Hall coefficient in an interacting electron gas

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The Hall conductivity in a weak homogeneous magnetic field, $\omega_c \tau \ll 1$, is calculated. We have shown that to leading order in $1/\epsilon_F \tau$ the Hall coefficient R_H is not renormalized by the electron-electron interaction. Our result explains the experimentally observed stability of the Hall coefficient in a dilute electron gas not too close to the metal-insulator transition. We avoid the currently used procedure that introduces an artificial spatial modulation of the magnetic field. The problem of the Hall effect is reformulated in a way such that the magnetic flux associated with the scattering process becomes the central element of the calculation.

I. INTRODUCTION

In recent experiments on the two-dimensional dilute electron gases the renormalization of the Fermi-liquid parameters is found to be significant.[1–4] These strong renormalizations are natural for a dilute gas with a large value of the ratio of the Coulomb energy to the Fermi energy. It was shown, however, that in a high mobility MOSFETs (metal-oxide-semiconductor field-effect transistor). The density of electrons n_H found from the Hall coefficient $R_H = 1/n_H ec$ coincides with that obtained from the Shubnikov–de Haas measurements n to within a few percent [see Fig. 4 in Ref. 5 and the inset in Fig. 4 in Ref. 3(b)]. This remarkable fact raises a question about the electron-electron (e-e) interaction renormalizations of the Hall conductivity σ_{xy} in a small magnetic field, $\omega_c \tau \ll 1$, where ω_c is a cyclotron frequency and τ is the free path time. Notice that while the e-e interaction is very strong, in other aspects this system is rather simple, because the effect of the crystalline lattice can be safely studied in the effective-mass approximation. Apart from impurities, one may consider the electron liquid as if it is in a translational invariant background. Therefore the influence of the interaction on the Hall coefficient should not be masked by unnecessary complications. The issue of the Hall coefficient renormalization has not been discussed much in the literature despite its obvious importance. We have in mind the renormalization of the leading in τ_{tr} terms in the conductivity tensor, i.e., of the Hall conductivity $\sigma_{xy} \propto B\tau_{tr}^2$, and the diagonal term $\sigma_{xx} = ne^2\tau_{tr}/m^*$. Here τ_{tr} is the transport relaxation time, and B is a magnetic field; notice that $R_H \equiv B^{-1}\sigma_{xy}/(\sigma_{xx})^2$.

In this paper we examine the effect of the e-e interaction on the Hall coefficient within the microscopical theory. We avoid the currently used procedure that introduces an artificial spatial modulation of the magnetic field. Based on an idea of the flux of a loop in a diagrammatic technique, we obtain a gauge invariant procedure for the calculations in the presence of a homogeneous magnetic field which makes it possible to conduct the analysis of σ_{xy} in a general manner. In particular, in the course of the calculations the translational invariance of the problem is maintained, and the usual rather involved procedure of extracting a constant B from the $\mathbf{q} \to 0$ limit of a singular expression for the vector potential

$\mathbf{A}(\mathbf{q})$ can be avoided.

In this paper, we show that the Hall coefficient for electrons with the quadratic spectrum, $\epsilon(p) = p^2/2m$, in the leading order in τ_{tr}, is not renormalized by the e-e interaction, i.e., $R_H = 1/nec$. [In fact, the cancellation of the e-e renormalization in the Hall coefficient holds for any spherically symmetric dispersion $\epsilon(|\mathbf{p}|)$; see Appendix B for comments.] This result explains the stability of the Hall coefficient R_H to the e-e interaction demonstrated in Refs. 5 and 3(b). Notice that the Altshuler-Aronov corrections to σ_{xx} and σ_{xy} teach us that there cannot be any general principle for the absence of renormalizations in R_H, because corrections to R_H of the order $1/\epsilon_F \tau_{tr}$ do exist.[6] Therefore the fact stated here about the cancellation of e-e renormalizations in the leading term of R_H should be proved specifically.

The renormalization of σ_{xy} can be also studied within the Fermi-liquid theory following the line given in Ref. 7. The magnetic field $\mathbf{B}$ comes into a transport equation in a combination with the velocity of a charge carrier as $(e[\mathbf{v_p} \times \mathbf{B}]\partial/\partial\mathbf{p})\delta n_{\mathbf{p}}$, where $\delta n_{\mathbf{p}}$ is the departure of the distribution function from equilibrium due to the applied electric field (see, e.g., Ref. 8). In the presence of the electron-electron interaction one may expect in addition terms like $(e[\mathbf{v_p} \times \mathbf{B}]\partial/\partial\mathbf{p})\langle f_{\mathbf{pp'}}\delta n_{\mathbf{p'}}\rangle$, or $\langle f_{\mathbf{pp'}}(e[\mathbf{v_{p'}} \times \mathbf{B}]\partial/\partial\mathbf{p'}) \times \langle f_{\mathbf{p'p''}}\delta n_{\mathbf{p''}}\rangle\rangle$, where $\langle\cdots\rangle$ means the average over the Fermi surface. Here $\langle f_{\mathbf{pp'}}\delta n_{\mathbf{p'}}\rangle$ is the response of the Fermi liquid to the departure of the distribution function from equilibrium.[7] This approach gives σ_{xy} renormalized by the e-e interaction which when combined with σ_{xx} yields the Hall coefficient R_H with no renormalization factors. It is, however, not clear that the derivation of the Hall conductivity in a weak magnetic field via the transport equation within the Fermi-liquid theory is accurate enough. For instance, consider the following process: the electric field leads to the departure of the distribution function from equilibrium; the electron liquid responds to this δn via the electron interaction; the magnetic field acts on the result of this response and transforms it into the Hall current involving again the e-e interaction. The nonfactorized part of such two time interaction process $\langle\langle(e[\mathbf{v_{p'}} \times \mathbf{B}]\partial/\partial\mathbf{p'})f_{np}(\mathbf{p};\mathbf{p'};\mathbf{p''})\delta n_{\mathbf{p''}}\rangle\rangle$ corresponds to nonpair correlations, and is beyond the scope of the Fermi-liquid theory. (Some doubts about the derivation

via the kinetic equation were expressed in the literature, e.g., Ref. 8, Sec. 7.12.) We will see, however, that this contribution is small by the parameter $1/\epsilon_F \tau_{tr}$. Therefore the present microscopical derivation justifies the applicability of the Fermi-liquid treatment of this problem.

In Sec. II the technicalities of the calculation of σ_{xy} in a constant magnetic field are given. In Sec. III the Hall conductivity of the noninteracting electrons is considered, and in particular a nontrivial problem of the Hall conductivity in a system with an arbitrary (nonshort range) impurity potential is studied. By studying this case we make the needed preparations for the analysis of the effects of the e-e interaction to the Hall conductivity [see Eqs. (23)–(25)]. The problem of the Hall effect in a weak magnetic field is reformulated in a way such that the magnetic flux associated with the scattering process becomes the central element of the calculation. In Sec. IV the role of the e-e interaction is analyzed. There, the main contribution to σ_{xy} is given by Eq. (35). The latter expression contains a product of two terms; one describes the scattering of the quasiparticles by impurities, while the other corresponds to the decoration of the current vertices by the e-e interaction. The derivatives over the momentum in Eq. (35) correspond to the vector product of the two coordinate differences, see Eq. (23), in the representation of the flux which leads to the skew action of the magnetic field.

In the concluding section for completeness we discuss R_H using the Fermi-liquid theory.[7] In Appendix C we reproduce the known answer (Ref. 9) for the weak localization corrections to σ_{xy} as an instructive example of the calculation within the new procedure.

II. DIAMAGNETIC AND FLUX CONTRIBUTIONS IN A HOMOGENEOUS MAGNETIC FIELD

To describe electrons moving in a magnetic field the magnetic vector potential **A** can be introduced through the extension of the momentum in the kinetic energy term: $\epsilon(\mathbf{p}) \rightarrow \epsilon(\mathbf{p} - (e/c)\mathbf{A})$. Although the analysis presented in this paper is valid for any spectrum with a spherical symmetry (see Appendix B), the consideration in the main text will be limited to the electron gas with a quadratic spectrum

$$H_0 = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2, \qquad (1)$$

where m is the conduction band mass (i.e., m is not renormalized yet by the electron-electron interaction), and we use the fact that for low concentrations of the electrons only the quadratic term in $\epsilon(p)$ is relevant.

Notice that the homogeneous magnetic field demands special care as the Fourier components of the vector potential, $\mathbf{A}(\mathbf{q}) = \int d^d r e^{-i\mathbf{q}\mathbf{r}} \mathbf{A}(\mathbf{r})$, are singular in this important case. Let us take the vector potential in the Landau gauge,

$$\mathbf{B} = B\hat{\mathbf{z}}; \quad A_x = 0, \quad A_y = Bx. \qquad (2)$$

Then the Fourier components of $\mathbf{A}(\mathbf{r})$ are

$$A_y(\mathbf{q}) = iB\frac{\partial}{\partial q_x}\delta^d(\mathbf{q}). \qquad (3)$$

When inserted directly into diagrams the expression for $A_y(\mathbf{q})$ being highly singular leads to complications. In order to circumvent this difficulty it was proposed in Ref. 10 to introduce a magnetic field modulated in space at some wave vector **q** in addition to a homogeneous electric field at finite frequency ω, and to study the $(\mathbf{q},\omega)$ response of the current. It has been argued in Ref. 10 that in the calculation of σ_{xy} one should perform the limiting procedure as follows: first extract the **q**-linear contributions related to the vector potential and arrange them into the gauge invariant combination $\mathbf{B}(\mathbf{q}) = i\mathbf{q} \times \mathbf{A}(\mathbf{q})$; next let $\mathbf{q} \rightarrow 0$ keeping $\mathbf{B}(\mathbf{q})$ to be finite; and, finally, perform the dc limit for the current response. The first step in this procedure requires combining different pieces into the gauge invariant combination, and this involves nontrivial cancellations amongst different diagrams. The outlined procedure is effective when a specific process is needed to be calculated (see, e.g., Refs. 6 and 9), but unfortunately it is unsuitable for a general analysis.

The method proposed below treats the magnetic field at $\mathbf{q} = 0$ *ab initio*, and at no stage of the calculation does the vector potential appear explicitly. In addition to the apparent gauge invariance the developed technique shows clearly that although the Hamiltonian in the magnetic field is not translational invariant, any measured quantity like conductivity can be calculated in an explicitly transitional invariant manner. The calculation is based on the fact that the Green's function in a constant magnetic field can be represented in the form[11-13]

$$\mathcal{G}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \exp[(ie/c)\Phi(\mathbf{r}_1, \mathbf{r}_2)]\tilde{\mathcal{G}}(\mathbf{r}_1 - \mathbf{r}_2, \tau), \qquad (4)$$

where the phase Φ is equal to

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = (\mathbf{r}_1 - \mathbf{r}_2)\mathbf{A}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}\right) \qquad (5)$$

and $\tilde{\mathcal{G}}$ is the core Green's function which contains the information about the Landau-level quantization. The function $\tilde{\mathcal{G}}$ is translational and rotational invariant and is also invariant with respect to different gauge representations of the magnetic field **B** (see Appendix A). Notice that in the presence of e-e interaction the representation (4) and (5) is still valid.

