Crossing of two Coulomb blockade resonances

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We investigate theoretically the transport of noninteracting electrons through an Aharanov-Bohm (AB) interferometer with two quantum dots (QD's) embedded into its arms. In the Coulomb blockade regime, transport through each QD proceeds via a single resonance. The resonances are coupled through the arms of the AB device, but may also be coupled directly. In the framework of the Landauer-Büttiker approach, we present expressions for the scattering matrix which depend explicitly on the energies of the two resonances and on the AB phase. We pay particular attention to the crossing of the two resonances.

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I. INTRODUCTION

The crossing of two Coulomb blockade resonances was studied in two recent experiments.[1,2] In both cases, two quantum dots (QD's) were embedded into the arms of an Aharanov-Bohm (AB) interferometer. By changing the parameters of the experiment (various gate voltages and the magnetic flux through the AB device), it was possible to study the crossing properties of two isolated Coulomb blockade resonances, one each due to one of the two QD's. In the present paper we present a theoretical framework for the analysis of both experiments.

Figure 1 shows a schematic representation of both experiments. The AB ring contains the two QD's labeled QDL and QDR, where L and R stand for left and right, respectively. The QD's are separated by barriers from the rest of the AB device. The latter consists of two parts. In Fig. 1, the lower (upper) part is labeled 1 (2). Both parts are coupled to the outside world by a number of leads. In Fig. 1, this number is two (three) for part 1 (part 2). In our theoretical treatment, the number of leads coupled to each part will be arbitrary. Typically, one of the leads coupled to part 1 (part 2) serves as source (sink) for the electrons. While the two QD's are not coupled directly to each other in the first experiment,[1] such a coupling does exist in the second experiment.[2] This coupling is indicated schematically by the dotted horizontal line representing the wire connecting QDL and QDR. In Ref. 2 the strength of that coupling was controlled by a further gate. Figure 1 does not show the plunger gates that make it possible to control the energies of the Coulomb blockade resonances in either QD. Thereby it is possible to have the energies of both Coulomb blockade resonances coincide. Experimentally, such crossings are seen in three-dimensional plots of the conductance versus the plunger gate voltages V_{L} and V_{R} applied on QDL and QDR, respectively. Each Coulomb blockade resonance corresponds to a ridge. The ridges of resonances in QDL (QDR) run essentially parallel to V_{R} (V_{L}). The crossing of two such ridges marks the crossing of two Coulomb blockade resonances. The coincidence of two resonances also affects the interference pattern of the transmission of an electron through the AB device. This pattern depends upon the magnetic flux Φ through the device. The flux is due to a homogeneous magnetic field perpendicular to the plane of the drawing. We are interested in weak magnetic fields only. (We recall that for a complete AB oscillation, the magnetic-field strength typically changes at most by several ten mT.) Therefore, we take into account only the AB phase due to the magnetic flux and neglect the influence of the magnetic field on the orbital motion of the electron. Gauge invariance then allows us to link the AB phase to the passage of the electron through a particular part of the AB device. In the absence of a direct coupling between the two QD's (i.e., without the dotted line in Fig. 1), we choose the barrier separating QDL from part 1. Whenever the electron leaves (enters) QDL for part 1 (from part 1), it picks up the phase factor $\exp(2i\pi\Phi/\Phi_0)$ $[\exp(-2i\pi\Phi/\Phi_0)]$, where Φ_0 is the elementary flux quantum. For brevity, we write the phase factor as $\exp(i\phi)$. In the presence of a direct link between the two QD's, the topology of the AB interferometer changes from that of a ring to that of a figure eight, and we use a different convention in Sec. V.

In Sec. II we define the Hamiltonian for the system. In Sec. III we use the Landauer-Büttiker approach and present the generic form of the scattering matrix which describes the experimental setup of Ref. 1. In Sec. IV, this scattering matrix is analyzed especially with regard to the crossing of two Coulomb blockade resonances. In Sec. V, we generalize our treatment to include the setup of Ref. 2. In Sec. VI we list the approximations and summarize our approach and results. Moreover, we address some of the approximations made. In particular, we discuss the neglect of the mutual Coulomb interaction between the two electrons which are added to the system as the resonances become populated, and that of the

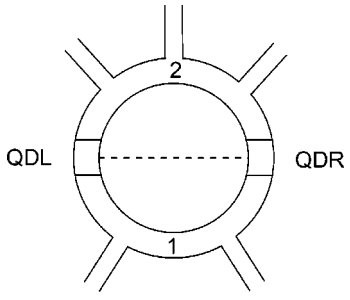

FIG. 1. Schematic representation of an AB interferometer with five external leads and two quantum dots labeled QDL and QDR embedded into its arms. The dotted line represents a link between the two quantum dots.

