Dynamical mean-field theory of quantum stripe glasses

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We present a many body approach for nonequilibrium behavior and self-generated glassiness in strongly correlated quantum systems. It combines the dynamical mean-field theory of equilibrium systems with the replica theory for classical glasses without quenched disorder. We apply this approach to study a quantized version of the Brazovskii model and find a self-generated quantum glass that remains in a quantum mechanically mixed state as $T \to 0$. This quantum glass is formed by a large number of competing states spread over an energy region which is determined within our theory.

I. INTRODUCTION

The formation of glasses upon cooling is a well known phenomenon for classical liquids. Even without quenched disorder, the relaxation times become so large that a frozen nonergodic state is reached before the nucleation into the crystalline solid sets in. The nucleation is especially easy to avoid if the material has many polymorphisms, as does, for example, SiO_2. The system becomes unable to reach an equilibrium configuration on the laboratory time scale and exhibits aging and memory effects. While extrinsic disorder is widely acknowledged to lead to glassy phenomena in quantum systems, can a similar self-generated glassiness occur for quantum liquids? Candidate materials for this behavior are strongly correlated electron systems which often exhibit a competition between numerous locally ordered states with comparable energies. Examples of such behavior are colossal magnetoresistance materials,[1–3] cuprate superconductors,[4–11] and likely low density electron systems[12] possibly close to Wigner crystallization.

The possibility of self-generated quantum glassiness in such systems and the nature of the slow quantum dynamics has been little explored. Within a purely classical theory a "stripe glass" state was recently proposed in Refs. 13 and 14. This proposal was stimulated by nuclear magnetic resonance[4–8] and μ-spin relaxation experiments[9–11] that found static or quasistatic charge and spin configurations similar to glassy or disordered systems. A prominent effect which reflects this slowing down was the "wipe-out" effect where below a certain temperature the typical time scales become so long that the resulting rapid spin lattice relaxation cannot be observed anymore.[5,6,8] More recently, the dynamical behavior of the stripe glass of Refs. 13 and 14, as determined by Grousson *et al.*,[15] was found to be in quantitative agreement with NMR experiments.[16] Finally, recent μ-spin experiments[10,11] analyzed the freezing temperatures as a function of the charge carrier concentration and disorder concentration and found a quantum glass transition which was insensitive to the amount of disorder added to the system. This latter experiment, which seems to support a generic explanation for glassiness, not caused by impurities, also demonstrated the need for a more detailed investigation of

the quantum regime of glassy systems. It is then important to develop appropriate theoretical tools to predict whether a given theoretical model for a quantum many body system will exhibit self-generated glassiness where the system forms a glass for arbitrary weak disorder.

In this paper we develop a general approach to self generated quantum glasses that combines the dynamical mean field theory (DMFT) of quantum many body systems[17–19] with the replica technique of Refs. 20 and 21 which describes classical glasses without quenched disorder. In addition, we also go beyond strict mean-field theory and estimate the spectrum of competing quantum states after very long times. This approach is applied to study a model with competing interactions which has a fluctuation induced first-order transition to a striped two-dimensional (2D)/lamellar (3D) phase. A quantum glass is shown to be in a quantum mechanical mixed state even at $T = 0$, formed by a large number of states which can be considerably above the true ground state. Using the concept of an *effective temperature*[22–24] for the distribution of competing ground states, our theory allows the investigation of glassy nonequilibrium dynamics in quantum systems using standard techniques of equilibrium quantum many body theory. On the mean field level, our theory is in complete agreement with the explicit dynamical non-equilibrium approach developed by Cugliandolo and Lozano[25] for quenched disordered spin glasses with entropy crisis, which includes weak long-term memory effects and the subtle interplay of aging and stationary dynamics.

We find that when quantum fluctuations are weak, i.e., for a small quantum parameter, c [for a definition, see Eq. (1) below], the glass transition resembles that of classical models[26] that exhibit a dynamical transition at a temperature T_A (where mean field theory becomes non-ergodic) and a Kauzmann entropy crisis at $T_K < T_A$ if equilibrium were to be achieved. The actual laboratory glass transition is located between these two temperatures and depends, for example, on the cooling rate of the system. Beyond a critical c, T_K and T_A merge and the transitions change character. Within mean-field theory, a discontinuous change of the relevant quantum mechanical states occurs at the glass transition. Even going beyond the mean field, the system remains in a mixed quantum state, though with a nonextensive number of relevant

quantum states. Still, at the quantum glass transition a discontinuous change of the density matrix to a usual quantum liquid occurs. Put another way, a glass in contact with a bath at $T=0$ is essentially a classical object, qualitatively distinct from a quantum fluid, enforcing the quantum glass transition to be discontinuous. Our results clearly support the later scenario, as can be seen in Fig. 2 .

In classical glass forming liquids an excess or configurational entropy with respect to the solid state due to an exponentially large number of metastable configurations emerges below T_A. Obviously, in the quantum limit a glass transition must be qualitatively different. Even a very large number of long lived excited states cannot compensate for their vanishing Boltzmann weight at equilibrium for $T=0$. An exponentially large ground state degeneracy, on the other hand, is typically lifted by hybridization, saving kinetic energy. In Ref. 27 it was then argued that for a glassy quantum system the Edwards-Anderson order parameter vanishes continuously at the quantum glass transition (in contrast to the classical behavior). On the other hand, in Ref. 28 it was concluded that in certain spin glasses with quenched disorder the glass transition at sufficiently small T is of first order.

The outline of this paper is as follows. In Sec. II we introduce a quantized version of the Brazovskii model of microphase separation. We then discuss the replica approach used in this paper as well as the dynamical mean-field theory employed for the solution of the problem in Secs. III and IV, respectively. Conclusions which are based on the approach developed here but which go beyond the strict mean-field limit are discussed in Sec. V. Finally we present a summary of our results in the concluding Sec. VI. The equivalence of the "cloned-liquid" replica approach and the Schwinger-Keldysh theory of non-equilibrium quantum systems is presented in the Appendix.

II. MODEL

We consider a Bose system with field $\varphi_{\mathbf{x}}$ governed by the action $\mathcal{S}[\varphi_{\mathbf{x}}]$ with competing interactions which cause a glass transition in the classical limit. Specifically we consider a system with action

$$\mathcal{S} = \frac{1}{2}\int d^3x \int d\tau \left[\left(\frac{\partial \varphi_{\mathbf{x}}}{\partial c\,\tau}\right)^2 + r_0^2 \varphi_{\mathbf{x}} + \frac{u}{2}\varphi_{\mathbf{x}}^4 \right.$$
$$\left. + 2q_0^{-2}([\nabla^2 + q_0^2]\varphi_{\mathbf{x}})^2 \right]. \tag{1}$$

Here, q_0 is a wave number which supports strong fluctuations for momenta with amplitude $|\mathbf{q}|=q_0$, i.e., modulated field configurations. The classical version of the model [Eq. (1)], was shown by Brazovskii[29] to give rise to a fluctuation induced first order transition to a lamellar or smectic state. Within equilibrium statistical mechanics, this ordered state gives the lowest known free energy. In Refs. 13 and 14 we demonstrated that, within nonequilibrium classical statistical mechanics, an alternative scenario is a self-generated glass. Instead of the transition to a smectic state, metastable solutions built by a superposition of large amplitude waves of wave number q_0, but with random orientations and phases

emerge.[30] Those form the stripe glass state discussed in Refs. 13 and 14. It is unclear so far whether the glassy solution occurs only if the ordered phase can be avoided by supercooling or whether there is a parameter regime where the glass is favored regardless of the cooling rate.

