Hysteretic *I*-*V* curves of superconducting nanowires

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Experimental *I*-*V* curves of superconducting MoGe nanowires show hysteresis for the thicker wires and none for the thinner wires. A rather quantitative account of these data for representative wires is obtained by numerically solving the one-dimensional heat flow equation to find a self-consistent distribution of temperature and local resistivity along the wire, using the measured *linear* resistance $R(T)$ as input. This suggests that the retrapping current in the hysteretic *I*-*V* curves is primarily determined by heating effects, and not by the dynamics of phase motion in a tilted washboard potential as often assumed. Heating effects and thermal fluctuations from the low-resistance state to a high-resistance, quasinormal regime appear to set independent upper bounds for the switching current.

I. INTRODUCTION

The *I*-*V* curves of superconducting MoGe nanowires of the order of a few hundred nanometers long, with diameters greater than ~ 10 nm and critical currents exceeding a fraction of a microampere, are strongly hysteretic, while thinner wires show relatively small nonlinearities and no hysteresis. An improved understanding of this hysteretic behavior is the primary objective of this paper.

Such a hysteresis is also seen in *macro*scopic superconducting filaments, where it has been explained in terms of self-heating hotspots, which can be used as the basis for hot-electron bolometers.[1] Hysteresis is also found in underdamped Josephson junctions, where it stems from the runaway and retrapping of the phase point in the tilted washboard potential of the Josephson junction (without regard to heating).[2] A superconducting nanowire forms a sort of hybrid case, in which phase-slip centers act qualitatively like weaklink Josephson junctions in series with the rest of the wire, but in which heating effects are especially important because of the difficulty in removing Joule heat through a long thin free-standing filament. That is, the quantum phase-slip process governs the local resistance of an element of the wire at a given current I and local temperature T, but classical heat flow determines the local temperature distribution generated by this heating by phase-slip processes. Thus, we must find a self-consistent solution for the generation and removal of heat.

The extent of hysteresis whether in a Josephson junction or nanowire is characterized by the difference between the *switching current* I_{sw}, and the *retrapping current* I_r. The switching current is the current level at which, with increasing bias current, the junction switches from a very low voltage state (with resistance due only to thermally activated or quantum tunneling phase slippage) to a more or less Ohmic resistive regime at higher current. The retrapping current is that at which, on decreasing current, the device switches back down to the zero (or low) voltage regime. The retrapping current is quite accurately reproduced from sweep to sweep, since it reflects the minimum current level at which the Ohmic resistive state is stable. The switching current, on the other hand, can be somewhat different on each successive

sweep, reflecting the stochastic nature of a fluctuation-induced switching process, but it has a reasonably well-defined average value. The switching current is sometimes referred to as the critical current, but we reserve the notation I_{co} for the maximum supercurrent allowed by the Ginzburg-Landau equations, without regard to stability. Accordingly, $I_{sw} < I_{co}$.

II. A SIMPLE MODEL FOR HEATING-CONTROLLED RETRAPPING CURRENT

Employing only classical heat-flow analysis together with the *linear* electrical resistance $R(T)$ measured with infinitesimal current in the resistive transition of a nanowire, one can construct a model for the heating-determined hysteresis of the wire at high current levels. Note that the use of the *measured* $R(T)$ in this way eliminates the need for any microscopic model of the phase-slip or other processes producing the heat; we simply assume that the local resistivity is primarily a function of the local T, independent of whether T is controlled externally or by internal heating. This assumption can be justified to some extent by the fact that T ranges from $\sim T_c/3$ all the way up to above T_c, while the barrier to phase-slip only varies by at most a factor of ~ 2 for the range of currents below I_{sw}, above which the system is nearly normal. Thus, the T dependence should be stronger than the explicit I dependence. In general, this model can be worked through only by computer modeling, and that will be demonstrated in Sec. III. However, an approximate analytic solution can be found for the limiting case of wires thick enough to have an abrupt transition of the local resistivity from the full normal value to zero, as in a macroscopic wire, when the local T falls through T_c. We give this solution here to serve as an introduction to the general numerical solution, which is required for wires thin enough to have a broadened resistive transition.