Coulomb interaction between each of these electrons and those on the dots. We also address the role of the spins of both quantum dots and of the two added electrons. Throughout the paper, we disregard temperature averaging for simplicity. Likewise, we disregard decoherence effects, although these are known to play some role in the actual experiments. We do so because part of the transport through the device is known to proceed coherently. Only this part will display a dependence on the AB phase. Moreover, decoherence has been thoroughly discussed in the literature, see, for instance, Ref. 3.

II. HAMILTONIAN

In defining the Hamiltonian of the system, we proceed in full analogy to Refs. 4 and 5. These papers addressed the AB phase for a single QD placed in one of the arms of an AB interferometer. (For a review of work on this problem, see Ref. 6.) We introduce fictitious barriers separating parts 1 and 2 of the AB device from the attached leads. Likewise, we consider parts 1 and 2 as separated from the two QD's. We impose boundary conditions on all these barriers such that as a result, we obtain self-adjoint single-particle Hamiltonians $H_{\rm lead}$ for the leads, H_1 and H_2 for the now separated parts 1 and 2, and $H_{\rm L}$ and $H_{\rm R}$ for the two QD's labeled QDL and QDR, respectively. Here $H_{\rm lead}$ possesses a continuous spectrum while the spectra of H_1, H_2, $H_{\rm L}$, and $H_{\rm R}$ are discrete. We label the leads attached to part 1 (part 2) by $s=1,\ldots,S$ $(t=1,\ldots,T)$. The transverse modes (channels) in lead s (t) are labeled $a=1,\ldots,N_s$ $(a=1,\ldots,N_t)$, and correspondingly for the creation and annihilation operators $c^\dagger$ and c. The associated energies are labeled ϵ. The eigenvalues of H_1 (H_2) are labeled E_{1j} (E_{2j}), with $j=1,\ldots,\infty$ and associated creation and annihilation operators $c_{1j}^\dagger$ $(c_{2j}^\dagger)$, and c_{1j} (c_{2j}). We assume that transport through either QD occurs in the Coulomb blockade regime where the intrinsic widths of individual resonances are small compared to their spacings. (The spacing includes, of course, the charging energy.) We also assume that the temperature is small in comparison with the spacings. Under these conditions, it is legitimate to assume that transport through either QD is dominated by a single Coulomb blockade resonance. We believe that this situation is met or nearly met in the experiments of Refs. 1 and 2. Thus, we admit only a single bound state with energy $E_{\rm L}$ $(E_{\rm R})$ in QDL (QDR), with associated creation and annihilation operators $d_{\rm L}^\dagger$ $(d_{\rm R}^\dagger)$ and $d_{\rm L}$ $(d_{\rm R})$. The energies $E_{\rm L}$ and $E_{\rm R}$ include the charging energies. Altogether, we have

$$H_{\rm lead}=\sum_{sa}\int d\epsilon\epsilon c_{sa}^\dagger(\epsilon)c_{sa}(\epsilon)+\sum_{ta}\int d\epsilon\,\epsilon c_{ta}^\dagger(\epsilon)c_{ta}(\epsilon),$$

$$H_1=\sum_j E_{1j}c_{1j}^\dagger c_{1j},$$

$$H_2=\sum_j E_{2j}c_{2j}^\dagger c_{2j},$$

$$H_{\rm L}=E_{\rm L}d_{\rm L}^\dagger d_{\rm L},$$

$$H_{\rm R}=E_{\rm R}d_{\rm R}^\dagger d_{\rm R}. \tag{1}$$

Hopping between the separate parts is induced by interaction terms containing tunneling matrix elements,

$$H_{\rm lead1}=\sum_{sa;j}\int d\epsilon[V_{sa;1j}(\epsilon)c_{sa}^\dagger(\epsilon)c_{1j}+{\rm H.c.}],$$

$$H_{\rm lead2}=\sum_{ta;j}\int d\epsilon[V_{ta;2j}(\epsilon)c_{ta}^\dagger(\epsilon)c_{2j}+{\rm H.c.}],$$

$$H_{1\rm L}=\sum_j\,(V_{1j;L}c_{1j}^\dagger d_{\rm L}+{\rm H.c.}),$$

$$H_{2\rm L}=\sum_j\,(V_{2j;L}c_{2j}^\dagger d_{\rm L}+{\rm H.c.}),$$

$$H_{1\rm R}=\sum_j\,(V_{1j;R}c_{1j}^\dagger d_{\rm R}+{\rm H.c.}),$$

$$H_{2\rm R}=\sum_j\,(V_{2j;R}c_{2j}^\dagger d_{\rm R}+{\rm H.c.}). \tag{2}$$

The direct coupling of QDL and QDR (dashed line in Fig. 1) is given by

$$H_{\rm LR}=V_{\rm LR}(d_{\rm L}^\dagger d_{\rm R}+{\rm H.c.}). \tag{3}$$

In the absence of any direct coupling between QDL and QDR $(V_{\rm LR}=0)$ we use gauge invariance to put the entire AB phase onto a single one of the barriers. Without loss of generality, we choose the barrier separating QDL and part 1. Then, all the matrices V in Eqs. (2) are real and symmetric except for $V_{1j;L}$ which obeys

$$V_{1j;L}\exp(-i\phi)=V_{L;1j}\exp(i\phi)=v_{1j;L} \tag{4}$$

with $v_{1j;L}$ real and symmetric. For $V_{\rm LR}\neq 0$, a modification is necessary and discussed in Sec. V below. The Hamiltonian H of the system is the sum of the terms defined by Eqs. (1)–(3). We have not considered the possibility of spin-orbit coupling on either QD.