The rotational invariance of the core function $\tilde{\mathcal{G}}$ is responsible for the fact that the Hall current originates either from the diamagnetic part of the current operator, or with the participation of the phase terms Φ. No contribution to σ_{xy} comes out from the core functions alone. This is because the calculation of σ_{xy} in the Kubo formula involves the averaging over the momenta of the nondiagonal current-current correlator $\langle J_x J_y \rangle$. Since the functions $\tilde{\mathcal{G}}$ are rotational invariant, the angular integration of the type $\langle p_x \cdots p_y' \rangle$ makes the core function's contribution to σ_{xy} zero if the scattering by impurities is simple enough and does not itself produce the skew effect on average. Then only the interference of the phase factor Φ or the diamagnetic current make it possible to achieve the skew action of the magnetic field to avoid the vanishing of the $\langle J_x J_y \rangle$ current correlator after the angular integration. (Notice in this connection that the Lorentz force

can be interpreted as a consequence of the phase accumulation in the magnetic field, Ref. 14.)

The diamagnetic part of the current operator, $-(e^2/mc)\mathbf{A}$, being nongauge invariant has to be completed into a gauge invariant contribution in any calculation of a measurable quantity. This completion comes out naturally here due to the differentiation of the phase factors in the Green's functions in the current operator,

$$J_\nu(\mathbf{r}_i) = (e/2mi) \lim_{\mathbf{r}_i' \to \mathbf{r}_i} (\nabla_{\mathbf{r}_i'}^\nu - \nabla_{\mathbf{r}_i}^\nu) \psi^\dagger(\mathbf{r}_i) \psi(\mathbf{r}_i')$$
$$- (e^2/mc) A^\nu(\mathbf{r}_i) \psi^\dagger(\mathbf{r}_i) \psi(\mathbf{r}_i). \quad (6)$$

Consider the current vertex at the point $\mathbf{r}_i$ connected to the rest of a diagram (e.g., for a current-current correlator) by a pair of Green's functions $\mathcal{G}(\mathbf{r}_1,\mathbf{r}_i)$ and $\mathcal{G}(\mathbf{r}_i,\mathbf{r}_2)$. Then the diamagnetic part results in

$$(e^2/mc)\mathbf{A}(\mathbf{r}_i)\mathcal{G}(\mathbf{r}_1,\mathbf{r}_i)\mathcal{G}(\mathbf{r}_i,\mathbf{r}_2),$$

whereas the first term yields

$$(e/2m)[\mathcal{G}(\mathbf{r}_1,\mathbf{r}_i)(-i)\nabla_{\mathbf{r}_i}\mathcal{G}(\mathbf{r}_i,\mathbf{r}_2) + i\nabla_{\mathbf{r}_i}\mathcal{G}(\mathbf{r}_1,\mathbf{r}_i)\mathcal{G}(\mathbf{r}_i,\mathbf{r}_2)].$$

To treat the phase factors in the Green's functions it is useful to split the $\mathbf{A}$ term into two half pieces and introduce the following relations:

$$[-i\nabla_{\mathbf{r}_i} - (e/c)\mathbf{A}(\mathbf{r}_i)]\mathcal{G}(\mathbf{r}_i,\mathbf{r}_2;\tau)$$
$$= \exp[(ie/c)\Phi(\mathbf{r}_i,\mathbf{r}_2)]$$
$$\times [-i\nabla_{\mathbf{r}_i} + (e/2c)(\mathbf{r}_i - \mathbf{r}_2) \times \mathbf{B}]\tilde{\mathcal{G}}(\mathbf{r}_i - \mathbf{r}_2;\tau), \quad (7)$$

and

$$[i\nabla_{\mathbf{r}_i} - (e/c)\mathbf{A}(\mathbf{r}_i)]\mathcal{G}(\mathbf{r}_1,\mathbf{r}_i;\tau)$$
$$= \exp[(ie/c)\Phi(\mathbf{r}_1,\mathbf{r}_i)]$$
$$\times [-i\nabla_{\mathbf{r}_1} - (e/2c)(\mathbf{r}_1 - \mathbf{r}_i) \times \mathbf{B}]\tilde{\mathcal{G}}(\mathbf{r}_1 - \mathbf{r}_i;\tau). \quad (8)$$

The terms on the right-hand side that contain B explicitly yield the diamagnetic contribution extended to gauge invariant combinations. Combining Eqs. (7) and (8) we obtain

$$J_{extended}^{diam} = (e\omega_c/4)[(\mathbf{r}_i - \mathbf{r}_2) \times \hat{\mathbf{B}} - (\mathbf{r}_1 - \mathbf{r}_i) \times \hat{\mathbf{B}}]$$
$$\times \tilde{\mathcal{G}}(\mathbf{r}_1 - \mathbf{r}_i)\tilde{\mathcal{G}}(\mathbf{r}_i - \mathbf{r}_2)R(\mathbf{r}_1 - \mathbf{r}_f, \mathbf{r}_2 - \mathbf{r}_f), \quad (9)$$

where $R(\mathbf{r}_1 - \mathbf{r}_f, \mathbf{r}_2 - \mathbf{r}_f)$ is the remaining part of a correlator ending at a point $\mathbf{r}_f$. Notice that $J_{extended}^{diam}$ is also translation invariant. In Eq. (9) the cyclotron frequency has been introduced in such way that its sign is that of the carriers: $\omega_c = eB/mc$.

The phase factor $\Phi(\mathbf{r}_1,\mathbf{r}_2)$ appearing in the Green's function $\mathcal{G}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ may be rewritten as the integral over the straight line:

$$\Phi(\mathbf{r}_1,\mathbf{r}_2) = (\mathbf{r}_2 - \mathbf{r}_1)\mathbf{A}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}\right) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{A}(\mathbf{r})d\mathbf{r}. \quad (10)$$

Then collecting the phase factors from all Green's functions of a given loop of a diagram one will obtain the so-called "flux of a loop" of a diagram. [The phase factors that appear on the right-hand side of Eqs. (7) and (8) after passing through the differentiation operator join other such factors from the rest of the Green's functions.] It follows from Eq. (10) that the phase associated with each loop in a diagram is proportional to the flux

$$\Phi_{loop} = \oint \mathbf{A}(\mathbf{r})d\mathbf{r}, \quad (11)$$

with the integral taken along the closed polygon corresponding to the loop. (The importance of the flux in the calculation of the Hall effect in the insulating state was emphasized in Refs. 15 and 16.) Notice that the polygon is oriented along the loop in the direction of the arrows of the Green's function. The magnetic flux through the polygon is gauge and translational invariant. It is equal to $\mathbf{BS}$, where $\mathbf{S}$ is the oriented area of a polygon. Depending on the situation one can decompose the polygon into the sum of different pieces. The decomposition can be done in a variety of ways, and a convenient choice may simplify the calculation. This is an advantage of this method. With the purpose of a general analysis of the Hall conductivity we will decompose the loop polygon in a diagram for the current-current correlator into the sum of triangles, and then use the fact that the flux through a triangle $(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ is equal to $(1/2)\mathbf{B} \cdot [(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_2)]$. The details of the decomposition procedure will be postponed to the next section [see Eqs. (20)–(23)], where it will be shown how the flux contribution can be analyzed for the calculation of the Hall conductivity in a system with an arbitrary (nonshort range) impurity potential.

Let us summarize. It follows from the symmetry of the problem that the Hall current may appear either from the diamagnetic part of the current operator, or through the phase factors of the Green's functions. Partially we have used the phase factors to extend the diamagnetic term in Eqs. (7)–(9). The rest of the phase factors contributions can be organized in the form of the fluxes of the loops. The main consequence of this structure is that if one is interested in σ_{xy} linear in the external magnetic field only, $\sigma_{xy} \propto B$, then it is enough for the calculations of σ_{xy} that the core Green's functions will be taken in the limit $B \to 0$, when $\tilde{\mathcal{G}}$ coincide with the Green's functions in the absence of the magnetic field: $\tilde{\mathcal{G}}_{B \to 0} = \mathcal{G}_{B=0}$. The latter fact enables us to reduce the calculation of σ_{xy} in a small magnetic field to a diagrammatic problem that uses the Green's functions in the absence of the magnetic field.

III. HALL CONDUCTIVITY FOR NONINTERACTING ELECTRONS IN THE PRESENCE OF A FINITE RANGE DISORDER

We first illustrate the use of Eq. (9) by deriving the Drude Hall conductivity for the simplest case of short-range impu-

rities. The conductivity will be calculated in the framework of the Kubo linear-response theory. The current-current correlator on the Matsubara frequencies is defined as

$$\Pi_{\alpha\beta}(i\omega_n) = -\int_0^\beta e^{i\omega_n\tau} \langle T_\tau J_\alpha(\tau) J_\beta(0) \rangle. \qquad (12)$$

The analytic continuation from the discrete frequencies on the upper complex half plane to the real axes yields the retarded correlator $\Pi_{\alpha\beta}^R(\omega)$. This correlator is directly related to the conductivity,

$$\sigma_{\alpha\beta}(\omega) = \frac{i}{\omega} \Pi_{\alpha\beta}^R(q=0,\omega). \qquad (13)$$

In the case of a short-range disorder the correlator (12) in the Drude approximation is equal to

$$\Pi_{xy}(r_i, r_f; \tau) = (e^2/4m^2) \left(\lim_{\mathbf{r}_i' \to \mathbf{r}_i} \{ -i\nabla_{\mathbf{r}_i'}^x - (e/c)A^x(\mathbf{r}_i') \} + i\nabla_{\mathbf{r}_i}^x - (e/c)A^x(\mathbf{r}_i) \right) \left(-i\nabla_{\mathbf{r}_f}^y - (e/c)A^y(\mathbf{r}_f) \right.$$
$$\left. + \lim_{\mathbf{r}_f' \to \mathbf{r}_f} \{ i\nabla_{\mathbf{r}_f'}^y - (e/c)A^y(\mathbf{r}_f') \} \right) \tilde{\mathcal{G}}(\mathbf{r}_f - \mathbf{r}_i, -\tau) \exp[(ie/c)\Phi(\mathbf{r}_f, \mathbf{r}_i)] \tilde{\mathcal{G}}(\mathbf{r}_i' - \mathbf{r}_f', \tau) \exp[(ie/c)\Phi(\mathbf{r}_i', \mathbf{r}_f')]. \qquad (14)$$

Using Eqs. (7) and (8), Eq. (14) can be rewritten as

$$\Pi_{xy}(\mathbf{r}_i, \mathbf{r}_f; \tau) = (e^2/4m^2) \left(\lim_{\mathbf{r}_i' \to \mathbf{r}_i} \{ -i\nabla_{\mathbf{r}_i'}^x + (e/2c)[(\mathbf{r}_i' - \mathbf{r}_f') \times \mathbf{B}]_x \} + i\nabla_{\mathbf{r}_i}^x - (e/2c)[(\mathbf{r}_f - \mathbf{r}_i) \times \mathbf{B}]_x \right) \left(-i\nabla_{\mathbf{r}_f}^y \right.$$
$$\left. + (e/2c)[(\mathbf{r}_f - \mathbf{r}_i) \times \mathbf{B}]_y + \lim_{\mathbf{r}_f' \to \mathbf{r}_f} \{ i\nabla_{\mathbf{r}_f'}^y - (e/2c)[(\mathbf{r}_i' - \mathbf{r}_f') \times \mathbf{B}]_y \} \right) \tilde{\mathcal{G}}(\mathbf{r}_f - \mathbf{r}_i, -\tau) \tilde{\mathcal{G}}(\mathbf{r}_i' - \mathbf{r}_f', \tau). \qquad (15)$$