Specifically, we consider a one-dimensional wire sample that is superconducting near the ends, where it is connected to presumably well-cooled electrodes at bath temperature $T_b < T_c$, but is normal in a self-heated central portion further from the electrodes. We take the superconducting region to have *zero resistance*, even though our data show that to be
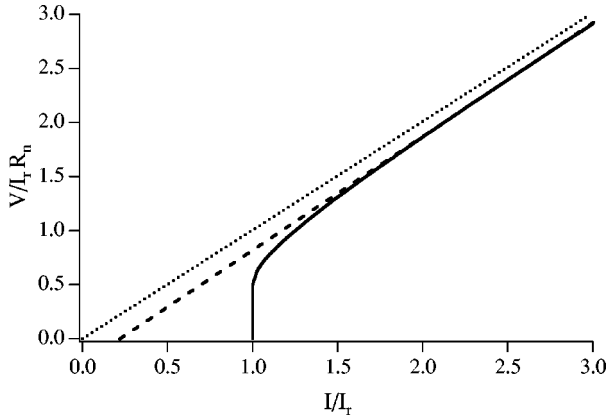
FIG. 1. Plot of the I-V curve (solid curve) predicted by Eq. (4), based on a simplified heating model. The retrapping current I_r is the lowest current at which a self-heating normal region can be sustained. At this point, V drops discontinuously to zero from half the normal value (dotted line). The dashed line illustrates extrapolation to an "excess current" from a current several times greater than I_r.

far from the truth for the thinner wires, to which this simplified version of the analysis definitely does not apply.

We take the wire to be of length L, with a normal region of length ℓ in the center; we define the normal fraction to be $x = \ell/L$. The normal resistance of the whole wire is denoted R_n. If a current I is applied, the total dissipated power is $I^2 R_n x$. This heat must be conducted to the electrodes through the superconducting segments on either side of the normal zone in the center. The lengths of these superconducting segments is $L(1-x)/2$, and the temperature difference across them is $(T_c - T_b)$ since the interface occurs where $T = T_c$. Equating the heat generated with that conducted away, we obtain our basic equation

$$x(1-x) = \frac{4K_s A(T_c - T_b)}{LI^2 R_n} \equiv \beta, \qquad (1)$$

where K_s is the (average) thermal conductivity in the superconducting region, and A is the cross-sectional area of the wire. The solution to this quadratic equation is

$$x = \frac{1 \pm \sqrt{1 - 4\beta}}{2}. \qquad (2)$$

Clearly this has no *real* solution unless $\beta \leq 1/4$, which corresponds via Eq. (1) to a *lower* limit for I. In other words, for any current below this limit, there is no solution with a normal metal resistive center, and the entire wire is superconducting. This current we identify as the retrapping current I_r, so that

$$I_r = 4 \sqrt{\frac{K_s A(T_c - T_b)}{LR_n}}. \qquad (3)$$

We can then write the voltage as $V = IR_n x$, or explicitly as

$$V = R_n \left| \frac{I + \sqrt{I^2 - I_r^2}}{2} \right|, \quad I > I_r, \qquad (4)$$

which is plotted in Fig. 1. This expression has a discontinuity between 0 and $R_n I_r/2$ at I_r, above which it asymptotically approaches the normal value $V = IR_n$. This is a reasonably good description of what is typically observed experimentally for the thicker wires, for which the approximation of zero resistance in the superconducting state is reasonably good. That is, upon reducing the current from well above I_c, the voltage slowly drops below the full normal value, and then drops to zero rather sharply from a voltage which approximates $R_n I_r/2$, i.e., half the fully normal voltage at that current, as predicted by Eq. (4), and shown in Fig. 1.

We should recall, however, that this result is not exact even for the assumed discontinuous resistive transition at low current because, in this calculation, only the thermal resistance of the superconducting domain is taken into consideration, neglecting that within the normal region. Also, K_s and K_n depend quite strongly on T, so the average value would be difficult to estimate, *a priori*. These limitations of the model can be largely rectified in a more detailed computational model, described in Sec III. That model computation accounts naturally for the fact that hysteresis is not observed in nanowires that are narrow enough to have linear resistance which is nearly independent of temperature. [Representative $R(T)$ curves can be seen in Fig. 2 of Ref. 3.]