We have been very explicit in the construction of H. The reason is that we wanted to show that H is a sum of single-particle Hamiltonians. This fact allows us to use the Landauer-Büttiker approach to describe transport through the system. The ensuing use of the scattering matrix enables us to display explicitly the phase dependence and energy dependence of the conductance coefficients. We have omitted the spins of, as well as any possible interaction between, the two electrons that will eventually populate the two resonances caused by $E_{\rm L}$ and $E_{\rm R}$. These points are taken up in Sec. VI.

III. SCATTERING MATRIX: RING TOPOLOGY

The transport through the device is described by the Landauer-Büttiker formula

$$I_s=\sum_{s'} G_{ss'}V_{s'}+\sum_{t} G_{st}V_t, s=1,\ldots,S,$$

$$I_t=\sum_{t'} G_{tt'}V_{t'}+\sum_{s} G_{ts}V_s, t=1,\ldots,T. \quad (5)$$

Here $I_s(I_t)$ is the current through lead s (lead t) and $V_s(V_t)$ is the voltage applied to that lead. The conductance coefficients $G_{ss'}$ are given by

$$G_{ss'}=\sum_{a=1}^{N_s}\sum_{a'=1}^{N_{s'}}[|S_{sa;s'a'}(E,\Phi)|^2-\delta_{ss'}] \quad (6)$$

and correspondingly for the index combinations $(st),(ts)$, and (tt'). The symbol $S_{sa;s'a'}(E,\Phi)$ denotes the element of the scattering matrix $S(E,\Phi)$ which connects channel a in lead s with channel a' in lead s' at energy E and magnetic flux Φ. Time-reversal symmetry requires the scattering matrix to obey the relation

$$S(E,\Phi)=S^T(E,-\Phi), \quad (7)$$

where T denotes the transpose.

The observable $\mathcal{O}$ describing a given experimental setup is determined by the experimental arrangement chosen (which of the leads are grounded, and in which of the leads is a current measured). For any such setup, $\mathcal{O}$ will be given as a rational function of the conductance coefficients G. This follows directly from Eq. (5). In order to present a general framework useful for the analysis of any such experiment, we focus attention on the scattering matrix S. With the help of the formulas for S given below, it is possible to work out the dependence of the G's and, hence, of $\mathcal{O}$ on the energies of the two Coulomb blockade resonances, and on the AB phase Φ.

It is possible to derive the form of S from the Hamiltonian H. This can be done along the lines of Refs. 4 and 7. We do not follow this course here because the explicit solution involves some lengthy algebra. Rather, we simply present the result which we believe to be intuitively obvious. In this and the following section, we focus attention on the ring topology and put $V_{\mathrm{LR}}=0$.

The scattering matrix S can be written as the product of three unitary matrices,

$$S(E,\Phi)=US^{(\mathrm{res})}(E,\Phi)U^T. \quad (8)$$

Without any coupling between each of the QD's and parts 1 and 2 of the AB device (this condition can be met experimentally by increasing the heights of the two barriers defining each QD), the resonant part $S^{(\mathrm{res})}(E,\Phi)$ is equal to the unit matrix, and $S(E,\Phi)$ is, thus, equal to UU^T. The form of the latter matrix follows from the observation that parts 1 and 2 are unlinked. A unitary scattering matrix $S^{(1)}$ ($S^{(2)}$) describes the nonresonant electron transport through unlinked part 1 (unlinked part 2). We assume that the energy dependence of both matrices is smooth over the energy interval defined by the widths of the two Coulomb blockade resonances introduced below. We accordingly neglect the energy dependences of both $S^{(1)}$ and $S^{(2)}$. Moreover, both matrices do not depend on the magnetic flux Φ, see the remark at the end of Sec. I. Time-reversal invariance then implies that both $S^{(1)}$ and $S^{(2)}$ are symmetric. Thus, we can write for $i=1,2$,