Notice that when in Eq. (15) the gradients from the J_x vertex act on the coordinate dependent terms in the J_y part the corresponding contributions cancel out. The part linear in B arising from the J_x vertex is equal to

$$\Pi_{xy}^{(x)}(\mathbf{r}_i, \mathbf{r}_f; \tau) = (e^2\omega_c/4m)[(\mathbf{r}_i - \mathbf{r}_f) \times \hat{\mathbf{B}}]_x$$
$$\times \{ [-i\nabla_{\mathbf{r}_f}^y \tilde{\mathcal{G}}(\mathbf{r}_f - \mathbf{r}_i, -\tau)] \tilde{\mathcal{G}}(\mathbf{r}_i - \mathbf{r}_f, \tau)$$
$$+ \tilde{\mathcal{G}}(\mathbf{r}_f - \mathbf{r}_i, -\tau)[i\nabla_{\mathbf{r}_f}^y \tilde{\mathcal{G}}(\mathbf{r}_i - \mathbf{r}_f, \tau)] \}, \qquad (16)$$

and the other term from the J_y vertex is

$$\Pi_{xy}^{(y)}(\mathbf{r}_i, \mathbf{r}_f; \tau) = (e^2\omega_c/4m)[(\mathbf{r}_f - \mathbf{r}_i) \times \hat{\mathbf{B}}]_y$$
$$\times \{ [i\nabla_{\mathbf{r}_i}^x \tilde{\mathcal{G}}(\mathbf{r}_f - \mathbf{r}_i, -\tau)] \tilde{\mathcal{G}}(\mathbf{r}_i - \mathbf{r}_f, \tau)$$
$$+ \tilde{\mathcal{G}}(\mathbf{r}_f - \mathbf{r}_i, -\tau)[-i\nabla_{\mathbf{r}_i}^x \tilde{\mathcal{G}}(\mathbf{r}_i - \mathbf{r}_f, \tau)] \}. \qquad (17)$$

[Notice that Eqs. (16) and (17) reproduce the structure of Eq. (9).] Now the transition $\tilde{\mathcal{G}}_{B\to 0} = \mathcal{G}_{B=0}$ can be performed in these expressions, and as all terms are translational invariant one may use the Fourier transformation. Then the coordinate difference $(\mathbf{r}_f - \mathbf{r}_i)$ leads to the differentiation of the Green's function with respect to momentum:

$$\Pi_{xy}^{(x)}(q=0,i\omega_n) = i(e^2\omega_c/4m)T\sum_{i\epsilon_n} \int \frac{d^d p}{(2\pi)^d}$$
$$\times [\mathcal{G}^A(\mathbf{p})\partial/\partial p_y \mathcal{G}^R(\mathbf{p})$$
$$- \mathcal{G}^R(\mathbf{p})\partial/\partial p_y \mathcal{G}^A(\mathbf{p})]p_y, \qquad (18)$$

and correspondingly

$$\Pi_{xy}^{(y)}(q=0,i\omega_n) = i(e^2\omega_c/4m)T\sum_{i\epsilon_n} \int \frac{d^d p}{(2\pi)^d}p_x$$
$$\times [\mathcal{G}^A(\mathbf{p})\partial/\partial p_x \mathcal{G}^R(\mathbf{p})$$
$$- \mathcal{G}^R(\mathbf{p})\partial/\partial p_x \mathcal{G}^A(\mathbf{p})]. \qquad (19)$$

The summation over the frequencies and momentum in Eqs. (18) and (19) leads to the standard answer: $\sigma_{xy} = \omega_c \tau \sigma_{xx}$. This part of the calculation is close to the one presented in Ref. 13.

Next we consider a nontrivial problem of the calculation of the Drude Hall conductivity for noninteracting electrons in the presence of a finite range disorder, i.e., for nonshort-range impurities. To the leading order in $1/\epsilon_F\tau$ the conductivity in this case are given by a set of ladder diagrams. To get σ_{xy} that is linear in the external magnetic field we consider separately the diamagnetic (extended) contribution, and the flux contribution that comes out from the phase factors accumulated by the Green's functions in a diagram. In the case of a finite range disorder the fermion loop does not degenerate to two retraced paths and the flux term becomes absolutely essential. (Naturally, we have to expand
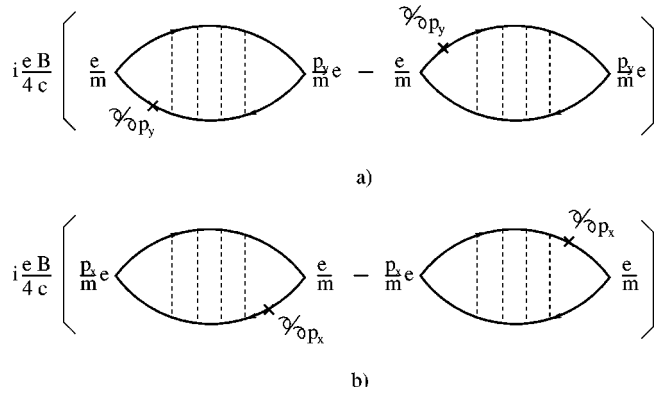
FIG. 1. The diamagnetic contributions: (a) from the J_x vertex, (b) from the J_y vertex.

FIG. 3. Flux contributions with no vertex differentiation involved and $\partial/\partial p_x$ is closer to p_x vertex than $\partial/\partial p_y$.

$\exp[(ie/c)\Phi_{loop}]$ and keep the linear term only as we are interested in σ_{xy} that is linear in B.)

The diamagnetic contribution is represented in Fig. 1. The cross means differentiation with respect to momentum. Only the Green's functions adjacent to the current vertex are differentiated. This contribution is obtained using the relations (7), (8), and (9) similar to the case of the short-range impurities.

To analyze the flux contribution to the Hall conductivity we decompose the loop polygon in a diagram for the current-current correlator into the sum of triangles. The decomposition will be done as follows: one of the vertexes of a n-vertex polygon is chosen as a reference point (let us call it $\mathbf{r}_1$), and then $\mathbf{r}_1$ is connected to all other vertexes. In result the polygon is decomposed into oriented triangles all having the reference point as their vertex. (As an example consider a loop presented in Fig. 2.) The flux through the polygon is therefore equal to the sum of the fluxes through each triangle:

$$\Phi_{loop}=\frac{1}{2}\mathbf{B}\cdot[(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_2)+(\mathbf{r}_3-\mathbf{r}_1)\times(\mathbf{r}_4-\mathbf{r}_3)$$
$$+\cdots],\tag{20}$$

which in turn is equal to

$$\Phi_{loop}=\frac{1}{2}\mathbf{B}\cdot[(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_2)+(\mathbf{r}_3-\mathbf{r}_2)\times(\mathbf{r}_4-\mathbf{r}_3)$$
$$+(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_4-\mathbf{r}_3)+\cdots].\tag{21}$$

Here we rewrote the vector differences $(\mathbf{r}_j-\mathbf{r}_1)$ as $[(\mathbf{r}_j-\mathbf{r}_{j-1})+(\mathbf{r}_{j-1}-\cdots)+(\mathbf{r}_2-\mathbf{r}_1)]$. Finally we add to Φ_{loop} a zero flux expression that does not change its value,
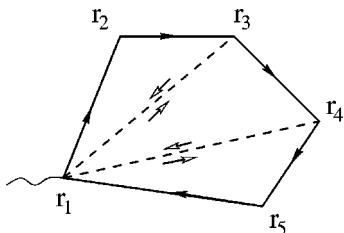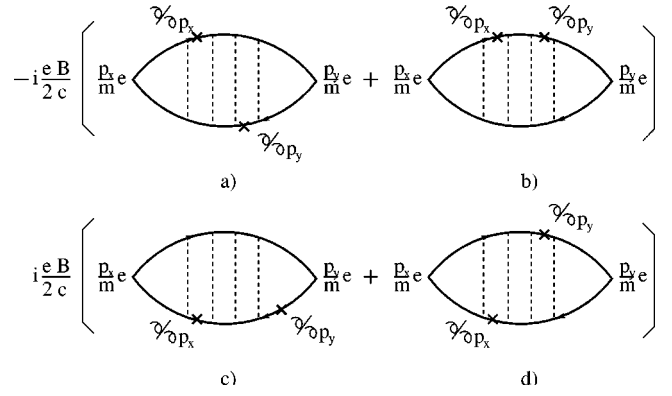

FIG. 2. Flux contribution.

$$0=\frac{1}{2}\mathbf{B}\cdot(\mathbf{r}_n-\mathbf{r}_{n-1}+\mathbf{r}_{n-1}-\cdots-\mathbf{r}_2+\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_1-\mathbf{r}_n).$$
$$\tag{22}$$

Now we are ready to state the general rule. After some reference point has been chosen the flux of the oriented polygon is given by

$$\Phi_{loop}=\frac{1}{2}\sum_{i\neq j}{}'\;(\pm)\mathbf{B}\cdot(\mathbf{r}_{i+1}-\mathbf{r}_i)\times(\mathbf{r}_{j+1}-\mathbf{r}_j),\tag{23}$$

where prime means that each pair (i,j), or (j,i) which is the same, enters the sum only once, and the pair is oriented with respect to the reference point in such a manner that its sign is "$+$" if we do not pass the reference point moving along the loop from r_i to r_j in the arrow direction. The sign is "$-$" otherwise.

After the flux contribution has been decomposed in pairs each pair supplies the diagram with a vector product of the two coordinate differences. When Fourier transformed these coordinate factors lead to the differentiation of the corresponding Green's function with respect to the momentum components p_x and p_y. There are also situations when one or two of $(\mathbf{r}_{i+1}-\mathbf{r}_i)$ are adjacent to the current vertexes. Then the Fourier transformation leads, in addition to the differentiation of the vertex-attached Green's functions, also to the differentiation of the vertex momentum.

Let the left-end current vertex be the p_x vertex, and let the p_y vertex be on the right end. The left-end located vertex will be chosen as the reference point with respect to which the orientation in Eq. (23) has been performed. First consider the terms represented in Figs. 3 and 4 when the vertices have not been differentiated. The difference between the two terms is that in Fig. 3 the differentiation $\partial/\partial p_x$ is closer to the left p_x vertex, whereas in Fig. 4 it is $\partial/\partial p_y$ that is closer to the p_x vertex. The ambiguous terms with $\partial/\partial p_x$ and $\partial/\partial p_y$ standing on equal distance, i.e., when they stand opposite to each other, are canceled out. Figure 5 represents schematically the flux contribution involving the differentiation of the p_y ver-
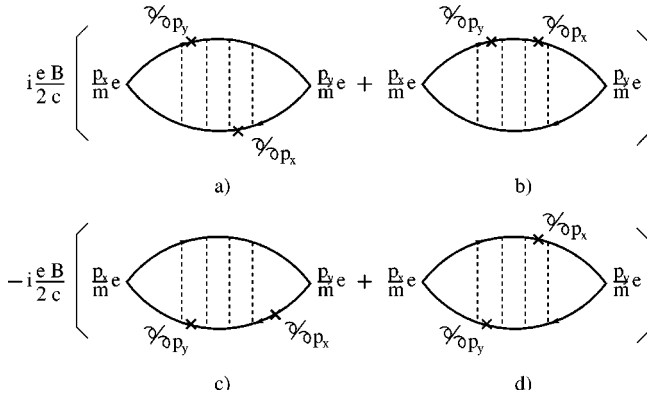
FIG. 4. Flux contributions with no vertex differentiation involved and $\partial/\partial p_y$ is closer to p_x vertex than $\partial/\partial p_x$.

tex at the right end. In Figs. 5(a) and (b) the $\partial/\partial p_x$ differentiation acts on each place in the loop. It should be noticed, however, that the momentum of the vertex originates from the gradients of the two Green's functions that are attached to the vertex [see, e.g., Eq. (14)]. Therefore the differentiations in Fig. 5 contain false nonflux contributions when $\partial/\partial p_x$ and $\partial/\partial p_y$ act on the same Green's function attached to the vertex. The compensation of the false terms is given in Fig. 6 where only the Green's functions closest to the p_y vertex are differentiated. Finally, in Fig. 7 the contribution involving the p_x vertex differentiation is represented. Here after cancellations amongst various terms combined into a full derivative only the two terms with the derivatives that act only on the Green's function closest to the p_x vertex survive. The apparent asymmetry between Figs. 5 and 6 compared to Fig. 7 is due to the fact that it was the p_x vertex that was chosen here as a reference point for the orientation of the pairs, rather than the p_y vertex.