A final qualitative remark: One can also study the nonlinear resistivity by measuring dV/dI vs I. An increase in I causes an increase in Joule heating and hence an increase in local T. Thus, if $R(T)$ increases with T ("superconducting behavior"), it will also increase with I, and the reverse for "insulating" samples for which dR/dT is negative. This simple observation correctly describes the sign of the initial change in dV/dI with increasing bias current for many samples.[4]

III. NUMERICAL MODELING OF HEAT-INDUCED HYSTERESIS

In this section we describe numerical simulations performed to model more accurately heat-induced hysteresis in superconducting nanowires. We use the relaxation method, in which the wire is broken up into a finite number of segments and the one-dimensional heat-flow equation is replaced by a finite-difference equation for each segment. Equating the heat generated with that conducted away, the finite-difference equation for segment i is

$$I^2 R(T_i) \frac{\Delta x}{L} = \frac{K_s(T_i) A [2T_i - T_{i-1} - T_{i+1}]}{\Delta x}, \qquad (5)$$

where $R(T_i)$ is the measured linear resistance of the wire of length L at T_i, $K_s(T_i)$ is the local thermal conductivity of the material in the wire at T_i, A is the cross-sectional area of the wire (assumed to be uniform), and Δx is the length of each segment. The wire is divided into N segments, and the end segments 0 and $N+1$ are used to impose the boundary conditions by fixing these two segments at the bath temperature T_b, i.e., at the temperature of the large area film electrodes at either end.[5]
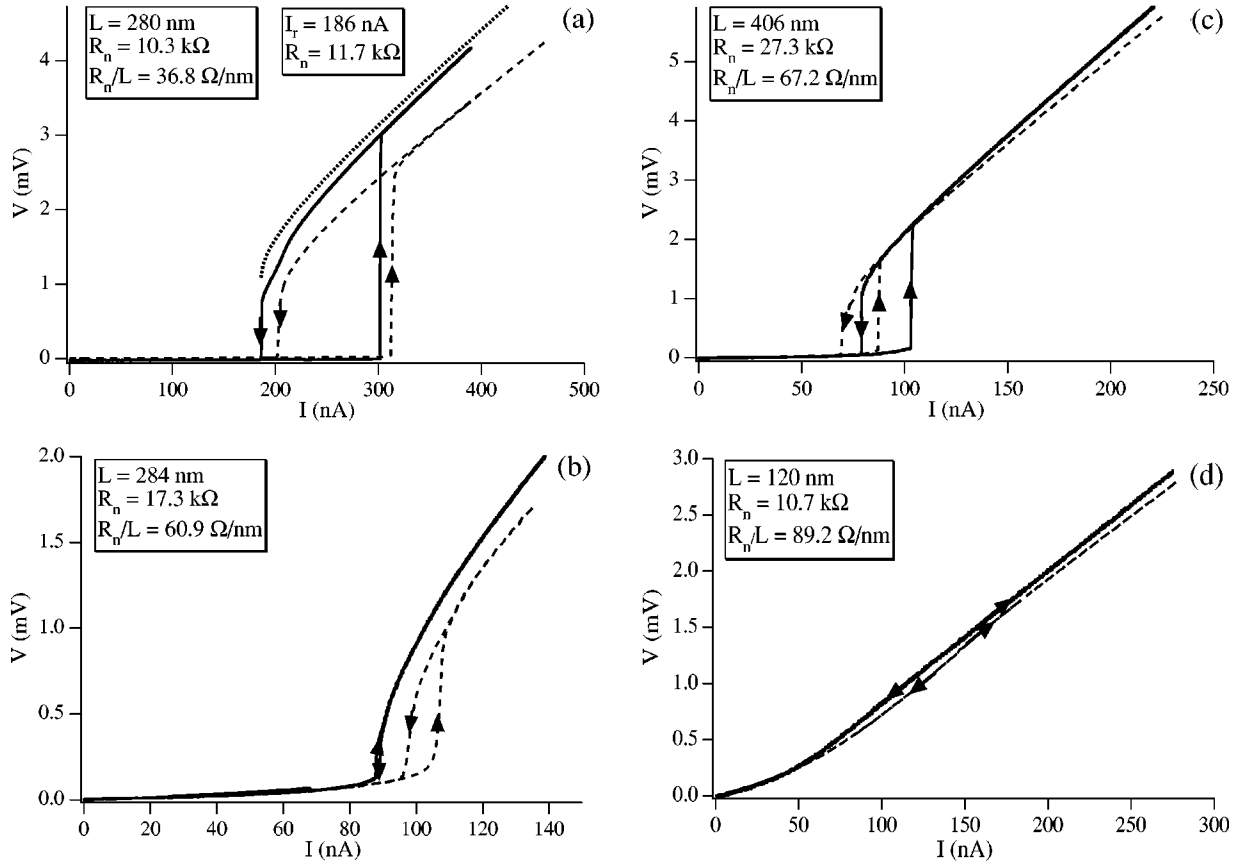
FIG. 2. Comparison of *I-V* characteristics measured at $T_b \approx 1.5$ K (solid curve) and simulated (dashed curve) of four representative superconducting nanowires of different length and cross-sectional area [proportional to $(R_n/L)^{-1}$]. Sample (a) is the thickest and sample (d) is the thinnest wire. Note the trend to less hysteresis with thinner nanowires, which stems from the weaker temperature dependence of the linear resistance of the thinner wires, which are only weakly superconducting. The dotted curve in (a) is the prediction of the analytic model, Eq. (4), which is only applicable to thick wires. The values of R_n and I_r fitted to the data are quoted in the second inset box.