$$S^{(i)}=U^{(i)}[U^{(i)}]^T. \quad (9)$$

Equation (9) holds for every unitary and symmetric matrix. As explained in Refs. 8 and 5 the unitary transformation $U^{(i)}$ accomplishes the transformation from the space of physical channels to the space of eigenchannels. We accordingly write the matrices $U^{(i)}$ explicitly in the form $U^{(1)}_{sa;\alpha}$ and $U^{(2)}_{ta;\beta}$. Here $U^{(1)}_{sa;\alpha}$ is the product of an orthogonal matrix $O^{(1)}_{sa;\alpha}$ which diagonalizes the symmetric matrix $S^{(1)}$ and of a diagonal matrix with entries $\exp(i\delta^{(1)}_\alpha)$, where the $\delta^{(1)}_\alpha$'s are the eigenphaseshifts of $S^{(1)}$, and similarly for $S^{(2)}$. The index α (β) runs from 1 to N_1 (N_2). Here the total number of channels N_1 in part 1 (N_2 in part 2) is given by $N_1=\Sigma_s N_s$ ($N_2=\Sigma_t N_t$). The matrix U is defined in the total space of $N=N_1+N_2$ channels. It is block diagonal and given by

$$U=\begin{pmatrix} U^{(1)} & 0 \\ 0 & U^{(2)} \end{pmatrix}. \quad (10)$$

Inspection of Eq. (8) shows that for $S^{(\mathrm{res})}=\mathbf{1}_N$, the unit matrix in N dimensions, the scattering matrix S is block diagonal and consists of the two matrices $S^{(1)}$ and $S^{(2)}$, as it must.

It is now obvious that $S^{(\mathrm{res})}$ differs from the unit matrix by terms that represent the two Coulomb blockade resonances, one each in QDL and QDR. Moreover, it is also clear that $S^{(\mathrm{res})}$ is defined in the space of eigenchannels of both $S^{(1)}$ and $S^{(2)}$. In this space, the coupling matrix elements $W_{\rho\mathrm{P}}$ describing the hopping of an electron from the resonance in QDP (with $P=L$ or R) to the eigenchannel ρ (with $\rho=\alpha$ for part 1 and $\rho=\beta$ for part 2) can be shown[8] to be real, save for the AB phase. We accordingly have for $\rho=1,\ldots,N$ and $P=L,R$

$$W_{\mathrm{P}\rho}=W^*_{\mathrm{P}\rho}=W_{\rho\mathrm{P}} \text{ unless } P=L \text{ and } \rho=\alpha, \quad (11)$$

while

$$W_{\mathrm{L}\alpha}\exp(i\phi)=W_{\alpha\mathrm{L}}\exp(-i\phi)=w_{\mathrm{L}\alpha} \quad (12)$$

with $w_{\mathrm{L}\alpha}$ real. We note that the $W_{\mathrm{P}\rho}$'s differ from but are linear in the $V_{\mathrm{P};sa}$'s and $V_{\mathrm{P};ta}$'s introduced in Sec. II.

We can now express $S^{(\mathrm{res})}$ in terms of the matrix elements $W_{\mathrm{P}\rho}$, and of the energies E_{P} of the two Coulomb blockade resonances. The latter can be varied experimentally by changing the plunger gate voltage on either QD. We observe that the matrices $W_{\mathrm{P}\rho}$ map the space of N eigenchannels onto the space of the two Coulomb blockade resonances, and vice versa for $W_{\rho\mathrm{P}}$. The matrix $S^{(\mathrm{res})}$ takes the form

$$S^{(\mathrm{res})}_{\rho\rho'}=\delta_{\rho\rho'}-2i\pi\sum_{\mathrm{PP}'} W_{\rho\mathrm{P}}[D^{-1}]_{\mathrm{PP}'}W_{\mathrm{P}'\rho'}. \quad (13)$$

The 2×2 matrix $D_{\mathrm{PP}'}$ has the form ($P=L,R$)

$$D_{\mathrm{PP}'}=\delta_{\mathrm{PP}'}[E-E_{\mathrm{P}}]+i\pi\sum_{\rho} W_{\mathrm{P}\rho}W_{\rho\mathrm{P}'}. \quad (14)$$

Equations (6),(8),(13), and (14) constitute the central result of this section. It is easy to check that the scattering matrix defined by these equations is unitary and obeys Eq. (7).

For the benefit of the reader, we rewrite the S matrix in a form that displays more clearly the physical role of the matrices $U^{(i)}$ with $i=1,2$. We define the complex coupling matrix elements

$$\overline{W}_{sa;\mathrm{P}}=\sum_{\alpha} U^{(1)}_{sa;\alpha}W_{\alpha\mathrm{P}},$$

$$\overline{W}_{ta;\mathrm{P}}=\sum_{\beta} U^{(2)}_{ta;\beta}W_{\beta\mathrm{P}}. \tag{15}$$