The total contribution. Let us collect all the terms together. Notice the cancellations of Fig. 6 with Fig. 1(b), and Fig. 7 with Fig. 1(a). Then Figs. 3(a), 3(b), and 5(a) on one hand and Figs. 3(c), 3(d), and 5(b) on the other can be packed in combinations such that everything on the right from the p_x derivative becomes a p_y derivative of the current operator:

$$\Pi_{xy}^{(3,5)} = i(\omega_c m e^2/2)[\mathcal{G}^A(\mathbf{p})\partial/\partial p_x \mathcal{G}^R(\mathbf{p})$$
$$- \mathcal{G}^R(\mathbf{p})\partial/\partial p_x \mathcal{G}^A(\mathbf{p})]J_x \partial/\partial p_y J_y. \quad (24)$$

(In the last expression we have omitted for brevity $T\Sigma_{i\epsilon_n}\int[d^d p/(2\pi)^d]$, as well as in Eqs. (25) and (26) below.)
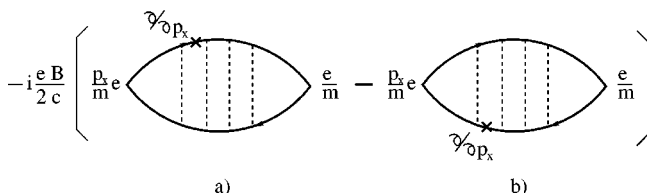

a)   b)

FIG. 5. Flux contributions with p_y vertex differentiation involved.

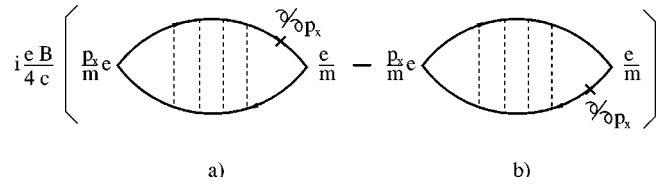

a)   b)

FIG. 6. Compensation of the false contributions of Fig. 5.

In addition, Figs. 4(a) and (b) on one hand and Figs. 4(c) and (d) on the other yield a similar combination,

$$\Pi_{xy}^{(4)} = -i(\omega_c m e^2/2)[\mathcal{G}^A(\mathbf{p})\partial/\partial p_y \mathcal{G}^R(\mathbf{p})$$
$$- G^R(\mathbf{p})\partial/\partial p_y \mathcal{G}^A(\mathbf{p})]J_x \partial/\partial p_x J_y. \quad (25)$$

Notice that finally we get a very simple rule: start with the derivative that stands at the left, and organize all the rest on the right as a derivative of the J_y current; arrange the signs depending on whether the derivative is on the left-to-right or the right-to-left segments of the loop, and depending on whether it is a $\partial/\partial p_x$ or $\partial/\partial p_y$ derivative [the change of the sign is in accord with the orientation rules for pairs formulated for Eq. (23)]. Remarkably, the diamagnetic and flux terms match each other to produce a rather simple form. (The Peierls substitution for the hopping matrix elements in the presence of a magnetic field is useful to discuss the diamagnetic contribution on an equal footing with the flux term.[15] In the continuous limit both terms originate from the phase factor $-ie/c\int \dot{\mathbf{r}}\mathbf{A}$ in the path-integral formulation of the motion of the electron in the presence of the vector potential.[17])

The Onsager relation, $\sigma_{xy}(\mathbf{B}) = \sigma_{yx}(-\mathbf{B})$, follows directly from the rule formulated above. To get $\sigma_{yx}(-\mathbf{B})$ from a given contribution to $\sigma_{xy}(\mathbf{B})$ one has to interchange derivatives $\partial/\partial p_x$ and $\partial/\partial p_y$ and reverse $\mathbf{B}$ to compensate the change of the sign.

Now we use a usual trick,[10] and interchange in the $\Pi_{xy}^{(4)}$ integration variables p_x and p_y. This leads to a convenient expression for Π_{xy}:

$$\Pi_{xy} = i\omega_c m \frac{e^2}{2}\left(\mathcal{G}^A(\mathbf{p})\frac{\partial}{\partial p_x}\mathcal{G}^R(\mathbf{p}) - \mathcal{G}^R(\mathbf{p})\frac{\partial}{\partial p_x}\mathcal{G}^A(\mathbf{p})\right)$$

$$\times\left(J_x\frac{\partial}{\partial p_y}J_y - J_y\frac{\partial}{\partial p_y}J_x\right). \quad (26)$$

The scattering by long-ranged impurities results in the renormalization of the current vertex by the ratio of the transport time τ_{tr} to the single-particle scattering time τ:

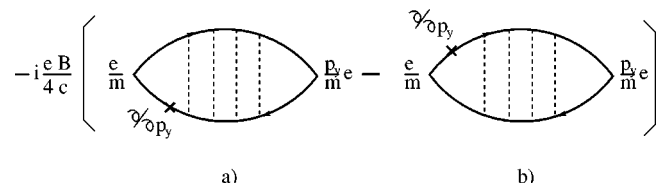$$J_\alpha = \frac{\tau_{tr}(|\mathbf{p}|)}{\tau}\frac{p_\alpha}{m}. \quad (27)$$

a)   b)

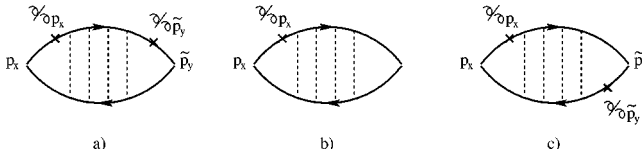FIG. 7. Flux contributions with p_x-vertex differentiation involved.

FIG. 8. Cancellation of the singular terms.

Fortunately, in the combination $(J_x \partial/\partial p_y J_y - J_y \partial/\partial p_y J_x)$ the terms of the type $\partial/\partial p_y \tau_{tr}(|\mathbf{p}|)$ cancel each other out. Then the standard integrations in Π_{xy} lead us to a natural conclusion that the answer for the finite-range impurities differs from that one for the short-range case by the substitution $\tau \rightarrow \tau_{tr}$:

$$\sigma_{xy} = \omega_c \tau_{tr} \sigma_{xx}. \tag{28}$$

Correspondingly, it follows from this conclusion that the Hall coefficient is independent on the impurity range. [Here one point remains to be cleared up. In the presence of disorder the Green's functions acquire a self-energy part whose imaginary part is $1/\tau$. Its real part shifts the chemical potential and hence is of no interest, but its dependence on the momentum near the Fermi energy may influence the result of the integration over momentum in Eq. (26). However, in the absence of a special structure of the scatterers, like in liquid metals, the sensitivity to the energy of the electron state is very small when $\epsilon_F \tau_{tr} \gg 1$, and this effect can be ignored.]

A comment may be in place here. It was important in the above derivation to keep the p derivatives in the form of combined expressions of the type $\partial/\partial p_y J_y$ rather than to consider separate terms. Separately these terms are larger or even more singular than their sum. Let us see how it works in a somewhat pathological but instructive example of short-range impurities. In this case the dressing of the vector vertex by the impurity lines does not give any effect because of the momentum averaging. The p_y derivative makes the vector averaging noneffective as it is illustrated in Fig. 8. Each of the three diagrams presented in Fig. 8 contains the singular propagator of the diffusion ladder diagrams (diffusion) that does not vanish after the averaging. This does not lead to any complications, however, as the sum of these singular terms vanishes identically. Indeed, the expression corresponding to the ending block (i.e., separated by the last vertical impurity line) when summed over the three diagrams turns out to be a derivative $\partial/\partial \tilde{p}_y (\mathcal{G}^R \tilde{p}_y \mathcal{G}^A)$, where $\tilde{\mathbf{p}}$ is the momentum circulating inside this block. The integration over $\tilde{\mathbf{p}}$ forces the whole contribution to be zero. As a result only the diagram that does not contain vertical impurity lines survives in the case of the short-range impurities as it is obvious when one considers these terms not separately but in the
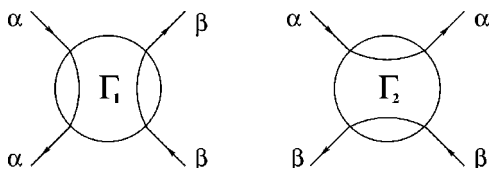
FIG. 9. The irreducible amplitudes of the e-e interaction Γ_1 and Γ_2.
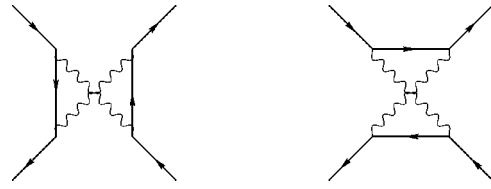
FIG. 10. Examples of the Γ_1 and Γ_2 amplitudes.

combined form $\partial/\partial p_y J_y$. The observation about the importance of keeping combined expressions in the calculation of σ_{xy} is of general character. Separate terms can be more involved than their total contribution, see, e.g., the discussion related to $\partial^2 \Sigma/\partial \xi^2$ following Eq. (36).

To conclude this section we notice that Eq. (26) reproduces the structure of the iterative solution of the transport equation for σ_{xy}. Let us compare Eq. (26) with the term $(e[\mathbf{v_p} \times \mathbf{B}]\partial/\partial_\mathbf{p})\delta n_\mathbf{p}$ that appears in the transport equation as a result of the action of the Lorentz force and describes the turn of the current. In the first bracket of Eq. (26) the derivative $\partial/\partial p_x$ yields the velocity $\mathbf{v_p}$, while $\partial/\partial p_y$ in the second bracket corresponds to the momentum derivative acting on the distribution function $\delta n_\mathbf{p}$. The derivation in this section was limited to the leading order in $1/\epsilon_F \tau_{tr}$. To get further, one has to study the dependence of $1/\tau_{tr}$ on the flux. The leading term of that kind arises due to the Cooperton corrections and is discussed in Appendix C.

IV. ELECTRON-ELECTRON INTERACTION

We are interested in the current-current correlator, and therefore the basic process to be considered is a sequential rescattering of electron-hole pairs. Diagrammatically this process is described by the particle-hole ladder sections alternating with the irreducible amplitudes of the e-e interaction Γ_1 and Γ_2. Amplitudes Γ_1 and Γ_2 differ in spin structure as it is schematically represented in Fig. 9. Examples of such amplitudes are given in Fig. 10.