In Eq. (1) we consider additional quantum fluctuations, characterized by the velocity c. Clearly, quantum fluctuations reduce the tendency towards a fluctuation induced first order transition. This can be seen by evaluating the $\langle \varphi^2(x,\tau)\rangle$ within the spherical approximation. In the classical limit $\langle \varphi^2(x,\tau)\rangle \sim Tq_0^2 r^{-1/2}$, with renormalized mass $r=r_0 + u\langle \varphi^2(x,\tau)\rangle$, the solution $r=0$ clearly does not exist. The same fluctuations which suppress the occurrence of a second order transition lead to a first order transition at the temperature where $r_0 \simeq uTq_0^2 r_0^{-1/2}$.[29] In the quantum limit the behavior is conceptually similar but fluctuations grow only logarithmically, $\langle \varphi^2(x,\tau)\rangle \sim q_0^2 \log(\Lambda/r)$. For an exponentially large correlation length $r^{-1/2}$ there should also be a fluctuation induced first order transition to a smectic, which might be related to the state proposed in Ref. 31 in the context of strongly correlated quantum systems. Another option, however, is the emergence of a stripe glass, even for large quantum fluctuations, which results in an amorphous modulated state instead. The investigation of this option will be the specific application of our theory. Before we go into specifics of the model [Eq. (1)], we develop a general framework for the description of self generated quantum glasses.

III. "CLONED LIQUID" - REPLICA APPROACH

Competing interactions of a glassy system cause the ground state energy as well as the excitations to be very sensitive to small additional perturbations. In order to quantify this we introduce, following Ref. 20, a static "ergodicity breaking" field ψ according to

$$\mathcal{S}_\psi[\varphi] = \mathcal{S}[\varphi] + \frac{g}{2}\int d\tau d^3x [\psi_{\mathbf{x}} - \varphi_{\mathbf{x}}(\tau)]^2,$$

and we take the limit $g\to 0$ eventually. The coupling between φ and ψ will bias the original energy landscape in the "direction" of the configuration ψ, enabling us to count distinct configurations. Adopting a mean-field strategy, we assume that even in the quantum limit ψ should be chosen as static variable, probing only time averaged configurations.

Introducing $\tilde{f}_\psi = -T\log Z_\psi$, with the biased partition function $Z_\psi = \int D\varphi\, e^{-\mathcal{S}_\psi}$, $\tilde{f}_\psi(T\to 0)$, corresponds to the ground state energy for a given ψ. If there are many competing ground states, it is natural to assume that $\tilde{f}_\psi$ determines the probability p_ψ for a given configuration. If we identify the actual state of the system we gain the information $S_c = -\lim_{g\to 0}\int D\psi\, p_\psi \log p_\psi$. Maximizing this configurational entropy S_c with respect to p_ψ yields the usual result

$$p_\psi \propto \exp(-\tilde{f}_\psi/T_{\text{eff}}), \tag{2}$$

where the effective temperature, T_{eff} is the Lagrange multiplier enforcing the constraint that the typical energy is $\tilde{F} = \lim_{g\to 0}\int D\psi\, p_\psi \tilde{f}_\psi$. T_{eff} is a measure for the width of the

energy region within which the relevant ground state energies can be found. If the system is glassy, $T_{\text{eff}} > T$ and a quantum mechanically mixed state results even as $T \to 0$. Thus, a glass will not be in a pure quantum mechanical state (characterized by a single wave function) even at zero temperature.

Introducing the ratio $m = T/T_{\text{eff}}$, it follows that $p_\psi \propto Z_\psi^m$, leading to

$$\tilde{F} = \frac{\partial [mF(m)]}{\partial m}$$

$$S_c = \frac{m}{T_{\text{eff}}} \frac{\partial F(m)}{\partial m},$$

where we introduced $F = \tilde{F} - T_{\text{eff}} S_c$ with

$$F(m) = -\lim_{g \to 0} \frac{T}{m} \log \int D\psi Z_m[\psi].$$

It is now possible to integrate out the auxiliary variable ψ, yielding an m-times replicated theory of the original variables φ with infinitesimal inter-replica coupling:

$$\mathcal{S} = \sum_{\alpha=1}^{m} \mathcal{S}[\varphi^\alpha] - g \sum_{\alpha,\beta=1}^{m} \int d^3x \, d\tau \, d\tau' \, \varphi_{\mathbf{x}}^\alpha(\tau) \varphi_{\mathbf{x}}^\beta(\tau'), \quad (3)$$

similar to a random field model with infinitesimal randomness g. The major difference here is that in systems with a tendency towards self-generated glassiness, the initial infinitesimal randomness g will self-consistently be replaced by an effective, interaction induced, self-generated randomness. Specifically this will be the off diagonal element of the self-energy in replica space.

At this point it is useful to discuss similarities and differences of the present approach if compared to the conventional replica approach of systems with quenched disorder. First, on a technical level, the replica index has to be analytically continued to $m = T/T_{\text{eff}} \lesssim 1$ and not to zero. This reflects the fact that slow metastable configurations do not equilibrate at the actual temperature, T, but at the effective temperature T_{eff}. In other words, the system has an essentially equal probability to evolve into states which are spread over a spectrum with width T_{eff}, even as $T \to 0$. If one applies the present approach to a system with explicit quenched disorder, where one can apply the conventional replica theory, and assumes replica symmetry, it turns out that m corresponds to the break point of a solution with one step replica symmetry breaking of the conventional replica approach and both techniques give identical results. Alternatively one might also consider the model [Eq. (1)] in the limit of an infinitesimal random field with width g and the break point of a one step replica symmetry breaking m solution. This implies that the present approach captures the essence of the glassy behavior in systems with two very distinct typical time scales. This will become particularly clear if we compare our results with the one obtained within the solution of the dynamical Schwinger-Keldysh theory in the Appendix. A more physical relationship between the two rep-

lica approaches can be obtained by realizing that the typical free energy of a frozen state, $\tilde{F}$, can also be written in the usual replica language via

$$\tilde{F} = \lim_{n \to 0} \frac{1}{n} (\overline{Z_\psi^n} - 1),$$

where the average is performed with respect to the distribution function $p_\psi \propto Z_\psi^m$. The distribution function of the self-generated randomness is non-Gaussian. For example, if $m \to 1$, $e^{-\mathcal{S}_\psi[\varphi]}$ can also be interpreted as the generating functional of the distribution p_ψ.[32] It is because this distribution is characterized by "colored noise" that we find a self-generated glassy state of the kind discussed here. Finally, if one replaces $([\nabla^2 + q_0^2]\varphi)^2$ in Eq. (1) by the usual $(\nabla\varphi)^2$ term (which is the $q_0 \to 0$ limit after appropriate rescaling of φ, c, u, etc.) there is no glass as $g \to 0$, making it evident that self-generated glassiness is ultimately caused by the uniform frustration of the finite q_0 problem where modulated configurations, $\varphi(\mathbf{x}) \propto \cos(\mathbf{q}_0 \cdot \mathbf{x})$, with $|\mathbf{q}_0| = q_0$ and arbitrary direction have low energy.