Since we do not know $K_s(T_i)$ directly, we first use the Wiedemann-Franz law to express the *normal* electronic thermal conductivity as $K_n(T_i) = L_0 T_i / \rho_n$ where L_0 is the Lorentz number. Thus, $K_n(T_i) = L_0 T_i L / A R_n$, where R_n is independent of temperature for an amorphous metal in the He temperature range. At T_c, $K_s(T_c)$ must be the same as $K_n(T_c)$. For temperatures below T_c, K_s is less than K_n, because there are fewer quasiparticles to carry the heat. As a crude approximation we assume a linear interpolation between the fully normal and fully superconducting values, $K_s/K_n = T/T_c$. (In making this estimate, we neglect heat conduction by phonons in both the nanowire and the carbon nanotube substrate, assuming large thermal interfacial resistance between wire and nanotube.) With this model the finite-difference equation becomes

$$I^2 R(T_i) \frac{\Delta x}{L} = \frac{L_0 T_i^2 L}{R_n T_c} \left[\frac{2T_i - T_{i-1} - T_{i+1}}{\Delta x} \right]. \quad (6)$$

Formally solving this equation for T_i and normalizing the temperatures to T_c, we have

$$t_i = \frac{[t_{i-1} + t_{i+1} + I^2 R(t_i) R_n (\Delta x/L)^2 / (L_0 t_i^2)]}{2}, \quad (7)$$

where $t = T/T_c$ and $L/\Delta x = N$, the number of segments in the wire. This equation can be solved by iteration, putting the t_i values for one iteration in the right hand side to compute the values for the next iteration, and continuing until stable values are obtained. For the resistance, $R(t_i)$, we use the measured[3] low-current resistance at t_i as a look-up table. For the strongly superconducting samples, such as the sample shown in Fig. 2(a), the low-current resistance becomes unobservably small at a temperature above the base temperature T_b. For these samples we used an analytical extrapolation to the data as a look-up table for temperatures below those for which measured data were available. Once the discrete function t_i has been worked out for a given current I and given boundary condition temperature T_b, we can calculate the total resistance and hence the voltage $V(I, T_b)$ across the wire by summing the resistance $R(t_i)/N$ for each segment.

We have considered 9 samples from Ref. 3. Figure 2 shows data at $T_b \approx 1.5$ K from four representative samples, chosen to cover the range of cross-sectional areas. For each sample, we show the simulated $V(I)$ as a dashed curve, as well as the (solid) measured $V(I)$ curve. For each simulation the wire was divided into 80 segments and the current step was 1 or 2 nA. The simulations were performed for both up

and down current sweeps. For the down current sweeps, we started the simulation with the temperature in the wire near T_c so the wire would be in the quasinormal state. Likewise, for the up sweeps, we started the simulation with the temperature of the wire at the base temperature so that the wire would be in the superconducting or low-resistance state. We also show the result of the analytic approximation (4) as a dotted curve in Fig. 2(a), which shows data for the thickest wire, for which Eq. (4) should be most nearly applicable.