Then S takes the form

$$S_{sa;s'a'}=S^{(1)}_{sa;s'a'}-2i\pi\sum_{\mathrm{PP}'} \overline{W}_{sa;\mathrm{P}}[D^{-1}]_{\mathrm{PP}'}\overline{W}_{\mathrm{P};s'a'},$$

$$S_{ta;t'a'}=S^{(2)}_{ta;t'a'}-2i\pi\sum_{\mathrm{PP}'} \overline{W}_{ta;\mathrm{P}}[D^{-1}]_{\mathrm{PP}'}\overline{W}_{\mathrm{P};t'a'},$$

$$S_{sa;ta'}=-2i\pi\sum_{\mathrm{PP}'} \overline{W}_{sa;\mathrm{P}}[D^{-1}]_{\mathrm{PP}'}\overline{W}_{\mathrm{P};ta'},$$

$$S_{ta;sa'}=-2i\pi\sum_{\mathrm{PP}'} \overline{W}_{ta;\mathrm{P}}[D^{-1}]_{\mathrm{PP}'}\overline{W}_{\mathrm{P};sa'}. \tag{16}$$

The matrix D has the same form as in Eq. (14), but can also be written as

$$D_{\mathrm{PP}'}=\delta_{\mathrm{PP}'}[E-E_{\mathrm{P}}]+i\pi\sum_{sa} \overline{W}_{\mathrm{P};sa}\overline{W}_{sa;\mathrm{P}'}$$

$$+i\pi\sum_{ta} \overline{W}_{\mathrm{P};ta}\overline{W}_{ta;\mathrm{P}'}. \tag{17}$$

Transformation (15) introduces complex matrix elements $\overline{W}$ which guarantee unitarity of S in the presence of the nondiagonal unitary matrices $S^{(1)}$ and $S^{(2)}$.

IV. ANALYSIS: CROSSING OF TWO RESONANCES

The effect of the two resonances that dominate the scattering matrix is contained entirely in the matrix D defined in Eq. (14). It is useful to display D in matrix form,

$$D=\begin{pmatrix} E-E_{\mathrm{L}}+(i/2)\Gamma_{\mathrm{L}} & (i/2)\Gamma_{\mathrm{LR}} \\ (i/2)\Gamma_{\mathrm{RL}} & E-E_{\mathrm{R}}+(i/2)\Gamma_{\mathrm{R}} \end{pmatrix}, \tag{18}$$

where

$$\Gamma_{\mathrm{L}}=2\pi\sum_{\rho} W_{\mathrm{L}\rho}W_{\rho\mathrm{L}},$$

$$\Gamma_{\mathrm{R}}=2\pi\sum_{\rho} W_{\mathrm{R}\rho}W_{\rho\mathrm{R}},$$

$$\Gamma_{\mathrm{LR}}=2\pi\sum_{\rho} W_{\mathrm{L}\rho}W_{\rho\mathrm{R}},$$

$$\Gamma_{\mathrm{RL}}=2\pi\sum_{\rho} W_{\mathrm{R}\rho}W_{\rho\mathrm{L}}. \tag{19}$$

Our explicit notation combined with Eqs. (11) and (12) shows that Γ_{L} and Γ_{R} are real, positive, and independent of the magnetic flux Φ, and that the only dependence on Φ occurs in Γ_{LR} and in Γ_{RL}. The latter two quantities are complex and related by

$$\Gamma_{\mathrm{LR}}=\Gamma^{*}_{\mathrm{RL}}. \tag{20}$$

We use Eqs. (18)–(20) to display the structure of certain elements of the scattering matrix S. We recall that S decays into two independent scattering matrices $S^{(1)}$ and $S^{(2)}$ whenever we have $W_{\mathrm{P}\rho}=0$ for all P, ρ. Parts 1 and 2 of the AB interferometer are linked only by the two Coulomb blockade resonances with energies E_{L} and E_{R}. If the two resonance energies are sufficiently different so that

$$|E_{\mathrm{L}}-E_{\mathrm{R}}|\gg\Gamma_{\mathrm{L}},\Gamma_{\mathrm{R}},|\Gamma_{\mathrm{LR}}|, \tag{21}$$

we can use perturbation theory in Γ_{LR} to invert D. Keeping only the lowest order terms in the expansion, we find

$$S_{tb;sa}=-2i\pi\sum_{\beta\alpha} U^{(2)}_{tb;\beta}\{W_{\beta\mathrm{L}}[E-E_{\mathrm{L}}+(i/2)\Gamma_{\mathrm{L}}]^{-1}W_{\mathrm{L}\alpha}$$

$$+W_{\beta\mathrm{R}}[E-E_{\mathrm{R}}+(i/2)\Gamma_{\mathrm{R}}]^{-1}W_{\mathrm{R}\alpha}\}U^{(1)}_{sa;\alpha}. \tag{22}$$