IV. DYNAMICAL MEAN-FIELD THEORY

Due to the mean-field character of the theory, it is appropriate to proceed by using the ideas of the DMFT for equilibrium many body systems[17–19] and assuming that the self-energy of our replicated field theory is momentum independent. Physically, ignoring the momentum dependence of the self-energy might be justified by the fact that a glass transition usually occurs in a situation of intermediate correlations, i.e., when the correlation length of the liquid state is slightly larger but comparable to the typical microscopic length scales in the Hamiltonian.[14] The free energy of the system can be expressed in terms of the Matsubara Green's function $G_{\mathbf{q}}^{\alpha\beta}(\omega_n) = \langle \varphi_{\mathbf{q}}^a(\omega_n) \varphi_{-\mathbf{q}}^b(-\omega_n) \rangle$ and the corresponding local (momentum independent) self energy $\Sigma^{\alpha\beta}(\omega_n)$ as

$$F(m) = \text{tr}(\Sigma G) - \text{tr} \log G + \Phi[G],$$

where the self energy is given by $\Sigma = \delta\Phi[G]/\delta G$, and $\Phi[G]$ is diagrammatically well defined for a given system.[33]

The difference from the usual equilibrium DMFT approach[17–19] is the occurrence of off diagonal elements in replica space, allowing us to map the system onto a local problem with the same interaction but dynamical "Weiss" field matrix. This brings us to an effective zero-dimensional theory similar to the mode-coupling theory of classical glasses.[34] We furthermore make the ansatz[20]

$$G_{\mathbf{q}}^{\alpha\beta}(\omega_n) = \mathcal{G}_{\mathbf{q}}(\omega_n)\delta_{ab} + \mathcal{F}_{\mathbf{q}}\frac{\delta_{n,0}}{T}, \quad (4)$$

with static off diagonal elements. A similar ansatz for the self-energy leads to the following two Dyson equations for the diagonal and off diagonal propagators:

$$\mathcal{G}_{\mathbf{q}}(\omega_n) = [\mathcal{G}_{0,\mathbf{q}}^{-1}(\omega_n) - \Sigma_{\mathcal{G}}(\omega_n)]^{-1},$$

$$\mathcal{F}_{\mathbf{q}} = \Sigma_{\mathcal{F}} \mathcal{G}_{\mathbf{q}}(0) [\mathcal{G}_{\mathbf{q}}^{-1}(0) - m\Sigma_{\mathcal{F}}/T]^{-1}. \qquad (5)$$

Glassiness is associated with finite values of the Edwards-Anderson order parameter $\mathcal{F}_{\mathbf{q}}$, whereas for $\mathcal{F}_{\mathbf{q}}=0$ we recover the traditional theory of quantum liquids. The structure of the Dyson equation already gives us crucial informations on the nature of the glass transition. From Eq. (5) it follows immediately that, contrary to the classical case where glassiness can occur with $m=1$,[13,14] in the quantum limit ($T \rightarrow 0$) the only way $\mathcal{F}_{\mathbf{q}}$ can be nonzero is to have $m \rightarrow 0$ such that $m/T = 1/T_{\mathrm{eff}} \neq 0$. This imposes a constraint on the replica symmetry breaking structure in the quantum limit of the replica approach developed in Ref. 20. Also, it is clear that $\Sigma_{\mathcal{F}}$ defines a new length scale of the problem that is associated with the glass transition.[14] In the classical glass transition the relation $m=1$ is satisfied at T_A. Then the two Dyson equations can be decoupled into an equilibrium, diagonal (in replica space) part, and a nonequilibrium, off diagonal part. This allows us to interpret $\Sigma_{\mathcal{G}}$ as related to the equilibrium correlation length and to associate $\Sigma_{\mathcal{F}}$ with the Lindemann length, associated with the typical length scale of wandering of defects of the equilibrium structure; see Ref. 14 for details. However, it follows from the structure of the self-energies that, in the quantum limit, where $m<1$ at T_A, the two Dyson equations cannot be decoupled anymore. In this case, the two self-energies will combined define a correlation length and a Lindemann length which are not independent, but rather closely intertwined.

We solved the impurity problem within a self-consistent large-N approach, i.e. we generalize the scalar field φ to an N-component vector and consider the limit of large N including first $1/N$ corrections. This approach was used earlier to investigate self generated glassiness in the classical limit.[13,14] In this limit it is also possible to solve the DMFT problem exactly,[35] demonstrating that glassiness found in the approximate large-N limit is very similar to the exact, finite-N theory and thereby supporting the applicability of the large N expansion.

Within the self consistent large-N approach, the diagonal and off diagonal element of the self-energy are given as

$$\Sigma_{\mathcal{G}}(\omega_n) = \Sigma_{\mathcal{G}}^H - \frac{2}{N}[\mathcal{F}D_{\mathcal{G}}(\omega_n) + D_{\mathcal{F}}\mathcal{G}(\omega_n)]$$

$$+ T\sum_m D_{\mathcal{G}}(\omega_n + \omega_m)\mathcal{G}(\omega_m),$$

$$(6)$$

$$\Sigma_{\mathcal{F}} = -\frac{2}{N}D_{\mathcal{F}}\mathcal{F},$$

with a Hartree contribution

$$\Sigma_{\mathcal{G}}^H = -uT\sum_m \mathcal{G}(\omega_m) - u\mathcal{F},$$

as well as

$$D_{\mathcal{G}}(\omega_n) = \frac{1}{u^{-1} + \Pi_{\mathcal{G}}(\omega_n) + 2\gamma\mathcal{F}\mathcal{G}(\omega_n)},$$

$$(7)$$

$$D_{\mathcal{F}} = \frac{-\gamma\mathcal{F}^2 D_{\mathcal{G}}^2(0)}{1 + \gamma\mathcal{F}^2 D_{\mathcal{G}}(0)/T_{\mathrm{eff}}},$$

and a bubble diagram $\Pi_{\mathcal{G}}(\omega_n) = \gamma T\Sigma_m\mathcal{G}(\omega_n + \omega_m)\mathcal{G}(\omega_m)$. Here $\mathcal{G}(\omega_m) = \int[d^3q/(2\pi)^3]\mathcal{G}_{\mathbf{q}}(\omega_n)$ and $\mathcal{F} = \int[d^3q/(2\pi)^3]\mathcal{F}_{\mathbf{q}}$ are the momentum averaged propagators. The DMFT is usually formulated on a lattice and it is possible to chose the same dimension (e.g., inverse energy) for the momentum dependent and momentum averaged propagator, by assuming the lattice spacing equal unity. In a continuum theory the role of the lattice spacing is played by the inverse upper cutoff of the momentum integration Λ, which enters our theory in Eqs. (7) through the constant $\gamma = \Lambda^{-3}$. In the case of the Hamiltonian [Eq. (1)], all momentum integrals are convergent as $\Lambda \rightarrow \infty$ and the scale which replaces the cut off is q_0, leading to $\gamma = q_0^{-3}$. We solved this set of self consistent equations numerically. Before we present the results we must discuss the stability of the ansatz, Eq. (4).