For all the samples, the simulated slope of the linear section of the $V(I)$ curve is somewhat smaller than the measured value. This could be due to our rigid boundary condition, which sets the end segments to the bath temperature T_b, rather than letting the temperature rise spread out into the film at the two ends.[5] Alternatively, our determination of the normal resistance of the wire from the $R(T)$ measurements at the point where the film went superconducting may introduce a small systematic error. Since our focus is on the hysteretic behavior, we do not consider this discrepancy to be a significant feature.

The sample shown in Fig. 2(a) has the largest cross-sectional area and, thus, is the most strongly superconducting, with a resistance that drops off quickly below T_c. In all the strongly superconducting samples studied, the predicted retrapping currents were close to the measured value, as seen in this figure. Further, we found that we could force the retrapping currents to agree completely with the data by making fairly modest adjustments in the estimated parameters such as the superconducting thermal conductivity. Qualitatively, this curve can be understood as follows: in the upsweep below the switching current I_{sw} the resistance is low because the temperature is low, so heating is small, forming a self-consistent situation, but on the downsweep, the resistance is high, and provides more heating, which self-consistently sustains the highly resistive (quasi-normal) state down to a lower current value I_r.

The samples shown in Figs. 2(b) and 2(c) have a narrower hysteretic region in the $V(I)$ curve, and our simulation also shows similar hysteretic behavior. Again, by modest adjustment of the estimated superconducting thermal conductivity, the simulations could be brought into better agreement with the measurements.

The sample shown in Fig. 2(d) has the smallest cross-sectional area and a completely nonhysteretic $V(I)$ curve. The simulated $V(I)$ curve is also nonhysteretic, and generally follows the measured curve.

Summing up, we have shown that the appearance of hysteresis in all the larger cross-sectional samples, and its absence in the thinner wires, can be explained rather quantitatively by static heating effects, using only the *measured* linear $R(T)$ as input data. Specifically, this analysis gives a good account of the measured retrapping currents I_r, and a reasonable account of the switching currents. We now turn to a more detailed consideration of the switching currents.

IV. ESTIMATION OF SWITCHING CURRENTS

The theoretical critical current density in Ginzburg-Landau (GL) theory is proportional to H_c/λ. Applied to a

filament thinner than λ and ξ, so that the current density is uniform, the critical current is simply proportional to the cross-sectional area, i.e., to L/R_n. Numerically, this GL value can be transformed[6] into the form

$$I_{co}(0) \approx 92\mu A \times [LT_c/R_n\xi(0)], \qquad (8)$$

using standard relations of GL and BCS theory [for the "dirty limit" $(l < \xi_o)$]. This expression conveniently gives a theoretical estimate of the critical current at $T \approx 0$ in terms of the measured length L, normal resistance R_n, transition temperature T_c, and coherence length $\xi(0)$, the latter being estimated from H_{c2} measured on similar material. For further numerical work, we insert representative values $\xi(0) = 7$ nm and $T_c = 5$ K, leading to

$$I_{co}(0) \approx 66\mu A \times (L/R_n), \qquad (8a)$$

with L/R_n in units of nm/Ω.

Next, we need to adjust this to account for the fact that our lowest temperature data are taken at $T \approx 1.5$ K, or $t = T/T_c \approx 0.3$, not $T = 0$. This temperature scaling is not straightforward, since Eq. (8) itself is only semiquantitative because its derivation involves combining GL relations, strictly valid only near T_c, with BCS expressions for $T \approx 0$. If we scale I_{co} by the "two-fluid" temperature dependence of $J_c \sim H_c/\lambda$, this introduces a factor of $(1-t^2)\sqrt{1-t^4}$, leading to a prefactor of 60 μA in Eq. (8a). [If we scaled by the leading order in $(1-t)$ of the GL temperature dependence valid near T_c, the numerical prefactor would be reduced by a factor of $(1-t)^{3/2}$ to 39 μA, but it seems inappropriate to use this form so far below T_c.] If instead, we make the smaller scaling up from $t=0$ to $t \approx 0.3$ by replacing $\xi(0)$ by $\xi(T) = \xi(0)/(1-t)^{1/2}$, the numerical prefactor becomes 55 μA. The differences among these various estimates gives a sense of their approximate nature. For concreteness, in further numerical work we take the value 55 μA, based on the coherence length, as a representative value. For any of these prefactor values, however, the experimental switching currents I_{sw} in the nanowires shown in Fig. 3 are found to be systematically considerably lower than the theoretical critical currents I_{co}, which are the maximum supercurrents that can be carried according to GL theory.