The two amplitudes on the right-hand side of Eq. (22) can be interpreted in terms of two paths of the electron on its way from part 1 to part 2. The electron may pass either through QDL (first term) or QDR (second term). As it passes through QDL, it picks up the AB phase contained in $W_{\mathrm{L}\alpha}$. This phase will affect the interference pattern due to the product of the amplitudes corresponding to the two paths. Whenever inequality (21) holds, the electron will not complete one or several loops within the AB ring as it passes from part 1 to part 2. It is instructive to consider also the terms of next order. These terms are given by

$$-2i\pi\sum_{\beta\alpha} U^{(2)}_{tb;\beta}\{W_{\beta\mathrm{L}}[E-E_{\mathrm{L}}+(i/2)\Gamma_{\mathrm{L}}]^{-1}\Gamma_{\mathrm{LR}}$$

$$\times[E-E_{\mathrm{R}}+(i/2)\Gamma_{\mathrm{R}}]^{-1}W_{\mathrm{R}\alpha}+W_{\beta\mathrm{R}}[E-E_{\mathrm{R}}$$

$$+(i/2)\Gamma_{\mathrm{R}}]^{-1}\Gamma_{\mathrm{RL}}$$

$$\times[E-E_{\mathrm{L}}+(i/2)\Gamma_{\mathrm{L}}]^{-1}W_{\mathrm{L}\alpha}\}U^{(1)}_{sa;\alpha}. \tag{23}$$

The path associated with the first amplitude leads the electron first through QDR and then through QDL, and vice versa for the second amplitude. With Γ_{LR} given by Eq. (19), we see that along the first path QDL can be reached from QDR either via part 1 or via part 2, and correspondingly for path 2. In the first (second) case, the AB phase does not (does) contribute to the scattering amplitude. This is correct because only in the second case does the electron complete a loop around the AB ring. A similar analysis of $S_{sa;s'a'}$ and of

$S_{tb;t'b'}$ shows that whenever inequality (21) holds, the scattering is dominated by $S^{(1)}$ and $S^{(2)}$, respectively. The consecutive passage through both Coulomb blockade resonances is strongly inhibited. We conclude that inequality (21) defines a fairly uninteresting regime of parameters of the problem.

Interest, therefore, focusses on the regime where this inequality does not hold and where our perturbation expansion is not appropriate. This is the regime where the two Coulomb blockade resonances may cross. We shall see that the crossing displays important features. Prior to calculating the exact result, it is useful to visualize the outcome in terms of a perturbation expansion in powers of Γ_{LR} and Γ_{RL}. This expansion generates terms of the same form as in formula (23), but of higher order in Γ_{LR} and Γ_{RL}. Each propagator $[E-E_{\mathrm{P}}+(i/2)\Gamma_{\mathrm{P}}]^{-1}$ occurring in the expansion signals a visit of the associated path to QDP with $P=L,R$. The intermittent factors Γ_{LR} and Γ_{RL} signal passage of the electron from QDR to QDL and vice versa. The passage may proceed via part 1 or part 2. Thus, the perturbation series stands for the infinite number of possibilities to connect the channels defined by the indices of the scattering matrix, by paths. These paths may loop around the AB ring a number of times, then change direction, loop again, change direction back etc. until the electron leaves the AB ring. The AB phase picked up by the electron is the sum of all such phases picked up in the individual loops and given in terms of the total number of completed counterclockwise loops minus the total number of completed clockwise loops. We are about to calculate the form of the scattering matrix by diagonalizing the matrix D. This procedure amounts to summing over all the paths just mentioned. This is why the AB phase will show up in the denominator of the result, see Eq. (27). (Experimentally, decoherence will actually limit the number of loops that contribute significantly to the amplitude, see Ref. 1).

We simplify the algebra by considering an AB ring that contains two perfectly identical QD's and which itself is perfectly symmetric about a vertical axis through the middle of Fig. 1. Then, $\Gamma_{\mathrm{L}}=\Gamma_{\mathrm{R}}=\Gamma$ (this defines the width Γ). We write the complex eigenvalues of the matrix D in the form $E-\varepsilon_i$ with $i=1,2$. Then

$$\varepsilon_{1,2}=\frac{1}{2}(E_{\mathrm{L}}+E_{\mathrm{R}}-i\Gamma)\pm\frac{1}{2}\sqrt{(E_{\mathrm{L}}-E_{\mathrm{R}})^2-|\Gamma_{\mathrm{LR}}|^2}. \quad (24)$$

Let us suppose that we change the resonance energies of both dots in such a way that $(E_{\mathrm{L}}+E_{\mathrm{R}})$ is kept fixed while $u=|E_{\mathrm{L}}-E_{\mathrm{R}}|$ decreases monotonically from an initially large value [in the sense of inequality (21)]. Then, the difference $|\varepsilon_1-\varepsilon_2|$ also decreases monotonically. Both resonances approach each other, retaining equal widths. The difference of resonance energies vanishes when $(E_{\mathrm{L}}-E_{\mathrm{R}})^2=|\Gamma_{\mathrm{LR}}|^2$: The two resonances coincide in energy and width. We deal with an exceptional point in the sense of Ref. 9. At this point, the system possesses only a single eigenfunction. As we decrease u further, the two resonances separate, retain equal resonance energies but acquire different widths. At $u=0$, the widths differ by $|\Gamma_{\mathrm{LR}}|$, the maximum amount possible.

