Adiabatic addition of the Chern-Simons flux, pair correlations, and particle statistics in two-dimensional electron systems

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Adiabatic introduction of Chern-Simons flux into a two-dimensional electron gas leads automatically to Laughlin correlations. In contrast to the mean-field composite fermion picture, only a single energy scale, the Coulomb scale, appears at the large values of the applied magnetic field. The Laughlin correlations partition the spectrum into bands separated by gaps that depend on pseudopotential coefficients $V(L_{12})$ of the Coulomb interaction. The adiabatic approach also suggests a simple intuitive picture for why the Jain wave functions are excellent trial functions for condensed states belonging to the Jain sequence.

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I. INTRODUCTION

The mean-field Chern–Simons (CS) picture[1] offers a simple intuitive way of understanding fractional quantum Hall systems.[2] Because it has been used with remarkable qualitative success in the interpretation of experimental data,[3] it has been accepted, almost without question. Many-body perturbation expansions about the CS mean field have been developed and applied with some success to both incompressible quantum liquid states[4] and to the compressible composite Fermion (CF) (a CF is an electron together with a CS flux tube) state which occurs at filling factor $\nu=1/2$.[5] Despite its qualitative success in describing the structure of the low-lying energy spectrum,[6] there are several perplexing questions related to why this simple mean-field CS picture works. The mean-field approximation introduces a second energy scale proportional to the cyclotron energy $\hbar\omega_c$ in addition to the Coulomb energy scale e^2/λ, where λ is the magnetic length. When $\hbar\omega_c \gg e^2/\lambda$, this second energy scale is large but totally irrelevant to the determination of the low-energy spectrum. In addition, there is no small parameter in the perturbation expansion to justify the use of simple many-body approximations like RPA, or even to guarantee convergence. Jain avoided the ''two energy scale problem'' in a simple way.[1,7] He constructed an antisymmetric product function for electrons at the mean-field CF filling factor ν^* equal to an integer, and multiplied it by the Jastrow factor $\mathcal{J}=\Pi_{i<j}z_{ij}^{2p}$, first introduced by Laughlin to obtain the $\nu=(2p+1)^{-1}$ incompressible states. He then took the projection of this wave function onto the lowest electron Landau level at filling factor $\nu=\nu^*(1+2p\nu^*)^{-1}$, and used this as a trial wave function. In this paper we address the question of why the mean-field CS picture is qualitatively so successful despite its apparent shortcomings, and why the Jain wave functions make such satisfactory trial wave functions. We do this by introducing the CS flux adiabatically instead of via a singular gauge transformation as done in most previous work.[4,5,8] Our emphasis is not on obtaining novel results, but on obtaining an intuitive physical picture of why these approaches are successful. The following results are found: (i) Laughlin pair correlations result automatically, but in the absence of Coulomb interactions, the energies are unchanged. This means that no new irrelevant energy scale is introduced. (ii) In contrast to the result of the gauge transformation, no change in particle statistic s occurs for any value of the CS flux. (iii) If one starts with an integral electron filling factor $\nu^*=n$ and adiabatically increases the dc magnetic field by a factor of $2n+1$, while at the same time adiabatically adding two CS flux quanta in the direction opposite to the dc field to each electron, Laughlin pair correlations are automatically introduced. The wave function for the pair i,j must contain a factor z_{ij}^2 to conserve the total flux through the pair orbit (i.e., the flux due to the dc field plus the CS flux). The resulting Jain-type wave function is almost totally contained within the subspace of the lowest Landau level at the higher magnetic field [i.e., $(2n+1)$ times the original field at filling factor $\nu^*=n$].