In Josephson junctions,[2] such "premature switching" is caused by a fluctuation which leads to a breakdown in the supercurrent and latching into a resistive state even at $I < I_{co}$. This only occurs if the junction is underdamped, so that the kinetic energy gained from running "down hill" in the tilted-washboard potential is not all dissipated, but enough remains to carry the representative point over the next "hill." In other words, once the phase point gets over a hill by a fluctuation, it keeps running, *provided that the damping is below some critical value*. It is not clear, however, to what extent this concept applies to our nanowires, because a metallic weak link junction is heavily damped and hysteresis normally arises only through heating effects. To move forward, we assume that heating will maintain the high voltage state once a fluctuation has allowed an initial phase slip to occur. More carefully, some finite rate of phase slip may be required to reach a threshold amount of Joule heating
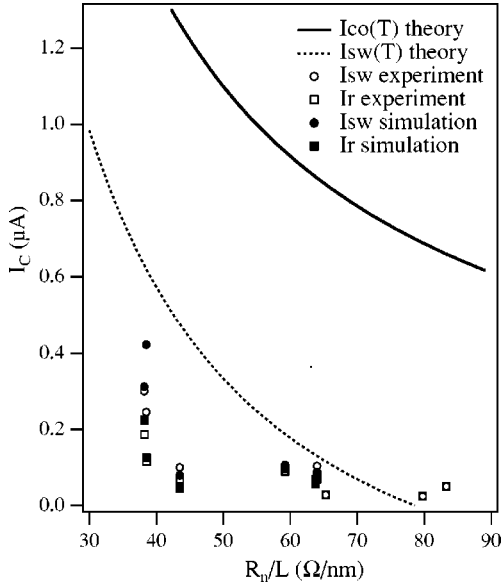
FIG. 3. Comparison of *measured values* of switching current I_{sw} and retrapping current I_r for nine samples, with: (a) the values of these two quantities *numerically simulated* using Eq. (7), and (b), the theoretical maximum supercurrent $I_{co}(T)$ at $T = 1.5$ K and the theoretically predicted ideal threshold $I_{sw}(T)$ using Eq. (11) for premature switching induced by thermal noise in the absence of heating and extrinsic noise. Equation (11) is expected to form an upper bound to the observed I_{sw}. The horizontal axis is proportional to (cross-sectional area)$^{-1}$.

sufficient to launch a self-sustaining latch into the resistive state. Such a refinement would only change the number in a logarithm, and hence make only a modest change in the result.

Before giving a detailed analysis of this process, however, we point out that stationary heating effects, as opposed to thermal fluctuations, can also cause "premature" switching in nanowires. Especially for the thinner wires, there is significant phase-slip resistance and hence heating even at low temperatures, so that the measuring current causes the temperature in the wire to rise above T_b. This temperature rise increases the phase-slip resistance, causing more dissipation, which can lead to a thermal runaway process above a certain current level, which would be identified as I_{sw}. This possibility is automatically included in the numerical simulations, and can give values of I_{sw} in agreement with experiment, which are much lower than those predicted by the model based on thermal fluctuations, as can be seen in Fig. 3. The numerical model used to simulate the I-V curves in Fig. 2 not only predicts the retrapping current into the low resistance state, but it also implicitly serves to calculate a switching onset triggered purely by these static heating effects, as distinct from the fluctuations which will form the basis for Eq. (11). The results for the simulated I_{sw} and I_r are plotted in Fig. 3 along with the measured values for a number of samples, and found to agree rather well, thus reinforcing the evidence for the importance of heating in determining the I-V curves of these nanowires.

We now return to the fluctuation model, recognizing that the *observed* I_{sw} will be the *lower* of the values determined

by stationary heating and by fluctuation effects. Ideally these two should be treated together, but for computational simplicity, we treat the fluctuation effect at T_b, assuming that, in this context, the static heating effect can be neglected.