A typical diagram for the Hall current correlator Π_{xy} includes the J_x-current vertex, several Γ amplitudes connected by the $R-R$, $A-A$, or $R-A$ sections and, finally, the J_y-current vertex. In essence, the correlator Π_{xy} may be evaluated following the same line as in the case of the long-range impurity scattering. There is, however, a complication related to Γ_1 insertions. In the case of noninteracting electrons we have dealt with a flux of a single loop extending from the J_x to the J_y vertex. An insertion of Γ_2 in a diagram leaves the number of loops to be the same, while the Γ_1 amplitude splits the loop into pieces and breaks the single loop structure. We restore the construction with the main flux loop having in mind to use the basic results of the previous
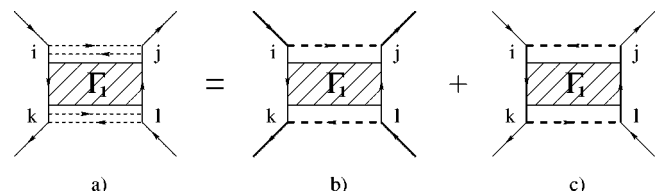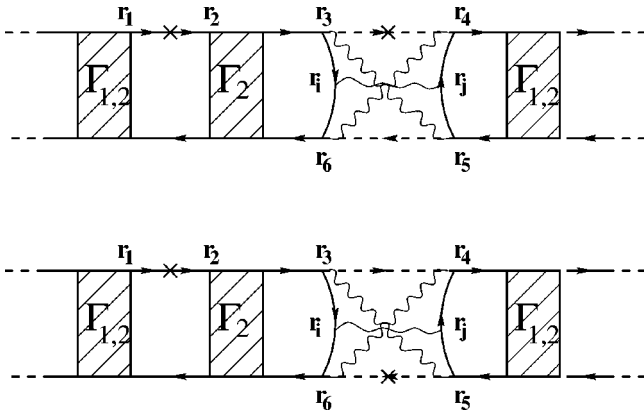

FIG. 11. The phase factors of the Γ_1 amplitude.

FIG. 12. The treatment of the dashed lines in the Γ_1 amplitude; the left standing derivative acts on a particle-hole section.

FIG. 13. A contribution to σ_{xy} that involves nonpair-correlation amplitudes.

section, i.e., Eqs. (24) and (25).

Consider an insertion of Γ_1 that splits a loop at the points i,j,k,l. To restore the construction with the main flux loop we attach the phase factor $i(e/c)(\mathbf{r}_j-\mathbf{r}_i)\mathbf{A}((\mathbf{r}_i+\mathbf{r}_j)/2)$ to the segment $\mathbf{r}_i\rightarrow\mathbf{r}_j$ in Fig. 11(a). The analogous operation will be done with the segment $\mathbf{r}_k\rightarrow\mathbf{r}_l$. Simultaneously the compensating phase factors with the opposite sign are attached to the lines going in the opposite directions, $\mathbf{r}_j\rightarrow\mathbf{r}_i$ and $\mathbf{r}_l\rightarrow\mathbf{r}_k$. In Fig. 11 all these segments are denoted by the dashed lines with arrows. These dashed lines carry phases but do not represent any Green's functions. Now we regroup the phase factors. It is possible because we are interested in σ_{xy} that is linear in B, and therefore we have to expand and keep only linear terms in the phase factors. The phases of the two dashed lines, $\mathbf{r}_i\rightarrow\mathbf{r}_j$ and $\mathbf{r}_k\rightarrow\mathbf{r}_l$, are used to obtain the flux of the main loop extending from the J_x to the J_y vertex [a fragment of this loop is presented in Fig. 11(b) where it is indicated by the thick lines]. The other two phase factors attached to the segments $\mathbf{r}_j\rightarrow\mathbf{r}_i$ and $\mathbf{r}_l\rightarrow\mathbf{r}_k$ are used to form a new flux loop encircling the brick of the Γ_1 amplitude in the direction opposite to the main flux loop. In Fig. 11(c) all the parts of the corresponding loop are given by the thick lines.

To get the contribution to Π_{xy} from the main flux loop we shall follow the derivation given in the previous section. Consider first the situation when the left standing derivative acts on a Green's function in a particle-hole section rather than on the Γ amplitudes. Let this left standing derivative be the p_x derivative. Then, the p_y derivative is supposed to act on everything on the right of the p_x derivative. The new element here is that the dashed lines added to restore the flux structure do not carry any Green's functions, but only the phase factors. Let us study the situation when the additional phase factors arise from one of the Γ_1 amplitudes with the dashed segments $\mathbf{r}_3\rightarrow\mathbf{r}_4$ and $\mathbf{r}_5\rightarrow\mathbf{r}_6$, as shown in Fig. 12. After the decomposition procedure the corresponding flux contribution is equal to

$$\Phi_{dashed}=\frac{1}{2}\mathbf{B}\cdot[(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_4-\mathbf{r}_3)+(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_6-\mathbf{r}_5)].$$
$$(29)$$

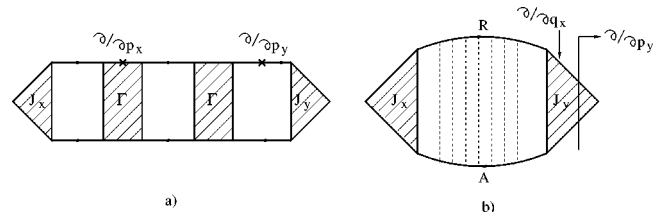Next, Φ_{dashed} can be rearranged as $\frac{1}{2}\mathbf{B}\cdot[(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_4-\mathbf{r}_5)+(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_6-\mathbf{r}_3)]$, and for any structure of the Γ_1 amplitude one obtains

$$\Phi_{dashed}=\frac{1}{2}\mathbf{B}\cdot[(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_4-\mathbf{r}_j+\mathbf{r}_j-\cdots-\mathbf{r}_5)$$

$$+(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_6-\mathbf{r}_i+\mathbf{r}_i-\cdots-\mathbf{r}_3)]. \quad (30)$$

It follows as a result of this rearrangement that after the Fourier transformation the operation of the p_y differentiation will move all along the Green's functions in the segments $\mathbf{r}_3\rightarrow\mathbf{r}_6$ and $\mathbf{r}_5\rightarrow\mathbf{r}_4$.

Since Eq. (30) is valid for any Γ_1 amplitude, we come to the conclusion that the dashed lines do not spoil the general structure of the discussed contributions, and every Green's function on the "vertical" lines, like $\mathbf{r}_3\rightarrow\mathbf{r}_6$ and $\mathbf{r}_5\rightarrow\mathbf{r}_4$, is p_y differentiated on the same footing as any other Green's function. Thus in the considered case the p_y derivative indeed acts on everything on the right to the p_x derivative, as it should be expected.

Now we continue with the analysis of the case of the left standing p derivative that acts on one of the Green's functions in the particle-hole section. The sections $R-R$ or $A-A$ need not to be considered as each of them yields two expressions that cancel each other out. By the $R-A$ section we assume a set of the impurity scattering ladder diagrams that modify a current vertex J by a factor τ_{tr}/τ. In the limit when the external frequency $\omega\rightarrow0$ only one $R-A$ section should be kept in the current-current correlator $\Pi_{\alpha\beta}(\omega)$, because each such section yields an additional power of ω. Differentiation of the Green's functions in the $R-A$ section splits the set of the ladder diagrams into two pieces that modify both vertices J_x and J_y by τ_{tr}/τ. As a result the contribution analogous to that represented in Figs. 3 and 5 reduces to Eq. (24), and in the same way the contribution analogous to Fig. 4 reduces to Eq. (25). Altogether we come back to Eq. (26) but with the current vertices and the Green's functions dressed by the e-e interaction.

The terms that have been discussed until now determine the leading contribution to $\sigma_{xy}\propto\tau_{tr}^2$. Its analysis will be completed in the end of this section, but first we turn to the other case when the left standing derivative, let it be the p_x derivative, is inside one of the interaction amplitudes Γ, while the p_y derivative acts on everything to the right from this amplitude [an example is presented in Fig. 13(a)]. As a candidate for a contribution of the order τ_{tr}^2 to σ_{xy} in the discussed process consider the case when the p_y derivative is applied to the $R-A$ section standing on the right from Γ.
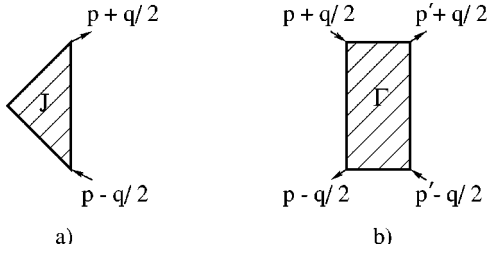
FIG. 14. The definition of the Fourier components of the current vertexes J and the interaction amplitude Γ.

This differentiation produces the singular terms $(\partial/\partial p_y \mathcal{G}^R)\mathcal{G}^A$ and $\mathcal{G}^R(\partial/\partial p_y \mathcal{G}^A)$ that separately yield the contributions $\propto \tau_{tr}^2$ but in the leading order they cancel each other out. The latter fact is obvious because we may take the integration over p_y by parts and then the $R-A$ section yields a contribution that is only $\propto \tau_{tr}$. A similar contribution appears when the $R-A$ section is located on the left from the amplitude Γ that is marked by the differentiation. Arranging both the terms in a way that the $R-A$ section becomes free from the differentiation the contribution to σ_{xy} in the discussed case can be presented as

$$\sigma_{xy} \propto \omega_c \tau_{tr} \tilde{J}_x (\partial q_x| \rightarrow \partial p_y - \partial q_y| \rightarrow \partial p_x) \tilde{J}_y . \quad (31)$$

Here the current vertices $\tilde{J}_{x,y}$ are dressed by the e-e interaction, but unlike J they do not contain the factor τ_{tr}/τ. The symbol $| \rightarrow$ [as well as the right-directed bar in Fig. 13(b)] indicates that the derivatives are ordered and the derivative over p acts on the right to the q derivative where q is an infinitesimal momentum in the current vertex J_y as shown in Fig. 14(a). The momentum q has been introduced to get opposite signs for the derivatives acting on oppositely directed lines in the current vertex. Since the parameters of the electron liquid change on the scale of the Fermi momentum this contribution is small compared to the leading one by the parameter $1/\epsilon_F \tau_{tr}$. The reason of this smallness is that in the discussed process the skew action of the magnetic field may develop only on an electron wavelength, while for the leading term the skew effect develops on a free path length. Notice that the processes just discussed involve the derivatives of the Γ amplitudes that generate terms with the non-pair electron correlations f_{np} that were mentioned in the Introduction. The discussed term may become significant if apart from the wavelength and the mean free path there is another length scale in the problem. For example, near the superconducting transition the fluctuations of the order parameter may contribute significantly and change the Hall coefficient.

The last contribution to Π_{xy} that remains to be analyzed is related to the flux loops encircling the Γ_1 amplitudes. Consider the loop encircling the amplitude $\Gamma_1(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$. The flux through the polygon $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ may be written as the sum of the fluxes through the two triangles:

$$\Phi_{\Gamma_1} = \mathbf{B} \cdot [(\mathbf{r}_1 - \mathbf{r}_2) \times (\mathbf{r}_4 - \mathbf{r}_1) + (\mathbf{r}_3 - \mathbf{r}_4) \times (\mathbf{r}_2 - \mathbf{r}_3)] . \quad (32)$$

We define the Fourier representation of $\Gamma_1(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ as follows [see Fig. 14(b)]:

$$\Gamma_1(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d p'}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d}$$

$$\times \exp\{i[(\mathbf{p} + \mathbf{q}/2)\mathbf{r}_1 - (\mathbf{p}' + \mathbf{q}/2)\mathbf{r}_2$$

$$+ (\mathbf{p}' - \mathbf{q}/2)\mathbf{r}_3 - (\mathbf{p} - \mathbf{q}/2)\mathbf{r}_4]\} \Gamma(\mathbf{p}, \mathbf{p}'; \mathbf{q}) . \quad (33)$$

Comparing Eq. (32) with Eq. (33) we obtain that the product $\Gamma_1 \Phi_{\Gamma_1}$ when Fourier transformed yields the following combination that enters the correlator Π_{xy}:

$$\Gamma_1 \Phi_{\Gamma_1} \rightarrow \hat{\mathbf{B}} \cdot \left[\frac{\partial}{\partial \mathbf{q}} \times \left(\frac{\partial}{\partial \mathbf{p}'} - \frac{\partial}{\partial \mathbf{p}} \right) \right] \Gamma_1(\mathbf{p}, \mathbf{p}'; \mathbf{q}) . \quad (34)$$

As the amplitude Γ_1 is taken in the limit $\mathbf{B}=0$ it is a scalar function of its momenta, and therefore the expression (34) vanishes.