Laughlin correlations mean that electrons avoid, as much as possible, pair states with the largest pair angular momentum (or smallest pair radius) thereby lowering their repulsive Coulomb energy. For N electrons confined to a spherical surface, the total angular momentum multiplets resulting from Laughlin correlated electrons, each with angular momentum l, can be obtained by addition of the angular momenta l^* of N composite Fermions,[6,9] where $l^*=l-(N-1)$. This set of total angular momentum multiplets is a subset of those formed from N electrons each with angular momentum l. For all incompressible states of the Jain sequence, only a single nondegenerate $L=0$ state (i.e., an integral CF filling ν^*) results. Empty states in a nearly filled CF angular momentum shell are Laughlin quasiholes, and filled states in a nearly empty first excited CF shell are quasielectrons. The quasiparticle (QP) angular momenta are $l_{\mathrm{QH}}=l^*$ and $l_{\mathrm{QE}}=l^*+1$. The Jain sequence of condensed states at $\nu^*=\pm1,\pm2,\ldots$ occurs at electron filling $\nu=\nu^*(1+2p\nu^*)^{-1}$, since $\nu^{*-1}=\nu^{-1}-2p$. They correspond to filled CF shells.

Sitko, Yi, and Quinn[10] proposed a CF hierarchy scheme to treat partially filled QP shells in which the mean-field CS transformation was reapplied to the QP's but not to the underlying filled shells. This CF hierarchy yielded condensed states at all odd denominator fractions, but it assumed that the CF QP's formed Laughlin correlated daughter states. Numerical calculations demonstrated that not all predicted fractions had incompressible $L=0$ ground states. The reason why rested on the behavior of the QP-QP pseudopotential, $V_{\mathrm{QP\text{-}QP}}(L_{12})$,[9] the interaction energy of an electron pair as a function of the total pair angular momentum. Only if $V_{\mathrm{QP\text{-}QP}}$ increased with L_{12} faster than $L_{12}(L_{12}+1)$ could Laughlin correlations among the QP's be applied. Quinn *et al.*

demonstrated[11] that the only Laughlin correlated spin polarized daughter states of the $\nu=1/3$ and $\nu=1/5$ Laughlin states occurred at $\nu_{QE}=1/5$ and at $\nu_{QH}=1/3$ and 1/7. This implied that the $\nu=4/11$ (QE's of the $\nu=1/3$ state at $\nu_{QE}=1/3$) and $\nu=4/13$ (QH's of the $\nu=1/3$ state at $\nu_{QH}=1/5$) states could not be spin polarized Laughlin correlated QP states. Non-spin polarized ground states, formation of QP-QP pairs, or other non-Laughlin-like correlations must be responsible for the minima in ρ_{xx} observed recently[12] at these and other unusual fractions.

II. CHERN-SIMONS GAUGE FIELD

The CF picture results from the introduction of a Chern-Simons gauge field produced by attaching fictitious magnetic flux tubes to each electron. The vector potential $\vec{a}(\vec{r})$ produced by a flux $\Phi=\alpha\phi_0=\alpha(hc/e)$ is given by

$$\vec{a}(\vec{r})=\Phi\int d^2r'\frac{\hat{z}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2}\rho(\vec{r}'), \tag{1}$$

where $\hat{z}$ is a unit vector normal to the two-dimensional (2D) layer, and $\rho(\vec{r})=\Psi^+(\vec{r})\Psi(\vec{r})$ is the quantum-mechanical density operator. This vector potential couples to the electron charge in the usual way. Although it introduces a phase factor into the quantum-mechanical wave function, $\vec{a}(\vec{r})$ has no effect on the classical equations of motion. This results from the fact that no two electrons can occupy the same position and that the magnetic field resulting from the CS flux is given by $\vec{b}(\vec{r})=\Phi\Sigma_i\delta(\vec{r}-\vec{r}_i)\hat{z}$, where $\vec{r}_i$ is the position of the ith electron. Of course, we assume that no electron interacts with its own flux tube.

The physics of introducing CS flux can be most simply understood for a two-electron system. The Hamiltonian for a pair of electrons in the presence of a dc magnetic field $\vec{B}=B\hat{z}$ separates into center of mass (CM) and relative (R) coordinate contributions. For the latter, the Hamiltonian in the symmetric gauge can be written

$$H_R=\frac{p^2}{2\mu}+\frac{qB}{2\mu c}l_z+\frac{q^2B^2}{8\mu c^2}r^2. \tag{2}$$