The local stability of the replica symmetric ansatz, Eq. (4) is determined by the lowest eigenvalue of the Hessian matrix $\delta^2 F/\delta G^{\alpha\beta}\delta G^{\gamma\delta}$. Proceeding along the lines of Ref. 36 and diagonalizing over the replica indices leads to the following matrix in momentum space:

$$\mathcal{M}_{\mathbf{q},\mathbf{q}'} = \delta(\mathbf{q} - \mathbf{q}')\mathcal{G}_{\mathbf{q}}^{-2}(0) + \mathcal{C}, \qquad (8)$$

where

$$\mathcal{C} = \frac{\delta^2\Phi}{\delta G^{\alpha\beta}\delta G^{\alpha\beta}} - 2\frac{\delta^2\Phi}{\delta G^{\alpha\beta}\delta G^{\alpha\delta}} + \frac{\delta^2\Phi}{\delta G^{\alpha\beta}\delta G^{\gamma\delta}}, \qquad (9)$$

with distinct α, β, δ, and δ. Diagrammatically, $\mathcal{C}$ is a sum of diagrams with four external legs with at least two distinct replica indices. Thus, as $g \rightarrow 0$ the constant $\mathcal{C}$ vanishes if $\mathcal{F}_{\mathbf{q}} \rightarrow 0$, an observation which will be relevant in our discussion of the nature of the zero temperature glass transition as a function of c. We find, for the lowest eigenvalue λ of $\mathcal{M}_{\mathbf{q},\mathbf{q}'}$,

$$\mathcal{C}^{-1} = \int \frac{d^3q}{(2\pi)^3}[\mathcal{G}_{\mathbf{q}}^{-2}(0) + \lambda]^{-1}. \qquad (10)$$

If $\lambda > 0$ the mean-field solution is stable, whereas it is marginal for $\lambda = 0$. The fairly simple result [Eq. (10)] for the stability of the replica structure is a consequence of the momentum independence of the self-energy within the DMFT which guarantees that $\Phi[G]$ only depends on the momentum averaged propagators. Otherwise $\mathcal{C}$ would depend on momentum, making the analysis of the eigenvalues of $\mathcal{M}_{\mathbf{q},\mathbf{q}'}$ in Eq. (8) much more complicated. Thus, the investigation of our problem within dynamical mean-field theory is not only convenient to obtain solutions for $\mathcal{G}_{\mathbf{q}}(\omega_n)$ and $\mathcal{F}_{\mathbf{q}}$, it is also crucial to make progress in the analysis of the stability of this

solution. We expect that it will be impossible to find a stable replica symmetric solution [Eq. (4)] once one goes beyond the DMFT.

In real physical systems the slow degrees of freedom relax on a finite time scale τ_α and the effective temperature $T_{eff} = T/m$ depends not only on the external parameters like T and pressure, but also on the cooling rate, or equivalently on the time, t_w, elapsed after quenching and thus on the events which, for a finite range system, can take the system to different states of the spectrum characterized by T_{eff}. Accounting for these effects goes beyond a mean-field treatment. However, for $t_w \ll \tau_\alpha$ mean field theory should apply. In fact, since within the mean-field approach the time τ_α is infinite, this is intrinsically the regime we are constrained to within the mean-field treatment. Then, the most important configurations of the order parameter are those which allow the system to explore the maximum number of ergodic regions. The best way to achieve this, without being unstable, is through interconnecting saddle points.[37] This leads to the marginal stability condition $\lambda = 0$, which we use to determine the effective temperature T_{eff}^0, which corresponds to the effective temperature right after a fast quench into the glassy state, i.e, $T_{eff}^0 = T_{eff}(t_w \ll \tau_\alpha)$:

$$\mathcal{C}(T_{eff}^0)^{-1} = \int \frac{d^3q}{(2\pi)^3} \mathcal{G}_{\mathbf{q}}^2(0, T_{eff}^0). \qquad (11)$$

Within the large-N approximation used to determine the propagator we can also evaluate the constant $\mathcal{C}$ of Eq. (9), leading to

$$\mathcal{C}(T_{eff}^0) = \frac{2}{N}[2D_{\mathcal{G}}^2(0)\Pi_{\mathcal{F}} - \mathcal{D}_{\mathcal{F}}], \qquad (12)$$

which is an implicit equation for T_{eff}^0. Note that, as we argued before, $\mathcal{C}$ vanishes for $\mathcal{F} \to 0$. Moreover, since the integral on the right hand side of Eq. (11) is bounded from above, $\mathcal{C}$ and therefore $\mathcal{F}_{\mathbf{q}}$ must vanish discontinuously at the transition even for $T = 0$. As explained above, from the structure of the Dyson equation in replica space it immediately follows that for c larger than some value c_A, T_{eff}^0/T must also jump discontinuously from $T_{eff}^0/T > 1$ in the glass state to $T_{eff}^0/T = 1$ in the quantum liquid state.

Together with Eq. (11) we have a closed set of equations which describe the quantum glass within mean-field theory. We have solved this set of equations for the model [Eq. (1)], and indeed found that there is a glassy state below a critical value for c. Most interestingly, for increasing c (i.e., increasing quantum fluctuations) the rapidly quenched quantum glass, which at some point becomes unstable, undergoes a discontinuous reorganization of the density matrix upon entering the quantum liquid state. It is crucial to solve this many body problem within some conserving approximation, i.e., based upon a given functional $\Phi(G)$, which must be simultaneously used to determine Σ and $\mathcal{C}$. The set of equations was solved numerically. The Matsubara frequency convolutions were calculated on the imaginary time axis via fast

fourier transform algorithm using 2^{15} frequencies. This accuracy is needed mostly to be able to find solutions of Eqs. (11) and (12).

The transition line between the liquid and glassy states in the (c, T) space is presented in Fig. 1. Note that the low temperature (quantum regime) behavior is qualitatively different from the classical stripe glass. In the quantum limit, T_A and T_K merge and the effective temperature at the transition $T_{eff}^0(T_A)$ is always larger than T_A, i.e., $m < 1$. Due to the reentrant character of the transition the quantum glass can also be reached by heating up the system. By generalizing Brazovskii's theory of the fluctuation induced first order transition to the quantum case, we found a similar reentrance behavior for this equilibrium transition, suggesting that this peculiar shape of the phase border is determined by the increasing relevance of fluctuations with wave vector $\mathbf{q} = |q_0|$ as one crosses over from a quantum to a classical regime (see the corresponding remarks made in Sec. II). Another way the quantum glass can be reached is of course via a "c quench." Note also that the reentrant behavior we find within our approach happens at the point where numerically, at the same time, $T_A - T_K$ vanishes (to within numerical precision), $m(T_A)$ starts falling with a larger derivative (see the inset of Fig. 3), and $\mathcal{F}(T_A)$ reaches a minimum (see inset of Fig. 2) before plateauing as $T \to 0$. This suggests that the reentrance behavior and the change of character of the transition are closely related. In Fig. 3 we show the dependence of the replica symmetry index $m = T/T_{eff}^0$ as a function of c for different temperatures.