Summarizing the analysis presented in this section we conclude that the leading contribution to σ_{xy} is determined by the following expression:

$$\Pi_{xy} = ieB \frac{e^2}{2c} \left(\frac{\tau_{tr}}{\tau} \right)^2 \left(\mathcal{G}^A(\mathbf{p}) \frac{\partial}{\partial p_x} \mathcal{G}^R(\mathbf{p}) - \mathcal{G}^R(\mathbf{p}) \frac{\partial}{\partial p_x} \mathcal{G}^A(\mathbf{p}) \right)$$

$$\times \left(\tilde{J}_x \frac{\partial}{\partial p_y} \tilde{J}_y - \tilde{J}_y \frac{\partial}{\partial p_y} \tilde{J}_x \right) . \quad (35)$$

The derivatives over the momentum in Eq. (35) correspond to the vector product of the two coordinate differences in the representation of the flux in Eq. (23). The current vertexes $\tilde{J}_{x,y}$ are not sensitive to the renormalizations by impurities due to $1/\epsilon_F \tau \ll 1$, and are determined by the Ward identity:[18]

$$\tilde{J}_\mu = \frac{p_\mu}{m} \left(1 + \frac{\partial \Sigma(p, \epsilon)}{\partial \xi_p} \right) , \quad (36)$$

where $\Sigma(p, \epsilon)$ is a self-energy of the Green's function in the presence of the e-e interaction. Equations (35) and (36) correspond to the solution of the transport equation for σ_{xy} for the interacting electrons [see the discussion of Eq. (26) concluding Sec. III]. Due to the specific structure of the last bracket in Eq. (35) the second derivative $\partial^2 \Sigma/\partial \xi^2$, which is not conventional in the Fermi-liquid theory, does not enter the final answer. Altogether we get the factor $[1 + \partial \Sigma(p,0)/\partial \xi_p]$ three times: two from the vertexes and one from the p_x derivatives of the Green's functions in the square brackets. The integration over momentum in Π_{xy} can be performed keeping the Green's functions in the pole approximation:

$$\mathcal{G}^{R,A}(\epsilon, \mathbf{p}) = \frac{[1 - \partial \Sigma(p_F, \epsilon)/\partial \epsilon]^{-1}}{\epsilon - \frac{m}{m^*} \xi_p \pm \frac{i}{2\tau}} , \quad (37)$$

where τ is the free path time of the quasiparticles that includes the renormalization by the e-e interaction, and

$$m/m^* = [1 + \partial\Sigma(p,0)/\partial\xi_p]/[1 - \partial\Sigma(p_F,\epsilon)/\partial\epsilon]. \quad (38)$$

The latter relation is a standard one for the microscopical theory of the Fermi liquid.[18] All three factors $[1 - \partial\Sigma(p_F,\epsilon)/\partial\epsilon]^{-1}$ created by the Green's functions in Eq. (35) together with the three factors $[1 + \partial\Sigma(p,0)/\partial\xi_p]$ can be combined into the cube of the physical combination m/m^*. After integration over ξ_p one power of m/m^* will be eliminated and ultimately we obtain

$$\sigma_{xy} = \omega_c^* \tau_{tr} \sigma_{xx} \quad (39)$$

with $\omega_c^* = eB/m^*c$ and $\sigma_{xx} = ne^2\tau_{tr}/m^*$. This result implies that no Hall coefficient renormalization develops due to the e-e interaction in the leading terms over $1/\epsilon_F\tau_{tr}$.

V. DISCUSSION

We have shown microscopically that the Hall coefficient in a weak magnetic field is not renormalized by the e-e interaction in the leading order in $1/\epsilon_F\tau_{tr}$. Although the terms that are not conventional in the Fermi-liquid theory appear in the intermediate stages, they do not enter the final answer. This remarkable fact is a direct consequence of the skew structure which arises in the Hall effect because of the magnetic flux.

The result about the cancellation of the Fermi-liquid renormalization corrections in the Hall coefficient differs from the one obtained previously in Refs. 10 and 19. It is formidable to make a comparison with the analysis of these papers because the distinction of the impurity scattering amplitudes from the e-e interaction amplitudes was not performed explicitly. These amplitudes have different structure in exchange of the frequency, however. Therefore any treatment lacking clear distinction between these amplitudes is potentially dangerous. The analysis of the Fermi liquid in Ref. 19 did not reproduce the well established results of the transport theory, because of the confusing treatment of the frequency integrations induced by the e-e interaction amplitudes. On the other hand, in Ref. 10 the authors were mainly concerned with the impurity scattering. One can check, in fact, that their factor $[1 + \partial\Sigma'(p,0)/\partial\epsilon(p)]^2$ in R_H differs negligibly from unity in the case of noninteracting electrons scattered by a random potential.

The absence of the renormalization in the Hall coefficient corresponds to the result that can be obtained within the phenomenological theory of the Fermi liquid. For completeness we reproduce it here following Sec. 3.6 in Ref. 7. In the Fermi-liquid theory the excited states of the interacting electrons are described by the gas of quasiparticles with the effective Hamiltonian written via the distribution function of the quasiparticles $n_p(\mathbf{r},t)$:

$$\bar{\epsilon}_p(\mathbf{r}) = \epsilon_p + \sum_{p'} f_{pp'} \delta n_{p'}(\mathbf{r}). \quad (40)$$

Here $(\mathbf{p},\mathbf{r})$ are classical variables, and the interaction term $f_{pp'} = \int d\mathbf{r}' \, f(\mathbf{pr},\mathbf{p}'\mathbf{r}')$ is assumed to be local. In the presence of the vector potential $\mathbf{A}(\mathbf{r},t)$ one should make a transition to the pair of the conjugate variables $(\mathbf{P},\mathbf{r})$:

$$\tilde{\epsilon}_P(\mathbf{r}) = \bar{\epsilon}_{P-(e/c)A}(\mathbf{r}); \quad \tilde{n}_P(\mathbf{r}) = n_{P-(e/c)A}(\mathbf{r}), \quad (41)$$

where the vector potential $\mathbf{A}(\mathbf{r},t)$ represents both magnetic and electric fields: $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$. In the phase space $(\mathbf{P},\mathbf{r})$ the flow of the quasiparticle density $\tilde{n}_P(\mathbf{r})$ satisfies the Liouville equation:

$$\frac{\partial\tilde{n}}{\partial t} + \frac{\partial\tilde{n}}{\partial\mathbf{r}}\frac{\partial\tilde{\epsilon}}{\partial\mathbf{P}} - \frac{\partial\tilde{n}}{\partial\mathbf{P}}\frac{\partial\tilde{\epsilon}}{\partial\mathbf{r}} = I_{coll}. \quad (42)$$

To make the dependence on the external fields explicit it is convenient to reexpress Eq. (42) in terms of $n(\mathbf{p},\mathbf{r},t)$:

$$\frac{\partial n}{\partial t} + \frac{\partial\bar{\epsilon}}{\partial p_\alpha}\frac{\partial n}{\partial r_\alpha} - \frac{\partial\bar{\epsilon}}{\partial r_\alpha}\frac{\partial n}{\partial p_\alpha} + eE_\alpha\frac{\partial n}{\partial p_\alpha} + \frac{e}{c}\left(\frac{\partial\bar{\epsilon}}{\partial\mathbf{p}}\times\mathbf{B}\right)_\alpha\frac{\partial n}{\partial p_\alpha}$$
$$= I_{coll}. \quad (43)$$

Next, it is useful to introduce the deviations from the global and local equilibrium defined as

$$\delta n_p(\mathbf{r}) = n_p(\mathbf{r}) - n^0(\epsilon_p), \quad \delta\bar{n}_p(\mathbf{r}) = n_p(\mathbf{r}) - n^0(\bar{\epsilon}_p(\mathbf{r})), \quad (44)$$

respectively. The two are related as follows:

$$\delta\bar{n}_p(\mathbf{r}) = \delta n_p(\mathbf{r}) - \frac{\partial n^0}{\partial\epsilon_p}\sum_{p'} f_{pp'}\delta n_{p'}(\mathbf{r}). \quad (45)$$

With the use of Eqs. (44) and (45), Eq. (43) yields after linearization

$$\frac{\partial\delta n}{\partial t} + eE_\alpha\frac{\partial n^0}{\partial p_\alpha} + v_\alpha\frac{\partial\delta\bar{n}}{\partial r_\alpha} + \frac{e}{c}(\mathbf{v}\times\mathbf{B})_\alpha\frac{\partial\delta\bar{n}}{\partial p_\alpha} = I_{coll}, \quad (46)$$

where $v_\alpha = p_\alpha/m^*$, and in the relaxation time approximation $I_{coll} = -\delta\bar{n}_p/\tau_{tr}$. The current density is also expressed through the deviation from the local equilibrium

$$J_\alpha(\mathbf{r}) = \sum_p \frac{p_\alpha}{m^*}\delta\bar{n}_p. \quad (47)$$

To study the Hall effect one should look for the stationary, homogeneous distribution $\delta\bar{n}_p$. The static limit of Eqs. (46) and (47) are identical to the corresponding equations for noninteracting electrons, if $m^* \to m$ and $\delta\bar{n} \to \delta n$. Therefore the transport coefficients are given by the free electron expressions with the substitution m to m^*, and $\sigma_{xy} = \omega_c^*\tau_{tr}\sigma_{xx}$.

It is clear form this discussion that since the Fermi-liquid theory uses a local functional with the distribution function depending on the classical variables $(\mathbf{p},\mathbf{r})$ it does not contain the nonpair-correlation contribution (31). The latter describes the influence of the flux phase on the interaction amplitudes and, obviously, is beyond the scope of the Fermi-liquid theory. Fortunately, this term is small by the parameter $1/\epsilon_F\tau_{tr}$.

To summarize, we have proved that the Hall coefficient is not renormalized by the e-e interaction in the leading order in $1/\epsilon_F\tau$. The result holds for not too low temperatures when

the logarithmic corrections from the Altshuler-Aronov effect can be ignored, i.e., when $(e^2/h\sigma_{xx})\ln(1/T\tau)\ll 1$. Furthermore it follows from the analysis of Ref. 20 that while as we show the leading term is robust, the temperature corrections to the Hall coefficient remain small up to the temperatures substantially lower than $1/\tau$. Combined with this observation the present analysis gives an explanation of the stability of the Hall coefficient R_H to the e-e interaction observed in Refs. 5 and 3(b) for the dilute electron gas in Si MOSFETs not too close to the metal-insulator transition.

APPENDIX A: SYMMETRY PROPERTIES

In this appendix the symmetry properties of the core Green's function $\tilde{G}$ are presented. A useful discussion of the symmetry properties of the Green's function in the presence of e-e interaction can be found in the classical papers[21,22] which study the quantum oscillations in a magnetic field.