The value of $|\Gamma_{\mathrm{LR}}|$ determines both, the value of u where the resonances coincide and the maximum difference of their widths. This value depends upon the AB phase Φ. Indeed, from Eq. (19) we have

$$|\Gamma_{\mathrm{LR}}|^2=4\pi^2\left[\left(\sum_\alpha w_{\mathrm{L}\alpha}w_{\alpha\mathrm{R}}\right)^2+\left(\sum_\beta W_{\mathrm{L}\beta}W_{\beta\mathrm{R}}\right)^2 + 2\cos\phi\sum_\alpha w_{\mathrm{L}\alpha}w_{\alpha\mathrm{R}}\sum_\beta W_{\mathrm{L}\beta}W_{\beta\mathrm{R}}\right]. \quad (25)$$

The value of $|\Gamma_{\mathrm{LR}}|^2$ oscillates periodically with magnetic flux Φ between the maximum value $4\pi^2(\Sigma_\alpha w_{\mathrm{L}\alpha}w_{\alpha\mathrm{R}}+\Sigma_\beta W_{\mathrm{L}\beta}W_{\beta\mathrm{R}})^2$ and the minimum value $4\pi^2(\Sigma_\alpha w_{\mathrm{L}\alpha}w_{\alpha\mathrm{R}}-\Sigma_\beta W_{\mathrm{L}\beta}W_{\beta\mathrm{R}})^2$. From Schwarz's inequality we conclude that the widths of the two resonances are always positive. To estimate the relative size of Γ and of $|\Gamma_{\mathrm{LR}}|$, we note from Eq. (19) that $\Gamma=\Gamma_{\mathrm{L}}=\Gamma_{\mathrm{R}}$ is a sum of squares while Γ_{LR} is a sum over terms that, aside from the AB phase, may have either sign. We expect that due to impurity scattering in parts 1 and 2 of the AB device, the $W_{\mathrm{P}\alpha}$'s are Gaussian random variables, see Ref. 5. As a consequence, we have $\Gamma\propto N$ while $|\Gamma_{\mathrm{LR}}|$ fluctuates strongly with a root-mean-square variance that grows like $\sqrt{N}$. Thus, the maximum difference of the widths of the two resonances is expected to be of the order of $\Gamma/\sqrt{N}$.

The possibility of complete coalescence of two resonances displayed above is a phenomenon that is opposite to the well-known Wigner-von Neumann level repulsion effect for bound states. The latter occurs whenever two bound states interact via a Hermitian interaction. Equation (18) shows that in the present case, we deal with resonances with complex resonance energies to begin with, and with a coupling that is due to a Hermitian interaction multiplied by i, the imaginary unit. This unusual form of interaction occurs because the two resonances are not coupled directly but via the open channels in parts 1 and 2. Both differences contribute towards a behavior that differs from standard Wigner–von Neumann level repulsion. Such behavior has been discussed previously in the literature. To the best of our knowledge, the coupling of two resonances was first studied explicitly by von Brentano *et al.*[10] in the context of nuclear physics. This work was followed by an experimental investigation.[11] Related work was published in Ref. 12. Recent work[13] has focussed on the properties of exceptional points.

To display the features of the exceptional point where $(E_{\mathrm{R}}-E_{\mathrm{L}})^2=|\Gamma_{\mathrm{LR}}|^2$ and where $\Gamma_{\mathrm{L}}=\Gamma_{\mathrm{R}}$, we consider two slightly asymmetric QD's for which the two resonance widths Γ_{L} and Γ_{R} are not exactly equal. Then, the eigenvalues $\varepsilon_{1,2}$ will never coincide exactly. This is seen from the expression of the discriminant which now has the value

$$\sqrt{[E_{\mathrm{L}}-E_{\mathrm{R}}-(i/2)(\Gamma_{\mathrm{L}}-\Gamma_{\mathrm{R}})]^2-|\Gamma_{\mathrm{LR}}|^2}. \quad (26)$$

Imagine now a change of the parameters of the system in such a way that the argument of the square root describes a closed loop in the complex plane around the exceptional point. This could be achieved as follows. We put $E_{\mathrm{R}}=-E_{\mathrm{L}}=|\Gamma_{\mathrm{LR}}|+\alpha/2, \Gamma_{\mathrm{L}}=-\Gamma_{\mathrm{R}}=\beta$ with α,β real and $|\alpha|$