In the classical limit, $c \to 0$, $\mathcal{F}_{\mathbf{q}}$ changes discontinuously at T_A, whereas T_{eff} changes continuously ($m = 1$), i.e., the relevant metastable states—within which a classical glassy
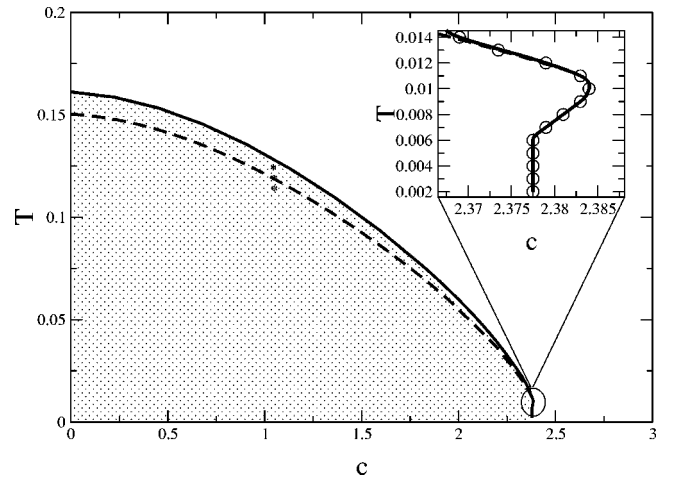

FIG. 1. DMFT Phase diagram for the model [Eq. (1)] within the one-loop self-consistent screening approximation. The solid line represents T_A whereas the dashed line represents T_K. The inset shows the low temperature region of the phase diagram where T_A and T_K have merged and the effective temperature at the transition is larger than $T [m(T_A) < 1]$. The line is simply a guide to the eye whereas the dots are the actual numerical result. The numerical simulations in the paper were carried out using the parameters $r_0 = -6$, $q_0 = 0.3$, and $u = 2\pi^2|r_0|$. The stars refer to the (T, c) points, of which the results in Fig. 4 are shown.
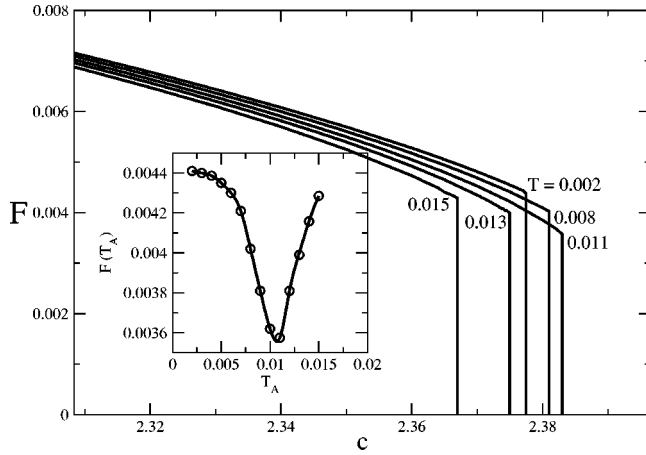
FIG. 2. Low temperature behavior of the Edwards-Anderson order parameter as a function of the intensity of quantum fluctuations. The temperatures are in the region displayed in the inset of Fig. 1. Inset: dependence of the order parameter at the transition temperature T_A.

FIG. 3. Dependence of the effective temperature as function of c for different temperatures. Inset: dependence of T_{eff}^0/T_A on T_A in the low temperature regime displayed on the inset of Fig. 1.

system is trapped—connect gradually to the relevant states which contribute to the liquid state partition function. A similar behavior in the quantum limit would imply that T_{eff} and therefore $\mathcal{F}_\mathbf{q}$ and $\mathcal{C}$ vanish continuously at the quantum glass transition. However, as discussed above, this is not possible if one uses Eq. (11) to determine the effective temperature. Thus, T_{eff} changes discontinuously and one might expect a nucleation of liquid droplets within the unstable glass state to be important excitations which cause a quantum-melting of the glass. Even though one can formally introduce, along the lines of Ref. 23, a latent heat $\delta Q = T_{\text{eff}} S_c$ at this transition, we do not know of a scenario which, within mean-field theory, allows this energy to be realized within the laboratory. The appearance of a first-order-like transition as one enters the quantum regime, though without reentrance behavior, was first pointed out in Ref. 28 in a related case of spin glasses with quenched disorder.

Finally, we make contact between our theory and the Schwinger-Keldysh approach used in Ref. 25, which gives a set of coupled equations for the symmetrized correlation function $C_{\mathbf{x},\mathbf{x}'}(\tau,t_w) = \frac{1}{2}\langle [\varphi_\mathbf{x}(t_w+\tau),\varphi_{\mathbf{x}'}(t_w)]_+\rangle$ and the retarded response function $G_{\mathbf{x},\mathbf{x}'}^r(\tau,t_w) = -i\theta(\tau)\langle [\varphi_\mathbf{x}(t_w+\tau),\varphi_{\mathbf{x}'}(t_w)]_-\rangle$, where $[,]_\pm \equiv AB \pm BA$. In the classical limit $C_{\mathbf{x},\mathbf{x}'}(\tau,t_w) = \langle \varphi_\mathbf{x}(t_w+\tau)\varphi_{\mathbf{x}'}(t_w)\rangle$ as usual. If t_w, is comparable to τ the dynamics is complex and depends on the nature of the initial state (aging regime). On the other hand, for t_w large compared to τ, $C(\tau,t_w)$ and $G^r(\tau,t_w)$ are expected to be dependent only on τ (stationary regime). Correspondingly, one can decompose the correlation function into aging and stationary contributions

$$C(\tau,t_w) = C_{\text{AG}}(\tau,t_w) + C_{\text{ST}}(\tau), \qquad (13)$$

and similarly for $G^r(\tau,t_w)$. Cugliandolo and Kurchan[24] showed within the mean-field theory of classical spin glasses that one cannot decouple the stationary dynamics from the aging regime. Instead, the system establishes a "weak long term memory" and one has to solve for the entire time de-
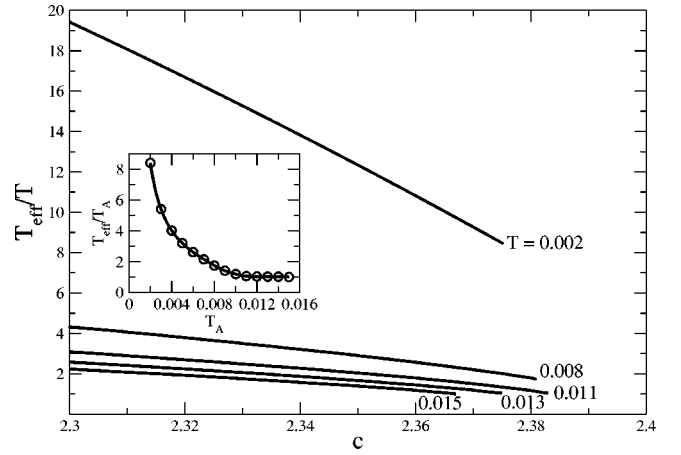
pendence. An elegant way to encode the aging dynamics is a generalized fluctuation dissipation theorem (FDT) $G_{\text{AG}}^r(\tau,t_w) = -T_{\text{eff}}^{-1}(\partial/\partial t_w)C_{\text{AG}}(\tau,t_w)$, with effective T_{eff}. This approach was generalized to the quantum case in Ref. 25.

As shown in the Appendix, we find a complete equivalence of the Schwinger-Keldysh theory of Ref. 25, if applied to the quantized Brazovskii model, with our replica approach if we identify (after analytical continuation to real time): $\mathcal{G}(t) = G_{\text{ST}}^r(t)$ and $\mathcal{F} = \lim_{t_w\to\infty}\lim_{t\to\infty}C_{\text{AG}}(t_w,t)$. T_{eff}^0, which follows from marginality [Eq. (11)], is identical to the one of the generalized FDT, when we assume $t_w \ll \tau_\alpha$.[38]

V. ASPECTS BEYOND THE STRICT MEAN-FIELD THEORY

In this section we discuss several physically significant conclusions one can draw from our theory which go beyond the strict mean-field limit. In particular we will discuss several aspects related to dynamical heterogeneity in glasses.