Here $\vec{r}=\vec{r}_1-\vec{r}_2$ and $\vec{p}=\frac{1}{2}(\vec{p}_1-\vec{p}_2)$ are the relative coordinate and momentum, respectively. The reduced mass and charge are $\mu=\frac{1}{2}m_e$ and $q=\frac{1}{2}e$, and $l_z=(\vec{r}\times\vec{p})_z$ is the z component of angular momentum. The solution of the Schrödinger equation can be written $\Psi_{nm}(\vec{r})=e^{im\phi}u_{nm}(\vec{r})$. The radial function satisfies the differential equation[13]

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}\right)+\frac{\hbar qBm}{2\mu c}+\frac{q^2B^2}{8\mu c^2}r^2\right]u_{nm}=E_{nm}u_{nm}. \tag{3}$$

The eigenvalues $E_{nm}=\frac{1}{2}\hbar\omega_c(2n+1+m+|m|)$, where m is any integer and $n=0,1,2,\ldots$. In most of the remainder of this paper we will consider only the lowest Landau level with $n=0$ and $m=0,-1,-2,\ldots$. The subscript n will simply be omitted in that case; then $E_m=\frac{1}{2}\hbar\omega_c$ and $u_m(r)=(r/2\lambda)^{|m|}\exp(-r^2/8\lambda^2)$, where λ is the magnetic length for a particle of charge e. Replacing the relative coordinate $\vec{r}$ by $-\vec{r}$ (or ϕ by $\phi+\pi$) corresponds to exchange of the pair of particles. Therefore for fermions (bosons) m must be an odd (even) integer. The function $|u_m(r)|^2$ has its maximum value at $r_m=2|m|^{1/2}\lambda$, the "semiclassical orbit" radius. This implies that the mth orbit encloses $2m$ flux quanta, i.e., $\pi r_m^2B=2|m|\phi_0$.

III. CHERN-SIMONS GAUGE TRANSFORMATION

Ever since the work of Aharanov and Bohm,[14] it has become customary to think of quantum-mechanical problems involving a vector potential $\vec{a}(\vec{r})$ within a region in which $\vec{b}(\vec{r})=\vec{\nabla}\times\vec{a}(\vec{r})$ vanishes in terms of a gauge transformation which alters the phase but not the amplitude of the wave function. A simple example[15] is that a charged particle confined to move on a circular path in the x-y plane. At the center of the path is a long thin solenoid oriented in the z direction. When the solenoid carries a current which produces a flux $\Phi=\alpha\phi_0$, the eigenfunction $\bar{\Psi}_m(\vec{r})$ can be written as follows:

$$\bar{\Psi}_m(\vec{r})=e^{-(ie/\hbar c)\int\vec{a}(\vec{r})\cdot d\vec{r}}\Psi_m(\vec{r}), \tag{4}$$

where $\Psi_m(\vec{r})=e^{im\phi}u_m(r)$ is the eigenfunction when $\alpha=0$, and in this case $|u_m(r)|^2=\delta(r-r_0)$ where r_0 is the radius of the circular path. Because $\vec{a}(\vec{r})=(\Phi/2\pi r)\hat{\phi}$, the phase factor in Eq. (4) is equal to $-i\alpha\phi$, where ϕ is the angular position of the particle along its path. This means $\bar{\Psi}_m(\vec{r})=e^{i(m-\alpha)\phi}u_m(r)$. For the relative coordinate of a pair (with $q=e/2$ and $\int\vec{a}\cdot d\vec{r}=2|\alpha|\frac{\phi_0}{2\pi}\phi$) the corresponding equation illustrates famous transmutation of statistics in two-dimensional (2D) systems.[8,16] Odd integral values of α change fermions to bosons and vice versa, while nonintegral values of α give rise to anyon statistics.