Conceptually, the fluctuation-based switching current I_{sw}, is the current at which the barrier to phase-slip has been reduced sufficiently by the current to allow a number of phase-slips adequate to launch self-sustaining heating to occur during the time interval Δt the system spends near that current value during the upsweep of current in a measurement of the "critical current." This switching current is a stochastic quantity, and its average value depends (logarithmically) on the sweep rate. To estimate it, we need to know how the barrier decreases with current, going to zero at the nominal unfluctuated critical current I_{co}. This issue is treated in the classic paper of Langer and Ambegaokar.[7] An explicit result for the current-dependent energy barrier is not given, but by numerical solution of their equations (3.13) and (3.23) it can be shown that the current-dependent barrier height can be approximated extremely well by the analytic expression

$$\Delta F(I) \approx \Delta F(0)(1 - I/I_{co})^{5/4}, \qquad (9)$$

where $\Delta F(0)$ is the barrier energy to phase slip for zero current. [It is worth noting that the corresponding expression for the more thoroughly studied case of a Josephson junction has the same form,[2] but an exponent of 3/2 instead of 5/4. Qualitatively, the two cases are very similar.] Clearly, Eq. (9) gives a linear decrease in barrier height for small I, but for I near I_{co} in the "premature switching regime," the barrier goes to zero as the 5/4 power of $(I_{co} - I)$. This 5/4 power leads to the typical switching current $\langle I_{sw} \rangle$ being depressed below I_{co} by the 4/5 power of $(k_B T / \Delta F)$, with a logarithmic factor depending on sweep rate. Specifically, adopting the expression for a Josephson junction,[2] but changing the exponent as described above, we estimate that the average switching current is given by

$$\langle I_{sw} \rangle = I_{co}\{1 - [(k_B T / \Delta F(0))\ln(\omega_p \Delta t / 2\pi)]^{4/5}\}. \quad (10)$$

For a wire we replace $\Delta F(0)$ by $\sqrt{6}(\hbar I_{co}/2e)$.[6] The attempt frequency in a Josephson junction is the junction plasma frequency ω_p. For a wire, it is probably better approximated by the reciprocal of the Ginzburg-Landau time ($\sim 10^{12}$ sec^{-1}) times L/ξ (to take account of the possibility of phase slip *anywhere* along the wire, as was done by McCumber and Halperin[8]). If macroscopic quantum tunneling is important at low temperatures, $k_B T$ might need to be replaced by something more like $k_B T_c$, but that would only change the magnitude of the correction by a factor of ~ 2. In any case, however, in comparing different wires of the same material at the *same* temperature, the depression of I_{sw} below I_{co} is proportional to I_{co} times $(I_{co})^{-4/5}$, i.e., to $(I_{co})^{1/5}$. The time interval Δt would depend on the sweep rate, but will be of order 1 s, unless more frequent fluctuation-induced phase slips are needed to generate sufficient heat to launch a self-sustaining hot spot. Since all other factors in Eq. (10) depend rather weakly on the nominal critical current, this implies

that it should be possible to describe the measured I_{sw} in terms of the theoretically expected I_{co} by the form

$$I_{sw} = I_{co}\{1 - (I_1/I_{co})^{4/5}\} = I_{co} - I_1^{4/5} I_{co}^{1/5}. \quad (11)$$

Here I_1 is a parameter that can be estimated using Eq. (10) to be

$$I_1 = [k_B T / \sqrt{6}(\hbar/2e)] \ln(\Delta t / \tau_{GL}). \quad (12)$$

The physical significance of I_1 is that, apart from the logarithmic factor, it is roughly the current level which shifts the barrier to phase slips by $\sim k_B T$. Taking $T = 1.5$ K, and $(\Delta t / \tau_{GL}) \approx 10^{12}$, we get $I_1 \approx 0.7$ μA. Since the factor involving attempt frequency, sweep rate, and the phase-slip rate that might be needed to trigger a thermal latch appears in a logarithm, our crude estimate of these values could be off by several orders of magnitude without significant effect compared to other uncertainties.