1. Translation invariance

A many-body system in a homogeneous magnetic field is described by the Hamiltonian

$$H=\sum_i \frac{1}{2m}\left(\mathbf{p}_i-\frac{e}{c}\mathbf{A}(\mathbf{r}_i)\right)^2+\frac{1}{2}\sum_{i\neq j} V(|\mathbf{r}_i-\mathbf{r}_j|). \quad \text{(A1)}$$

In this case the vector potential depends linearly on the coordinates, and we introduce a matrix $A^{\mu\nu}$ following the considerations of Ref. 12,

$$A^\mu(\mathbf{r})=A^{\mu\nu}r_\nu. \quad \text{(A2)}$$

The conjugate vector potential is defined through its transpose $\tilde{A}^\mu(\mathbf{r})=\tilde{A}^{\mu\nu}r_\nu$ in such a way that the gauge invariant relation $\tilde{A}^{\mu\nu}-A^{\mu\nu}=B\varepsilon_{\mu\nu}$ holds. The generators of the magnetic translations are defined as

$$T^\mu=\sum_i \left(p_i^\mu-\frac{e}{c}\tilde{A}^\mu(\mathbf{r}_i)\right), \quad \text{(A3)}$$

where the sum is taken over all electrons. Consider the operator of a finite translation $S_\mathbf{a}=\exp(-ia_\mu T^\mu)$. The Matsubara Green's function $\mathcal{G}(\mathbf{r}_1,\mathbf{r}_2;\tau)=-\langle T_\tau\psi(\mathbf{r}_1,\tau)\psi^\dagger(\mathbf{r}_2)\rangle$ remains unchanged under the canonical transformation $\psi\to\psi'=S_\mathbf{a}^{-1}\psi S_\mathbf{a}$, as the generators (A3) commute with the Hamiltonian (A1). Consider now this transformation explicitly. Using the Baker-Hausdorff identity $\exp(A+B)=\exp(A)\exp(B)\exp(-[A,B]/2)$ the $S_\mathbf{a}$ may be decomposed as

$$S_\mathbf{a}=\exp(-i\mathbf{a}\cdot\mathbf{P}_{tot})\exp\left(i\frac{e}{c}\mathbf{a}\cdot\tilde{\mathbf{A}}_{tot}\right)$$
$$\times\exp\left[-\frac{ie}{4c}a_\mu a_\nu(\tilde{A}^{\nu\mu}+\tilde{A}^{\mu\nu})\right], \quad \text{(A4)}$$

where the $\mathbf{P}_{tot}$ is the total momentum and $\tilde{\mathbf{A}}_{tot}$ is the sum of the conjugate vector potentials felt by all the particles in the system. Then the transformed operator ψ' becomes

$$\psi'(\mathbf{r})=\psi(\mathbf{r}-\mathbf{a})\exp\left(\frac{ie}{c}\mathbf{a}\tilde{\mathbf{A}}(\mathbf{r})\right) \quad \text{(A5)}$$

and correspondingly

$$\psi'^\dagger(\mathbf{r})=\psi(\mathbf{r}-\mathbf{a})\exp\left(-\frac{ie}{c}\mathbf{a}\tilde{\mathbf{A}}(\mathbf{r})\right). \quad \text{(A6)}$$

Relations (A5) and (A6) imply that

$$\mathcal{G}(\mathbf{r}_1,\mathbf{r}_2;\tau)=\mathcal{G}(\mathbf{r}_1-\mathbf{a},\mathbf{r}_2-\mathbf{a};\tau)\exp\left(\frac{ie}{c}\mathbf{a}\tilde{\mathbf{A}}(\mathbf{r}_1-\mathbf{r}_2)\right). \quad \text{(A7)}$$

It follows then from the relation (A7) that the core Green's function defined in Eq. (4) is translation invariant. In the presence of disorder the statement is still valid for averaged Green's functions. (The average over disorder generates translation invariant two-body "interaction" terms when one introduces $n\to 0$ replicas of the system, or uses other standard tricks.)

2. Gauge invariance

It has been shown in Ref. 21 that Green's functions are transformed as

$$\exp\left(i\frac{e}{c}[\chi(\mathbf{r}_1)-\chi(\mathbf{r}_2)]\right)\mathcal{G}_\mathbf{A}(r_1,r_2;\tau)=\mathcal{G}_{\mathbf{A}+\nabla\chi}(r_1,r_2;\tau) \quad \text{(A8)}$$

under the gauge transformation $\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\nabla\chi$. Clearly, the integral $(ie/c)\int_{\mathbf{r}_2}^{\mathbf{r}_1}\mathbf{A}d\mathbf{r}$ in the exponential prefactor in Eq. (4) changes exactly in a way as to cancel out the factor appearing in the left-hand side of Eq. (A8). This cancellation ensures the gauge invariance of the core Green's function.

3. Rotational invariance

We consider the transformation properties under rotations in the $x-y$ plane in the circular gauge,

$$\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}. \quad \text{(A9)}$$

In this gauge the Hamiltonian (A1) takes explicitly the form which contains rotational invariant terms only. Similarly to $S_\mathbf{a}$ the operators $S_\phi=\exp(-i\phi L_z)$ can be introduced. As S_ϕ commute with the Hamiltonian, the Green's function remains invariant under the action of S_ϕ which rotate the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ in $\mathcal{G}(\mathbf{r}_1,\mathbf{r}_2,\tau)$. The phase factor $(ie/c)\int_{\mathbf{r}_2}^{\mathbf{r}_1}\mathbf{A}d\mathbf{r}$ is

also unchanged in the course of rotation in the circular gauge, and therefore the core Green's function $\tilde{G}$ should be rotational invariant as well. Since $\tilde{G}$ is gauge invariant the statement holds for any gauge.

APPENDIX B: SPHERICALLY SYMMETRIC SPECTRUM

Transport properties in the case of a complicated band structure are nonuniversal. For this reason we are not going to extend the present analysis of the renormalization of σ_{xy} by e-e interaction for an arbitrary spectrum. However, the obtained result can be readily extended to R_H in the case of a general but spherically symmetric spectrum $\epsilon(|\mathbf{p}|)$. (Such spectrum can naturally appear if the degrees of freedom in some energy shell are integrated out in the course of the Renormalization Group treatment.) In the presence of the magnetic field the corresponding part of the Hamiltonian is given by the gauge invariant extension of $\epsilon(|\mathbf{p}|)$ with a subsequent symmetrization required to make the Hamiltonian Hermitian. The latter is needed since different components of the velocity operator do not commute. To get a symmetrized extension of the kinetic term we expand $\epsilon(|\mathbf{p}|)$ in the Taylor series,

$$\epsilon(|\mathbf{p}|)=\sum_{m,n} \epsilon_{m,n}p_x^m p_y^n. \tag{B1}$$

The symmetrization procedure leads to

$$\epsilon(|\mathbf{p}|)\Rightarrow\sum_{m,n} \epsilon_{m,n}(C_{m+n}^n)^{-1}\sum_{P} \left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)_{P[1]}$$

$$\times\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)_{P[2]}\cdots\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)_{P[m+n]}, \tag{B2}$$

where the sum is taken over all inequivalent permutations P, in such a way that altogether there are m and n factors $[\mathbf{p}-(e/c)\mathbf{A}]_x$ and $[\mathbf{p}-(e/c)\mathbf{A}]_y$ respectively, and $P[l]=x,y$. The current operator is determined by the variation of the energy with respect to the vector potential:

$$J_{\mu}=-\delta\langle H\rangle/\delta A_{\mu}. \tag{B3}$$

Now we follow the main text after Eq. (6). Consider the diagrammatic element related to the current operator at the point $\mathbf{r}_i$ after the contractions with the operators $\psi(\mathbf{r}_1)$ and $\psi^{\dagger}(\mathbf{r}_2)$ have been performed. Using Eqs. (B2) and (B3) and integrating by parts we obtain for this element

$$J_{\mu}^{m,n}=\epsilon_{m,n}\sum_{P} \sum_{l}{}' (C_{m+n}^n)^{-1}\left(-i\nabla_{\mathbf{r}_1}-\frac{e}{c}\mathbf{A}\right)_{P(l-1)}$$

$$\cdots\left(-i\nabla_{\mathbf{r}_1}-\frac{e}{c}\mathbf{A}\right)_{P(1)}G(\mathbf{r}_1-\mathbf{r}_i)\left(-i\nabla_{\mathbf{r}_i}-\frac{e}{c}\mathbf{A}\right)_{P(l+1)}$$

$$\cdots\left(-i\nabla_{\mathbf{r}_i}-\frac{e}{c}\mathbf{A}\right)_{P(m+n)}G(\mathbf{r}_i-\mathbf{r}_2). \tag{B4}$$

Here for each permutation all terms with the projection μ are varied to get the current J_{μ}; correspondingly the sum Σ' is taken over all l with $P[l]=\mu$. In the absence of the vector potential the current operator yields $\partial\epsilon/\partial\mathbf{p}$ as expected. To obtain the diamagnetic term in its extended form [see Eq. (9) above] it is needed to apply Eq. (4) and to pass the terms $[-i\nabla-(e/c)\mathbf{A}]$ through the phase factors using Eqs. (7) and (8). The result is

$$J_{\mu}^{m,n}=\epsilon_{m,n}\sum_{P} \sum_{l}{}' (C_{m+n}^n)^{-1}\exp\{(ie/c)[\Phi(\mathbf{r}_1,\mathbf{r}_i)$$

$$+\Phi(\mathbf{r}_i,\mathbf{r}_2)]\}\left(-i\nabla_{\mathbf{r}_1}-\frac{e}{2c}(\mathbf{r}_1-\mathbf{r}_i)\times\mathbf{B}\right)_{P(l-1)}$$

$$\cdots\left(-i\nabla_{\mathbf{r}_1}-\frac{e}{2c}(\mathbf{r}_1-\mathbf{r}_i)\times\mathbf{B}\right)_{P(1)}\tilde{G}(\mathbf{r}_1-\mathbf{r}_i)$$

$$\times\left(-i\nabla_{\mathbf{r}_i}+\frac{e}{2c}(\mathbf{r}_i-\mathbf{r}_2)\times\mathbf{B}\right)_{P(l+1)}$$

$$\cdots\left(-i\nabla_{\mathbf{r}_i}+\frac{e}{2c}(\mathbf{r}_i-\mathbf{r}_2)\times\mathbf{B}\right)_{P(m+n)}\tilde{G}(\mathbf{r}_i-\mathbf{r}_2). \tag{B5}$$

Here the gradients not acting on the Green's functions are canceled out due to the symmetrization procedure. At this stage the analysis of the main text can be extended up to the Eq. (35), where now

$$\tilde{J}_{\mu}=\frac{\partial\epsilon(p)}{\partial p_{\mu}}\left(1+\frac{\partial\Sigma(p,\epsilon)}{\partial\xi_p}\right). \tag{B6}$$

The result (39) still holds with

$$\omega_c^*=\frac{\epsilon'(p)}{p}\left(\frac{1+\partial\Sigma(p,0)/\partial\xi_p}{1-\partial\Sigma(p_F,\epsilon)/\partial\epsilon}\right)\frac{eB}{c}. \tag{B7}$$

It follows from Eq. (B7) that the renormalization corrections to R_H are canceled out.