Because the many body Hamiltonian resulting from the gauge transformation contains the CS vector potential $\vec{a}(\vec{r})$ given by Eq. (1) added to the vector potential $\vec{A}(\vec{r})$ of the dc magnetic field, it is considerably more complicated. Simplification results from replacing the density operator appearing in $\vec{a}(\vec{r})$ by its "mean-field" value n_s, the equilibrium electron concentration which is independent of position. The resulting Hamiltonian is the sum of single-particle Hamiltonians containing an effective magnetic field $B^*=B+\alpha\phi_0n_s$. In fractional quantum Hall systems this mean-field CS picture leads to integral CF filling $|\nu^*|=1,2,\ldots$ at electron filling factor ν given by $\nu=\nu^*/(1+|\alpha|\nu^*)$. Here α is the CS flux per electron, and negative values of ν^* correspond to states with B^* in the opposite direction from B. For $|\alpha|=2$ and values of $l^*=l-(N-1)$ close to $\frac{1}{2}(N-1)$, the quasihole (quasielectron) angular momentum is $l_{QH}=l^*$ ($l_{QE}=l^*+1$). The lowest band of energy states for any applied magnetic field is obtained by addition of the angular momenta of the quasiparticles required for the given value of l and N. It is worth noting that for a Laughlin state (e.g., $\nu=1/3$) $B^*=\nu B$ and the CF magnetic length is $\lambda^*=\nu^{-1/2}\lambda$. The semiclassical CF orbit has a radius

$r_m^* = \nu^{-1/2} r_m$, and since $\pi r_m^{*2} B^* = \pi r_m^2 B$, it encloses the same total flux as the original electron orbit. However, in the eigenvalue E_{nm}, the electron cyclotron frequency ω_c must be replaced by $\omega_c^* = \nu \omega_c$, and this gives rise to the irrelevant energy scale that cannot affect the low-energy states when $\hbar \omega_c \gg e^2/\lambda$.

IV. ADIABATIC ADDITION OF CHERN-SIMONS FLUX

Instead of making a gauge transformation we can start with an initial electron pair state and slowly increase the value of $\alpha\phi_0$ from zero to a final value where α is an even integer. The single-particle Hamiltonian is

$$H_i = \frac{1}{2m}\left(\vec{p}_i + \frac{e}{c}[\vec{A}(\vec{r}_i) + \vec{a}(\vec{r}_i)]\right)^2, \tag{5}$$

and the pair Hamiltonian is $H_1 + H_2$. In Eq. (5) $\vec{a}(\vec{r}_1) = (\alpha\phi_0/2\pi r)\hat{\phi}$, where $\vec{r} = |\vec{r}_1 - \vec{r}_2|$, and $\hat{\phi}$ is a unit vector in the direction of increasing angular coordinate ϕ. It is apparent that $\vec{a}(\vec{r}_1) = -\vec{a}(\vec{r}_2)$. When the two-particle Hamiltonian is separated into CM and relative coordinate contributions, the resulting H_R is identical to that in Eq. (2), except that B is replaced by $\tilde{B} = B + 2\alpha\phi_0/\pi r^2$. If we assume that the eigenfunction for the lowest Landau level is of the form $\tilde{\psi}_m = e^{im\phi} w_m(r)$, then the new radial function $w_m(r)$ satisfies Eq. (3) if $\tilde{B}$ replaces B. It is not difficult to regroup the terms in the resulting equation to demonstrate that $w_m(r) = u_{m+\alpha}(r)$. Thus the exact pair wave function is $\tilde{\Psi}_m = e^{im\phi} u_{m+\alpha}(r)$. In contrast, $\overline{\Psi}_m = e^{i(m-\alpha)\phi} u_m(r)$, obtained via the gauge transformation, does not change the radial function at all, but alters the phase of the angular part of the wave function. In the adiabatic approach, the pair wave function with angular quantum number m remains in state m as $|\alpha|$ is slowly increased from zero to a final value of $|\alpha| = 2$. This means there is no change in particle statistics as α increases continuously with time. The radial function (particle "orbit") does change, starting from $u_m(r)$ and ending with $u_{m+\alpha}(r)$. The reason for this expansion of the orbital wave function is that the time rate of change of the CF flux gives rise, through Faraday's law, to an induced electric field $\vec{\varepsilon}$ along the orbit. The response of an electron to crossed $\vec{\varepsilon}$ and $\vec{B}$ fields is to move in the direction $\vec{\varepsilon} \times \vec{B}$, expanding or contracting the orbit to conserve the flux enclosed. Starting from the $m = -1$ electron pair state, which encloses three flux quantum of the field $\vec{B}$, and adiabatically adding $|\alpha| = 2$ CS flux quanta oriented opposite to $\vec{B}$ increases the orbit size from $r_1 = 2\lambda$ to $\sqrt{3} r_1$ since $|m + \alpha| = 3$. This orbit encloses one flux quantum ($+3$ from the B field and -2 from the CS flux tube). The orbit is exactly the same size as the mean-field CF orbit in the field B^*, but the energy is unchanged. Only when Coulomb interactions are included do the pair states with larger pair orbits have lower energy. Therefore no new and irrelevant energy scale is introduced by the adiabatic addition of the CS flux.