Going beyond mean-field theory, for $t_w \gg \tau_\alpha$, the marginal stability cannot be sustained, since the system can save free energy via droplet formation,[26] which drives the system towards equilibrium. As discussed in Ref. 26, the free-energy gain is due to a gain in configurational entropy which is inhibited within mean-field theory due to infinitely large barriers, but possible in finite subsystems where transitions between distinct metastable states become allowed. Thus, the glass might be considered as consisting of a mosaic pattern built of distinct mean-field metastable states. The size of the various droplets forming the mosaic is determined by a balance of the entropic driving force (proportional to the volume of the drop) and the surface tension.[26]

One way to account phenomenologically for such a behavior within our theory is to assume that T_{eff} becomes a time dependent quantity[23] and that the exploration of phase space allows the system to "cool down" its frozen degrees of freedom by realizing configurational entropy. One would naturally expect $T_{\text{eff}}(t_w)$ to decrease towards a value T_{eff}^∞

until either $\lim_{N \to \infty} S_c(T_{\text{eff}}^{\infty})/N = 0$ or $T_{\text{eff}}^{\infty} = T$. In the former case there is no longer any extensive entropic driving force which favors the exploration of phase space, whereas in the latter case the system has reached equilibrium but with, in general, a finite remaining configurational entropy. These two regimes are separated by the Kauzmann temperature where $\lim_{N \to \infty} S_c(T_{\text{eff}}^{\infty})/N = 0$ at $T_{\text{eff}}^{\infty} = T$ simultaneously.[26] We cannot, of course, calculate the explicit time dependence of $T_{\text{eff}}(t_w)$ here. However, we can parametrically study $S_c(T_{\text{eff}})$ versus T_{eff}, i.e. keep the replica value m an open parameter of the theory and analyze whether the trends for the variation of $S_c(T_{\text{eff}})$ at fixed temperature T are sensible. Since we determined m previously by the marginality condition, an effective temperature $T_{\text{eff}} < T_{\text{eff}}^0$ implies that the replicon eigenvalue is different from zero. We found numerically that $\lambda < 0$, reflecting an instability of our solution likely related to some kind of dynamical heterogeneity. We argue that this heterogeneity is different for temperatures close to T_A and T_K.

We first discuss the behavior below but close to T_A in the classical regime where marginality gives a continuous change of m, i.e. $T_{\text{eff}}^{\infty}(T_A) = T$. In this regime the system is close to equilibrium and it should be sensible to consider small fluctuations around the mean field solution. Such small fluctuations are then dominated by the eigenvectors of the replicon problem which correspond to the lowest eigenvalue. From Eq. (8) we can easily determine the momentum dependence of this eigenvector, which can be interpreted as the fluctuating mean square of the long time correlation function $\mathcal{F}_{\mathbf{q}}$ away from its mean-field value. It is given by

$$\Psi_{\mathbf{q}} = \Psi_0 \frac{q_0^4}{\mathcal{G}_{\mathbf{q}}^{-2} - \lambda}, \qquad (14)$$

where Ψ_0 is a normalization constant. At marginality, these modes in correlation function space are massless and thus easy to be excited. The typical length scale of these correlation function fluctuations are determined by $\mathcal{G}_{\mathbf{q}=\mathbf{q}_0}^{-2}$, i.e., are confined to a length scale determined by $\Sigma_{\mathcal{G}}^{-1/2}$. Since this length is not the actual correlation length, but rather determined by the shorter Lindemann length discussed on Ref. 14, the wandering of defects seem to set the scale on which the dynamics close to T_A evolves. Thus correlations over the Lindemann length are fluctuating in space. Even though the fluctuating object is characterized by a rather short scale, its fluctuations are, of course, correlated over larger distances, as characterized by the nonlinear susceptibility $\chi^{\alpha\beta\gamma\delta} = \langle \varphi_\alpha(x)\varphi_\beta(x)\varphi_\gamma(x)\varphi_\delta(x) \rangle$. $\chi^{\alpha\beta\gamma\delta}$ is the inverse of the Hessian and thus diverges if $\lambda \to 0$. Thus we conclude that close to T_A this heterogeneity is driven by the correlation function fluctuations of shape $\Psi_{\mathbf{q}}$ of Eq. (14). We believe that this "Goldstone"-type heterogeneity is very similar in character to the recent interesting approach to heterogeneity resulting from the assumption of a local time reparametrization invariance.[39] This is supported by the close relation between the reparametrization invariance and marginality, as shown in Ref. 40. In addition, the approach of Ref. 39 considers fluctuations relative to the mean field solution with

stiffness proportional to the (mean field) Edwards-Anderson parameter. Thus, small fluctuations caused by the marginality of the mean-field solution are considered, similar to the ones given in Eq. (14). Further away from T_A such a linearized theory is likely to break down because additional eigenvectors, not related to the marginal eigenvalue, become relevant and non-Gaussian fluctuations come into play. This should always be the case where m is not close to unity, i.e., in the classical regime for temperatures below T_A and everywhere in the quantum regime. One might expect droplet physics to become important then.[26]

In both cases it is useful to analyze the evolution of the spectrum of states of this formally unstable theory, particularly if the replica structure of the theory remains unchanged. As shown in Fig. 1, we found numerically for the model, Eq. (1), that for $t_w \gg \tau_\alpha$, three different possible final situations result, depending on the relation between the bath temperature T and the Kauzmann temperature T_K: (1) $T_A > T > T_K$, T_{eff} will relax until it reaches T, but an excess entropy will remain. (2) $T = T_K$, T_{eff} will relax until it reaches T, in a state with zero configurational entropy. (3) $T < T_K$, T_{eff} will relax until all the excess entropy vanishes, but the system remains in a non-equilibrium state with $T_{\text{eff}}^{\infty} > T$. In this case there are still many (even though less than exponentially many) states, distributed according to a Boltzmann function with an effective temperature T_{eff}^{∞}. Their energies $\widetilde{f}_\psi$ differ by nonextensive amounts in a range of order $T_{\text{eff}}^{\infty} < T_{\text{eff}}^0$. At a critical value of the quantum parameter slightly below c_A we find that T_A and T_K merge and the nature of the glass transition changes. The system is either in a quantum fluid or in a nonequilibrium frozen state and the identification of the glass transition using equilibrium techniques alone becomes impossible. Note, however, that in the quantum glass regime, even at temperatures T arbitrarily close to T_A, for $t_w \gg \tau_\alpha$ we always obtain $\lim_{N \to \infty} S_c(T_{\text{eff}}^{\infty})/N = 0$ for $T_{\text{eff}}^{\infty} > T$ (see the dashed curve in Fig. 4).