APPENDIX C: WEAK LOCALIZATION CORRECTIONS TO THE HALL COEFFICIENT

The weak localization corrections originate from the quantum interference, and diagrammatically are known to be related to a set of the ladder diagrams in the particle-particle channel which is called "the cooperon." As a doubly charged object the cooperon in the magnetic field acquires the form

$$C_B(\mathbf{r}_i,\mathbf{r}_f)=\exp[(i2e/c)\Phi(\mathbf{r}_i,\mathbf{r}_f)]\tilde{C}(\mathbf{r}_i-\mathbf{r}_f), \quad \text{(C1)}$$

where Φ is given by Eq. (5) and $\tilde{C}$ has the same symmetry properties as the corresponding function $\tilde{G}$ in Eq. (4). The latter fact can be seen from the Dyson equation for the cooperon in the magnetic field

$$C_B(\mathbf{r}_i,\mathbf{r}_f)=\frac{\delta(\mathbf{r}_i-\mathbf{r}_f)}{2\pi\tau\nu}+\frac{1}{2\pi\tau\nu}\int d\mathbf{r}'\,\mathcal{G}^R(\mathbf{r}_i,\mathbf{r}')$$
$$\times\mathcal{G}^A(\mathbf{r}_i,\mathbf{r}')C(\mathbf{r}',\mathbf{r}_f). \quad \text{(C2)}$$

Using Eqs. (4) and (C1) one gets from Eq. (C2) an integral equation for $\tilde{C}$,

$$\tilde{C}(\mathbf{r}_i,\mathbf{r}_f)=\frac{\delta(\mathbf{r}_i-\mathbf{r}_f)}{2\pi\tau\nu}+\frac{1}{2\pi\tau\nu}\int d\mathbf{r}'\,\tilde{\mathcal{G}}^R(\mathbf{r}_i-\mathbf{r}')\tilde{\mathcal{G}}^A(\mathbf{r}_i-\mathbf{r}')$$
$$\times\exp[(2ie/c)\Phi(\mathbf{r}_i,\mathbf{r}',\mathbf{r}_f)]\tilde{C}(\mathbf{r}',\mathbf{r}_f), \quad \text{(C3)}$$

where $\Phi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ is the magnetic flux through the triangle with vertexes $(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$:

$$\Phi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\frac{1}{2}\mathbf{B}\cdot[(\mathbf{r}_1-\mathbf{r}_3)\times(\mathbf{r}_2-\mathbf{r}_1)]. \quad \text{(C4)}$$

Equation (C3) contains invariant ingredients only, and this ensures that $\tilde{C}$ has the same symmetry properties as $\tilde{G}$. One can readily derive the following equation for $\tilde{C}$:

$$\left[D\left(-i\nabla-\frac{2e\mathbf{B}}{2c}\times\mathbf{r}\right)^2+|\omega_n|\right]\tilde{C}(\mathbf{r},\omega_n)=\frac{\delta^d(\mathbf{r})}{2\pi\tau^2\nu}. \quad \text{(C5)}$$

As it has been already explained in the main text for σ_{xy} linear in B it is enough to keep $\tilde{G}$ and $\tilde{C}$ in the limit $B=0$. At the vanishing magnetic field $\tilde{C}$ turns into the singular propagator

$$C(\mathbf{Q},i\omega_n)=\frac{1}{2\pi\tau^2\nu}\frac{1}{|\omega_n|+DQ^2}. \quad \text{(C6)}$$

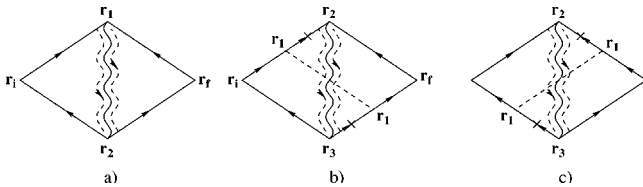In this Appendix we will assume that the scattering potential is short ranged and that the electrons have an arbitrary

FIG. 15. Weak localization corrections to σ_{xy}. The wavy line denotes the cooperon, the dashed wavy lines carry phases, the dashed line denotes the scattering by impurities.

but spherically symmetric spectrum $\epsilon(p)$. Consider first the diagram presented in Fig. 15(a). With the use of Equation (9) the diamagnetic contribution from the J_x vertex can be written as

$$\Pi_{xy}^x(\mathbf{q}=0,i\omega_n)$$

$$=i\frac{e^3B}{4c}T\sum_{i\epsilon_n}\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dQ}{(2\pi)^d}\varepsilon_{\mu\nu}\frac{\partial^2\epsilon}{\partial p_x\partial p_\mu}$$

$$\times\left(\mathcal{G}^A(\mathbf{p})\frac{\partial}{\partial p_\nu}\mathcal{G}^R(\mathbf{p})-\mathcal{G}^R(\mathbf{p})\frac{\partial}{\partial p_\nu}\mathcal{G}^A(\mathbf{p})\right)$$

$$\times C(\mathbf{Q})\mathcal{G}^R(-\mathbf{p}+\mathbf{Q})\mathcal{G}^A(-\mathbf{p}+\mathbf{Q})v_y(-p+Q),$$

$$\text{(C7)}$$

where $v_\alpha(p)=\partial\epsilon/\partial p_\alpha$ is the velocity and $\partial^2\epsilon/\partial p_\alpha\partial p_\beta$ is the inverse mass tensor. Together with the J_y vertex contribution a symmetric combination

$$[(\partial^2\epsilon/\partial p_x^2)(\partial\epsilon/\partial p_y)^2-2(\partial^2\epsilon/\partial p_x\partial p_y)(\partial\epsilon/\partial p_x)(\partial\epsilon/\partial p_y)$$
$$+(\partial^2\epsilon/\partial p_y^2)(\partial\epsilon/\partial p_x)^2]$$

arises. Exactly the same combination appears also in the calculation of σ_{xy} in the Drude's approximation. In the spherically symmetric case this combination reduces to $(2/d)v_F^2m^{-1}$, where $m^{-1}=v_F/p_F$. [The fact that the discussed combination contains the first derivative of $\epsilon(p)$ only was important for the derivation of $R_H=1/nec$ for the arbitrary spectrum $\epsilon(|\mathbf{p}|)$, see Eq. (B7).]

To evaluate the flux term, we split the phase of the cooperon into two equal parts indicated by the dashed lines in Fig. 15(a). Then the flux of this diagram becomes a sum of the fluxes through the two triangles:

$$\Phi_a=\Phi(\mathbf{r}_i,\mathbf{r}_1,\mathbf{r}_2)+\Phi(\mathbf{r}_f,\mathbf{r}_1,\mathbf{r}_2). \quad \text{(C8)}$$

When the Fourier transformation is performed the coordinate differences in Φ_a lead to the differentiation with respect to momentum. To get a nonvanishing contribution to σ_{xy} one of the differentiation should be applied to a current vertex. It turns out that the diamagnetic and flux terms are identical. Together they yield

$$\delta\sigma_{xy}^a=3\omega_c\tau\delta\sigma_{xx}, \quad \text{(C9)}$$

where $\omega_c=(eB/c)m^{-1}$, and $\delta\sigma_{xx}$ is the weak localization correction to the longitudinal conductivity.

For the terms presented in Figs. 15(b) and (c) the diamagnetic contributions are absent. The flux term of the diagram 15(b) may be represented most economically as

$$\Phi_b=\Phi(\mathbf{r}_i,\mathbf{r}_1,\mathbf{r}_3)+\Phi(\mathbf{r}_f,\mathbf{r}_2,\mathbf{r}_1)-2\Phi(\mathbf{r}_1,\mathbf{r}_3,\mathbf{r}_2). \quad \text{(C10)}$$

The advantage of this representation is that the first two fluxes are similar to the diamagnetic terms and do not contribute to σ_{xy}. The remaining flux term leads to

$$\Pi_{xy}^b(\mathbf{q}=0,i\omega_n)=-i\frac{e^3B}{c}\left(\frac{1}{2\pi\nu\tau}\right)T\sum_{i\epsilon_n}\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dp'}{(2\pi)^d}\int\frac{d^dQ}{(2\pi)^d}v_x(p)\mathcal{G}^A(p)\mathcal{G}^R(p)\frac{\partial\mathcal{G}^A(|-\mathbf{p}+\mathbf{Q}|)}{\partial(-p+Q)_x}$$

$$\times C(Q)\mathcal{G}^A(|-\mathbf{p}'+\mathbf{Q}|)\mathcal{G}^R(|-\mathbf{p}'+\mathbf{Q}|)\frac{\partial\mathcal{G}^A(p')}{\partial p'_y}v_y(-\mathbf{p}'+\mathbf{Q}). \tag{C11}$$

The differentiated Green's functions are indicated by crosses in Figs. 15(b) and (c). Similarly, the flux term of the diagram 15(c) is

$$\Pi_{xy}^c(\mathbf{q}=0,i\omega_n)=i\frac{e^3B}{c}\left(\frac{1}{2\pi\nu\tau}\right)T\sum_{i\epsilon_n}\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dp'}{(2\pi)^d}\int\frac{d^dQ}{(2\pi)^d}v_x(p)\mathcal{G}^A(p)\mathcal{G}^R(p)\frac{\partial\mathcal{G}^R(|-\mathbf{p}+\mathbf{Q}|)}{\partial(-p+Q)_x}$$

$$\times C(Q)\mathcal{G}^A(|-\mathbf{p}'+\mathbf{Q}|)\mathcal{G}^R(|-\mathbf{p}'+\mathbf{Q}|)\frac{\partial\mathcal{G}^R(p')}{\partial p'_y}v_y(-\mathbf{p}'+\mathbf{Q}). \tag{C12}$$

To proceed further, we rearrange the sum $\tilde{\Pi}_{xy}=\Pi_{xy}^b+\Pi_{xy}^c$ to the following form:

$$\tilde{\Pi}_{xy}(\mathbf{q}=0,i\omega_n)$$

$$=i\frac{e^3B}{c}\left(\frac{1}{2\pi\nu\tau}\right)T\sum_{i\epsilon_n}\int\frac{dQ}{(2\pi)^d}C(Q)$$

$$\times\int\frac{d^d\tilde{p}}{(2\pi)^d}\frac{1}{d}v^2[\mathcal{G}_R^3(\tilde{p})\mathcal{G}_A(\tilde{p})+\mathcal{G}_A^3(\tilde{p})\mathcal{G}_R(\tilde{p})]$$

$$\times\int\frac{d^dp}{(2\pi)^d}\frac{1}{d}v^2[\mathcal{G}_R^3(p)\mathcal{G}_A(p)-\mathcal{G}_A^3(p)\mathcal{G}_R(p)]. \tag{C13}$$

One should be cautious with the last integral as $\tau(\mathcal{G}_R^3\mathcal{G}_A-\mathcal{G}_A^3\mathcal{G}_R)=2i\xi(\mathcal{G}_R\mathcal{G}_A)^3$ is an odd function in ξ. We have therefore to perform an expansion in ξ to get a nonvanishing result. This leads us to an integral

$$\frac{2i}{d}\int d\xi\frac{d(\nu v^2\tau^{-1})}{d\xi}\xi^2(\mathcal{G}_R\mathcal{G}_A)^3$$

$$=\frac{4i}{d}\int d\xi\left[(\nu\tau^{-1})\frac{\partial^2\epsilon(p)}{\partial p^2}+v^2\frac{d(\nu\tau^{-1})}{d\xi}\right]\xi^2(\mathcal{G}_R\mathcal{G}_A)^3. \tag{C14}$$

Unlike the expressions discussed above, the integrand in Eq. (C14) depends on the specific form of the spectrum, i.e., it is not universal. For $\epsilon(p)=p^2/2m$ and $d=2$ the diagrams of Figs. 15(b) and (c) yield $\delta\sigma_{xy}^{(b+c)}=-\omega_c\tau\delta\sigma_{xx}$. Then the weak localization correction to the Hall coefficient,

$$\frac{\delta R_H}{R_H}=\frac{\delta\sigma_{xy}}{\sigma_{xy}}-2\frac{\delta\sigma_{xx}}{\sigma_{xx}}, \tag{C15}$$

vanishes as it was first shown by H. Fukuyama.[9]

The possibility of an economic representation of the flux is the essential advantage of the method. The expressions (C7), (C11), and (C12) have been obtained without any intermediate steps here.

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