If the pair wave function for the relative motion in the orbital m is multiplied by the pair function for the CM in the orbital $m = 0$, one obtains

$$\overline{\Psi}_m(1,2) = z_{12}^{|m|} \exp\left[-\frac{r_1^2 + r_2^2}{4\lambda^2}\right]. \tag{6}$$

Here $z_{12} = re^{-i\phi}$ and $\lambda^2 = \hbar c/eB$ even though the relative and CM coordinates have different charges and different magnetic lengths. This is just the Laughlin wave function for a pair since m must be an odd integer. It can also be obtained by constructing the antisymmetric product of single-particle wave functions in the two smallest orbitals (i.e., $z_{12}\exp[-(r_1^2 + r_2^2)/4\lambda^2]$) and multiplying by the Jastrow factor $(z_{12})^{m-1}$. Here, of course, $m-1$ must be an even integer.

V. TRIAL FUNCTIONS

It is obvious that the behavior of a single-electron pair can be generalized to obtain the Laughlin wave function for the $\nu = m^{-1}$ state of an N electron system. You simply start with the N electron system at the filling factor $\nu = 1$ and adiabatically increase the dc magnetic field by a factor of 3 while at the same time adding adiabatically two CS flux quanta per electron in the direction opposite to the dc field. The $\nu = 1$ wave function

$$\overline{\Psi}_1 = \prod_{i<j} z_{ij} \exp\left(-\sum_i \frac{r_i^2}{4\lambda_1^2}\right), \tag{7}$$

which is obtained from the antisymmetrized product function $\mathcal{A}\Pi_{i=1}^N \psi_{m=i-1}(z_i)$, is simply multiplied by the Jastrow factor z_{ij}^2 for each pair, exactly as Laughlin proposed, while the magnetic length λ_ν changes from λ_1 to $\lambda_{1/3}$. Of course, in this case $\psi_m(z) = z^m \exp(-r^2/4\lambda_1^2)$, where $z = re^{-i\phi}$, and m takes on the values $0,1,\ldots,N-1$ appropriate for N particles filling the N states of the lowest Landau level.

The same idea can be applied to states of the Jain sequence. As an example, consider the $\nu = 2/5$ state. Start with the N electron system at filling factor $\nu = 2$. The wave function describing this state is simply the antisymmetric product of the single-particle functions $\psi_{nm}(\vec{r}) = e^{im\phi}\rho_{nm}(r)$, for E_{nm} in the lowest and first excited Landau levels. The radial function ρ_{nm} are given by $\rho_{nm}(r) = r^{|m|} L_n^{|m|}(r^2/2\lambda^2) e^{-r^2/4\lambda^2}$, and the associated Laguerre polynomials appearing in the product function are $L_0^{|m|}(x^2) = |m|!$ and $L_1^{|m|}(x^2) = (|m|+1)!(|m|+1-x^2)$. The main effect of adiabatically adding the CS flux is to introduce Laughlin correlations. This can be accounted for by introducing a Jastrow factor z_{ij}^2 for every pair as suggested by Jain.[1,7] The quantum numbers of the single-particle wave functions appearing in the antisymmetrized product function are unchanged, but the magnetic length appearing in these functions changes from λ_2 (appropriate at the filling factor $\nu = 2$) to $\lambda_{2/5}$ (appropriate ar $\nu = 2/5$) as the dc magnetic field is changed adiabatically from B_2 to $B_{2/5} = 5B_2$. It is worth noting that the maximum value of the exponent of r_i appearing in the transformed wave function is $2(N-1) + (N_\phi - 1) + 2$. The three terms arise from the Jastrow factor, the maximum value of $|m|$ for filling factor $\nu = 2$, and from the associated Laguerre polynomial $L_1^{|m|}(r^2/2\lambda^2)$, respectively. We require this maximum value to be equal to $\tilde{N}_\phi - 1$, where