Because the 1/5 power varies so slowly, the subtracted term in Eq. (11) varies by only a factor of ~ 1.2 over the entire range of samples shown in Fig. 3, and the result looks roughly like subtracting a constant from I_{co} to get I_{sw}. Inspection of the I_{sw} data in Fig. 3, compared with the plotted theoretical estimate of $I_{co}(T)$ from (8a) (using the estimated temperature-corrected prefactor of 55 nA) shows that such a dependence gives a rather quantitative *upper bound* for the switching currents found experimentally. This makes sense, since Eq. (11) considers only the effect of ideal thermal noise at T_b without heating, whereas switching to the resistive state could be triggered at a *lower* current by heating or by any sort of extrinsic noise. Similarly, if quantum phase slips are important, the effect would be similar to an increased temperature, and would also lower the observed I_{sw}.

An implication of Eq. (11) is that, for $I_{co} < I_1$, it would yield a *negative* value of I_{sw}. Presumably this nonphysical result implies that there is no premature switching or hysteresis for wires thin enough to have values of I_{co} smaller than this crossover value. This is qualitatively consistent with our observations that hysteresis is only seen in the thicker wires. Using the estimate of I_1 found from Eq. (12), this criterion yields $I_{co} \approx 0.7$ μA for this crossover current. Given the crudeness of our numerical estimates, this is certainly in reasonable agreement with the empirical value below which the wires in Fig. 3 are not hysteretic, i.e., have $I_{sw} \approx I_r$.

V. CONCLUDING SUMMARY

We have examined the effect of Joule heating in long thin superconducting nanowires on their I-V characteristics by solving a one-dimensional heat-flow equation for a range of sample parameters. This was done using both a simple analytic approximation for a limiting case and a numerical solution for the general case. We have compared these simulations with measured I-V curves, and found generally good agreement. The thicker wires show hysteresis and the thinner wires do not. The difference stems from the different *linear* $R(T)$ dependences, which in turn reflect the dependence of the phase-slip rate on wire thickness. Because the phase-slip rate depends *exponentially* on wire thickness, this thickness dependence in $R(T)$ dominates the simple linear thickness dependence of the thermal conductance. These classical heating effects, based on measured $R(T)$ data instead of a microscopic model of phase-slip rates, appear to be able to account surprisingly well for the hysteretic nature of the I-V curves, without explicit reference to the dynamics of phase motion in a tilted washboard potential as is used to discuss hysteresis in underdamped Josephson junctions. Heating can account directly for the value of the retrapping current, below which there is insufficient Joule heat to sustain a temperature above $\sim T_c$. The switching current is bounded by the effect of thermal fluctuations about the zero (or low) voltage state, but may be further depressed by extrinsic noise, quantum tunneling, or non-negligible Joule heating in the low-resistance state. Joule heating probably provides the mechanism for latching into the resistive state. Quantitatively, our analysis accounts reasonably accurately for the depression of the measured I_{sw} and I_r below the theoretically estimated I_{co}. We infer that heating effects are important for understanding all aspects of the hysteresis found in the I-V curves of superconducting nanowires.

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[1] The basic hot-spot model is outlined in W. J. Skocpol, M. R. Beasley, and M. Tinkham, J. Appl. Phys. **45**, 4054 (1974). A representative recent paper from the hot-electron bolometer literature [H. F. Merkel *et al.*, IEEE Trans. Microwave Theory Tech. **48**, 690 (2000)] uses a detailed thermal balance model to obtain the hot-spot length profile, and uses this to predict I-V

characteristics, conversion gain, and fluctuation noise. Another recent paper [D. W. Floet *et al.*, Appl. Phys. Lett. **73**, 2826 (1998)] treats a Nb microbridge between Au contacts, in which the focus is on how proximity effect and Andreev reflection affect optimization as a heterodyne mixer. By contrast, the work reported in the present paper is simplified by dealing with freely suspended filaments with no thermal contact with the substrate (except at the ends), but complicated by the fact that there is no generally accepted theory for the role of quantum phase-slips in causing resistance in superconducting nanowires.

[2] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996), pp. 202–210.

[3] C. N. Lau, N. Markovic, M. Bockrath, A. Bezryadin, and M.

Tinkham, Phys. Rev. Lett. **87**, 217003 (2001), and unpublished data on other samples.

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[5] In a more precise analysis, this thermal boundary condition would be replaced by inserting a thermal ''spreading resistance'' between the end of the wire and the assumed bath temperature T_b.

Since this effectively simply increases the length of the wire by an amount approximately equal to its width, we ignore this small correction in the interest of simplicity.

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