VI. CONCLUSION

In summary, we have presented an approach to self-generated quantum glasses which enables the counting of competing ground state energies (quasiclassically of long lived metastable states) in interacting quantum systems. Technically very similar in form to the traditional quantum many body theory of equilibrium systems, it allows one to investigate whether a given system exhibits self-generated glassiness as a consequence of the frustrating interactions. Slow degrees of freedom are assumed to behave classically and are shown to equilibrate to an effective temperature which is nonzero even as $T \to 0$ and which characterizes the width and rigidity of the energy landscape of the competing states of the system. Applied to the specific model [Eq. (1)], we do find a glass below a critical value for the quantum fluctuations. Using a marginality criterion to determine T_{eff}, we can generally show that quantum glass transitions are bound to be discontinuous transitions from pure to mixed quantum states. This leads to the interesting question of how quantum melting of the nonequilibrium quenched states can
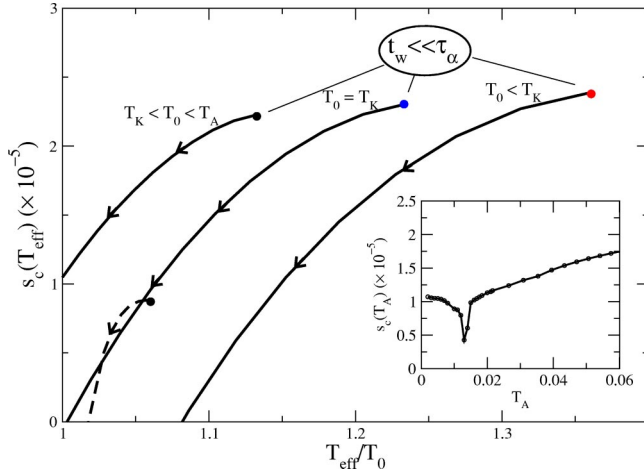
FIG. 4. Parametric plot of the relaxation of T_{eff} and S_c for varying t_w after quenches to the three points indicated by stars in the phase diagram Fig.1. In addition, S_c is shown for one point right at the quantum glass transition with $T_A = 0.01 > T$. The direction of evolution of the waiting time t_w is indicated by the arrows. The system can relax from a state with marginal stability with T_{eff}^0 to three different final states, as explained in the text. Inset: configurational entropy at the transition point as function of T_A.

occur via nucleation of the corresponding quantum liquid state. Even going beyond mean-field theory by assuming a time dependent effective temperature, we find that T_{eff} saturates (at least for extremely long times) at a value $T_{\text{eff}}^{\infty} > T$. Finally we made connection to the dynamical approach for non-equilibrium quantum many body systems of Ref. 25, which shows that our theory properly takes into account the effects of aging and long term memory. We believe that the comparable simplicity of our approach allows it to apply our technique to a wide range of interesting problems in strongly interacting quantum systems.

ACKNOWLEDGMENTS

APPENDIX: SCHWINGER-KELDYSH FORMALISM

In this section we apply the Schwinger and Keldysh close-time path Green function formalism applied to quantum glasses without quenched disorder. We follow closely Cugliandolo and Lozano,[25] who applied the technique to spin glasses with quenched disorder. In this formalism response and correlation functions are treated as independent objects, coupled through a set of equations called Schwinger-Dyson equations. A perturbation theory scheme can be set up by considering the generalized matrix Green function

$$\widetilde{G}_{\mathbf{x},\mathbf{x}'}(t,t') = \begin{pmatrix} 0 & G_{\mathbf{x},\mathbf{x}'}^a(t,t') \\ G_{\mathbf{x},\mathbf{x}'}^r(t,t') & C_{\mathbf{x},\mathbf{x}'}(t,t') \end{pmatrix} \quad (A1)$$

and self-energy

$$\widetilde{\Sigma}_{\mathbf{x},\mathbf{x}'}(t,t') = \begin{pmatrix} 0 & \Sigma_{\mathbf{x},\mathbf{x}'}^a(t,t') \\ \Sigma_{\mathbf{x},\mathbf{x}'}^r(t,t') & \Sigma_{\mathbf{x},\mathbf{x}'}(t,t') \end{pmatrix},$$

where

$$C_{\mathbf{x},\mathbf{x}'}(t,t') = \frac{1}{2} \langle [\varphi_{\mathbf{x}}(t_w + \tau), \varphi_{\mathbf{x}'}(t_w)]_+ \rangle,$$

$$G_{\mathbf{x},\mathbf{x}'}^{r(a)}(t,t') = \mp i\theta(\pm(t-t')) \langle [\varphi_{\mathbf{x}}(t), \varphi_{\mathbf{x}'}(t')] \rangle.$$

In analogy to conventional perturbation theory, the components of $\widetilde{G}$ and $\widetilde{\Sigma}$ obey the Schwinger-Dyson equations

$$G_{\mathbf{q}}^r = G_{0\mathbf{q}}^r + G_{0\mathbf{q}}^r \otimes \Sigma_{\mathbf{q}}^r \otimes G_{\mathbf{q}}^r, \quad (A2)$$

$$C_{\mathbf{q}} = G_{\mathbf{q}}^r \otimes [G_{0\mathbf{q}}^{r-1} \otimes C_{0\mathbf{q}} \otimes G_{0\mathbf{q}}^{a-1} + \Sigma_{\mathbf{q}}] \otimes G_{\mathbf{q}}^a. \quad (A3)$$

We use $\otimes$ to distinguish the matrix product over time (the time convolution) where $A \otimes B(t,t') \equiv \int ds A(t,s) B(s,t')$ from the scalar (element-wise) product, where $AB(t,t') = A(t,t')B(t,t')$.

The first Schwinger-Dyson equation (A2) is similar to conventional perturbation theory. However it is coupled to the second equation of $C(t_1,t_2)$ which admits nontrivial solutions. For simplicity, we choose an initial condition such that $C_0 = 0$ and Im $G_0^r = 0$. This leads to thermalization of G^r (fulfillment of the FDT) in the absence of nonlinearities, so it is an appropriate condition. Let us call $t_2 = t_w$ and $t_1 = t_w + \tau$, i.e., the correlations between instants separated by τ are measured after the waiting time t_w has elapsed. The glassy dynamics appears in a regime of $t_w \to \infty$ and $\tau \to \infty$. To proceed we assume that in this limit all correlation functions can be decomposed into a slow, non-time-translation-invariant, aging part (AG) and a fast, time translation invariant, stationary part (ST) like in Eq. (13).

In this approach the stationary term decays on a characteristic time scale τ_β and it represents the correlations between degrees of freedom which are in equilibrium with the thermal reservoir at temperature T, i.e., retarded and Keldysh correlation functions are related by the FDT. The aging part, on the other hand, depends weakly on t_w (aging phenomena) and varies slowly on τ in a characteristic large time scale $\tau_\alpha \gg \tau_\beta$, which allows us to neglect its τ derivatives. Moreover, as in Ref. 25, we enforce a relationship between G_{AG}^r and C_{AG} by defining an effective temperature T_{eff} at which the long time correlations thermalize through a generalized FDT relation

$$G_{\text{AG}}^r(\tau,t) = -\frac{1}{T_{\text{eff}}(t)} \frac{\partial}{\partial t} C_{\text{AG}}(\tau,t). \quad (A4)$$

For $t \ll \tau_\alpha$ we can neglect the time dependence of T_{eff}, leading to a constant $T_{\text{eff}}(t) \to T_{\text{eff}}$.