$\tilde{N}_\phi = 5N_\phi$ is the degeneracy of the lowest Landau level at the magnetic field for $\nu = 2/5$. Equating $2N + N_\phi$ to $\tilde{N}_\phi = 5N_\phi$ gives $N = \frac{2}{5}\tilde{N}_\phi$ or $\nu = N/\tilde{N}_\phi = 2/5$. This means that the new wave function corresponds to a filling factor $\nu = 2/5$. As Jain has demonstrated, these wave functions lie mostly within the Hilbert space of the lowest Landau level at the magnetic field $B_{2/5}$. The most accurate evaluation of the energy spectrum for the Jain states requires the projection of the trial wave functions onto the Hilbert space of the lowest Landau level at the higher magnetic field. Jain and Kamilla[7] developed an efficient technique for obtaining the projected trial functions, and demonstrated their excellent overlap with the exact eigenfunctions for finite-size systems.

VI. SUMMARY AND CONCLUSIONS

We have demonstrated that the adiabatic addition of an even number of CS flux quanta per electron results in Laughlin correlations. This selects from the set of total angular momentum multiplets, a subset that has smaller Coulomb repulsion and therefore lower energy. In the absence of Coulomb interactions, the energies of the selected subset are unchanged by the presence of the CS flux, so no new and irrelevant energy scale is introduced. For incompressible quantum Hall states only a single $L = 0$ state belongs to the subset, and it corresponds an integral number $|\nu^*|$ of filled CF angular momentum shells.

This picture gives an intuitive feeling for why the CF model describes states with filling close to that of the incompressible liquid states. It also gives some insight into why the Jain trial functions are so good. They can be obtained without any mean-field approximation by adiabatically increasing the dc magnetic field and at the same time adiabatically adding an opposing CS flux. This leaves the integrally filled N particle wave function unchanged except for two effects, first the magnetic length is changed from that of the original magnetic field to the smaller magnetic field associated with the fractional filling; second the Jastrow factor z_{ij}^{2p} (where $2p$ CS flux quanta are added) multiplies the wave function obtained from the antisymmetrized product function after the change in magnetic length and replacement of $|m|$ in $L_1^{|m|}$ by $|m| + 2p$.

One might well ask about the $\nu = 1/2$ state where the RPA has been applied to interactions beyond the mean field. There is no small parameter to justify using the RPA. As pointed out in Ref. 5 and studied in detail by Simon and Halperin,[17] Landau Fermi-liquid corrections can be inserted by hand. We prefer thinking of Ref. 5 in terms of Silin's[18] generalization of the Landau theory to an interacting electron liquid. Long-range Coulomb interactions are accounted for through the self-consistent Hartree field (equivalent to the RPA), and short-range correlation effects through the Landau interaction $f(\vec{p},\vec{p}')$ among quasiparticles. The Landau-Silin kinetic equation contains an effective mass m^*, renormalized by the Coulomb interactions, and a cyclotron frequency $\omega_c = eB/m^*c$. In the Landau theory m^* is equal to $(1 + A_1)m$, where m is the bare band mass and A_1 is the $n = 1$ coefficient of the expansion of the spin symmetric interaction function in Fourier series in the angle between $\vec{p}$ and $\vec{p}'$.[19] The expansion coefficients A_n depend on magnetic field and filling factor, and they are very difficult to evaluate from first principles. However, the spin-independent response functions[17,19] contain denominators of the form $[\omega - n(1 + A_n)\omega_c]$. This causes the fundamental $n = 1$ cyclotron resonance to occur at the bare cyclotron frequency eB/mc as required by Kohn's theorem. Though the RPA by itself is not valid for the CS gauge interactions, the Landau-Silin kinetic equation with phenomenological interaction coefficients A_n should be a reasonable way to evaluate the response of a strongly interacting 2D composite Fermion liquid in the presence of a magnetic field.

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