unpublished).

[37] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).

[38] H. B. Dwight, *Tables of Integrals and other Mathematical Data* (Macmillan, New York, 1961).

[39] V. Fock, Z. Phys. **47**, 446 (1928); C. G. Darwin, Cambridge Philos. Soc. **27**, 86 (1931).

[40] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill, New York, 1968).

[41] F. Bolton, Phys. Rev. Lett. **73**, 158 (1994).

[42] F. Bolton, Solid State Electron **37**, 1159 (1994).

[43] A. Harju, V. A. Sverdlov, R. M. Nieminen, and V. Halonen, Phys. Rev. B **59**, 5622 (1999); F. Pederiva, C. J. Umrigar, and E. Lipparini, *ibid.* **62**, 8120 (2000); M. Eto, cond-mat/0101345 (unpublished).

[44] E. R. E. Yang, A. H. MacDonald, and M. D. Johnson, Phys. Rev. Lett. **71**, 3194 (1993).

[45] K. Varga, P. Navratil, J. Usukura, and Y. Suzuki, Phys. Rev. B **63**, 205308 (2001).

[46] P. Hawrylak and D. Pfannkuche, Phys. Rev. Lett. **70**, 485 (1993).

[47] F. Bolton, Phys. Rev. B **54**, 4780 (1996).

[48] J. Kainz, S. A. Mikhailov, A. Wensauer, and U. Rössler, Phys. Rev. B **65**, 115305 (2002).