Using the above assumptions, the first Schwinger-Dyson equation (A2) in the stationary regime can be solved by a Fourier transform, which gives

$$\mathcal{G}^r_{\text{ST}\mathbf{q}}(\omega) = \frac{1}{\mathcal{G}^{r-1}_{0\mathbf{q}}(\omega) - \Sigma^r_{\text{ST}}(\omega)}. \tag{A5}$$

Here we will explore solutions where $G^r_{\text{AG}}(\tau, t_w)$ is small but finite. This corresponds to the weak long term scenario where, even though $\lim_{\tau\to\infty}\lim_{t\to\infty}G^r_{\text{AG}}(\tau, t)\to 0$, the integral $\lim_{t\to\infty}\int_0^t dt' G^r_{\text{AG}}(\tau, t')$ is still finite. In other words, we consider that the system keeps a vanishingly small memory of what happened in the past which, when accumulated over long times, gives a finite contribution to the dynamics. Thus, following along the lines of Ref. 25, for the aging regime we get

$$G^r_{\text{AG}\,\mathbf{q}}(\tau, t_w) = \Sigma^r_{\text{AG}}(\tau, t_w)\mathcal{G}^r_{\text{ST}\,\mathbf{q}}(0)^2. \tag{A6}$$

Analogously, the second Schwinger-Dyson equation (A3) can be solved for the same two regimes, yielding

$$G^K_{\text{ST}\,\mathbf{q}}(\omega) = \Sigma^K_{\text{ST}}(\omega)\mathcal{G}^r_{\text{ST}\,\mathbf{q}}(\omega)^2 \tag{A7}$$

and

$$C_{\text{AG}\,\mathbf{q}}(\tau, t_w) = \frac{\Sigma_{\text{AG}}(\tau, t_w)\mathcal{G}^r_{\text{ST}\,\mathbf{q}}}{\mathcal{G}^r_{\text{ST}\,\mathbf{q}}(0)^{-1} - \dfrac{1}{T_{\text{eff}}}\Sigma^K_{\text{AG}}(\tau, t_w)}. \tag{A8}$$

Note that if we identify the real time correlation functions with the inter-replica correlation functions $\mathcal{F}$ and $\mathcal{G}$ as follows:

$$\lim_{\tau\to\infty}\ \lim_{t_w\to\infty} C_{\text{AG}}(\tau, t_w) \equiv \mathcal{F} \tag{A9}$$

$$\mathcal{G}^r_{\text{ST}}(\omega) \equiv \mathcal{G}(i\omega_n \to \omega + i\delta), \tag{A10}$$

we get that the Schwinger-Dyson equations (A5) and (A8) are exactly equivalent to the replica equations in 5, provided that the self energies in the two schemes are also equivalent, i.e., $\Sigma_{\text{AG}} = -\Sigma_{\mathcal{F}}$ and $\Sigma^r_{\text{ST}}(\omega) = \Sigma_{\mathcal{G}}(i\omega_n \to \omega + i\delta)$. To prove this last requirement, we apply the same one-loop self-consistent screening perturbative scheme for the real time DMFT self energies $\tilde{\Sigma}$. This gives

$$\tilde{\Sigma} = \begin{pmatrix} 0 & C\otimes D^a + D\otimes G^a \\ C\otimes D^r + D\otimes G^r & D^a\otimes G^a + D^r\otimes G^r + D\otimes C \end{pmatrix}$$

where dressed interactions are given by

$$D^r = u + u\Pi^r\otimes D^r, \tag{A11}$$

$$D = u\Pi^r\otimes D + u\Pi\otimes D^a, \tag{A12}$$

and the polarization bubbles are given by the elementwise products (no time or space convolutions)

$$\Pi^r = \gamma C G^r, \tag{A13}$$

$$\Pi = \gamma[(G^a)^2 + (G^r)^2 + C^2]. \tag{A14}$$

The stationary and aging contributions to the self energies and polarizations can be calculated analogously by using definitions (A13) and (A14) and separating the stationary from the aging contribution according to their asymptotic time behavior. This gives

$$\Sigma^r_{\text{ST}} = C_{\text{AG}}\otimes D^r_{\text{ST}} + D_{\text{AG}}\otimes G^r_{\text{ST}} + C_{\text{ST}}\otimes D^r_{\text{ST}} + D_{\text{ST}}\otimes G^r_{\text{ST}}, \tag{A15}$$

$$\Sigma_{\text{AG}} = D_{\text{AG}}\otimes G_{\text{AG}}, \tag{A16}$$

and

$$\Pi^r_{\text{ST}} = \gamma(C_{\text{AG}}G^r_{\text{ST}} + C_{\text{ST}}G^r_{\text{ST}}) \tag{A17}$$

and

$$\Pi_{\text{AG}} = \gamma C^2_{\text{AG}}. \tag{A18}$$

It is easy to verify accordingly that Π_{ST} and Π^r_{ST} are related by FDT at the temperature T and Π_{AG} and Π^r_{AG} are related by the classical FDT at a temperature T_{eff}. The last term in Eq. (A15) is exactly the Matsubara convolution on (6). The remaining terms are calculated as follows:

$$\mathcal{D}^r_{\text{ST}}(\omega) = \frac{1}{u^{-1} + \Pi^r_{\text{ST}}(\omega)}, \tag{A19}$$

$$D_{\text{AG}} = \frac{-\Pi_{\text{AG}}\mathcal{D}^r_{\text{ST}}(\omega = 0)}{\mathcal{D}^r_{\text{ST}}(\omega = 0)^{-1} + \dfrac{1}{T_{\text{eff}}}\Pi_{\text{AG}}}. \tag{A20}$$

Using relations (A9) and (A10) and performing the usual Matsubara sums, we get, $\Pi_{\mathcal{F}} = \Pi_{\text{AG}}$ and $\Pi_{\mathcal{G}}(i\omega_n \to \omega + i0) = \Pi^r_{\text{ST}}(\omega)$ which yields $D_{\text{AG}} = D_{\mathcal{F}}$ and $D_{\mathcal{G}}(i\omega_n \to \omega + i0) = \mathcal{D}^r_{\text{ST}}(\omega)$. Therefore, Eqs. (A19), (A20), (A15), and (A16) together prove that $\Sigma_{\text{AG}} = \Sigma_{\mathcal{F}}$ and $\Sigma^r_{\text{ST}}(\omega) = \Sigma_{\mathcal{G}}(i\omega_n \to \omega + i\delta)$, and that there is a complete connection between replica and Schwinger-Keldysh formalisms. There is still one more independent equation, namely, Eq. (A6), in the Schwinger-Keldysh formalism which bares no analog on the Dyson equations for the inter-replica correlation functions. However, if we integrate over $\mathbf{q}$ both sides of Eq. (A6) and substitute the definition $\Sigma^r_{\text{AG}} = C_{\text{AG}}\otimes D^r_{\text{AG}} + D_{\text{AG}}\otimes G^r_{\text{AG}}$ into Eq. (A6) with $D^r_{\text{AG}} = \Pi^r_{\text{AG}}\mathcal{D}^2_{\text{ST}}(0)$, we obtain

$$[2\mathcal{D}^2_{\text{ST}}(0)\Pi^K_{\text{AG}} - D^K_{\text{AG}}]\int\frac{d^dq}{(2\pi)^d}\mathcal{G}^2_{\text{ST}\,\mathbf{q}}(0) = 1, \tag{A21}$$

which is exactly the marginality condition of the replica approach. This proves our previous statement that the saddle point condition gives the dynamical behavior at the time scales $t_w \to \infty$ and $\tau \to \infty$ with $t_w \ll \tau_\alpha$. In this limit the effective temperature is not yet in equilibrium. In the opposite limit one cannot disregard the time dependence of $T_{eff}(t)$ and it is thus not possible to write a closed form like Eq. (A21). Nevertheless, the equilibrium approach based on the replica trick, together with the droplet relaxation picture enables us to access the behavior of $T_{eff}(t_w)$ in the time limit $t_w \gg \tau_\alpha$.

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