$\ll|\Gamma_{LR}|, |\beta| \ll |\Gamma_{LR}|$. The discriminant becomes approximately equal to $\sqrt{2|\Gamma_{LR}|(\alpha - i\beta)}$. Changing α from a small negative to a small positive value while keeping $\beta > 0$ fixed and small, then keeping α fixed and changing β from its small positive value to a small negative one, then keeping β fixed and changing α back to its original value and doing, finally, the same for β yields a rectangle in the complex plane with the exceptional point in its interior. While under this operation the phase of $(\alpha - i\beta)$ changes by 2π, the phase of $\sqrt{2|\Gamma_{LR}|(\alpha - i\beta)}$ changes only by π: Under this operation, the two eigenvalues $\varepsilon_{1,2}$ are interchanged, and so are the two eigenfunctions, including an additional phase factor.[13] In comparison with the work of Refs. 13, the present system seems to offer an additional degree of freedom in terms of the AB phase. The latter determines the value of $|\Gamma_{LR}|$, the location of the exceptional point, and the form of the two eigenfunctions as linear combinations of the two QD states.

Unfortunately, all these appealing features have no bearing on the properties of the scattering matrix S. This is because the energy E is always real. As a consequence, we can never reach the exceptional point, and the determinant of the matrix D never vanishes for real values of E. The two eigenfunctions of D remain distinct. In view of the recent interest in exceptional points, we have, nevertheless, felt that a discussion of this topic is appropriate in the present context.

We return to the symmetric case. The matrix D can be diagonalized by a matrix A so that $D = A^{-1}(E\mathbf{1}_2 - \varepsilon)A$ where ε denotes the diagonal matrix $\mathrm{diag}(\varepsilon_1, \varepsilon_2)$. Using this form in Eq. (13), we obtain

$$S^{(\mathrm{res})}_{\rho\rho'} = \delta_{\rho\rho'} - 2i\pi \sum_{\mathrm{PP_1P'}} W_{\rho\mathrm{P}} A^{-1}_{PP_1} [E - \varepsilon_{P_1}]^{-1} A_{P_1P'} W_{\mathrm{P'}\rho'} . \tag{27}$$

Inserting this matrix into Eq. (6) yields the conductance coefficients and, hence, the dependence of any observable on the AB phase. The AB phase appears explicitly not only in the eigenvalues $\varepsilon_{1,2}$ but also in the matrix A and, of course, in some of the $W_{\rho\mathrm{P}}$'s. The matrix A can easily be calculated. Details are not given here.

For the sake of completeness, we discuss the limitations of a two-lead experiment. These limitations have played a role in previous studies of AB devices.[6] We recall that the scattering matrix S is unitary and obeys $S^T(E, -\Phi) = S(E, \Phi)$. It follows that, in general, we have $S_{sa;s'a'}(E, -\Phi) = S_{s'a';sa}(E, \Phi)$ and correspondingly for the lead indices (s,t) and (t,t'). The cases where the two lead indices coincide are special and yield $S_{sa;sa'}(E, -\Phi) = S_{sa';sa}(E, \Phi)$ and $S_{ta;ta'}(E, -\Phi) = S_{ta';ta}(E, \Phi)$. For the conductance coefficients, this means that $G_{ss'}(E, -\Phi) = G_{s's}(E, \Phi)$, $G_{tt'}(E, -\Phi) = G_{t't}(E, \Phi)$, and $G_{st}(E, -\Phi) = G_{ts}(E, \Phi)$, while $G_{ss}(E, -\Phi) = G_{ss}(E, \Phi)$ and $G_{tt}(E, -\Phi) = G_{tt}(E, \Phi)$. Unitarity then shows that for a two-lead experiment the G's are even in Φ, while this is not the case for the off-diagonal G's when we deal with more than two leads. This conclusion, first drawn by Büttiker,[14] is seen to be quite general and not affected by the topology of our AB ring with two QD's. A simplified two-lead situation was studied in Ref. 17.

V. FIGURE-EIGHT TOPOLOGY

If the two QD's are connected by a wire, the topology differs from that of a ring analyzed so far. The necessary modifications are quite straightforward, however. The AB ring is divided into two parts by the wire connecting the two QD's. Let Φ_1 (Φ_2) be the flux through the lower (upper) part. In a manner completely analogous to Eqs. (11) and (12), we put $\Phi_1(\Phi_2)$ onto $W_{\alpha\mathrm{L}}$ ($W_{\beta\mathrm{L}}$). With $\phi_i = 2\pi\Phi_i/\Phi_0$, $i = 1,2$, Eqs. (11) and (12) are thus replaced by

$$W_{\mathrm{R}\rho} = W^*_{\mathrm{R}\rho} = W_{\rho\mathrm{R}} , \tag{28}$$

while

$$W_{\mathrm{L}\alpha}\exp(i\phi_1) = W_{\alpha\mathrm{L}}\exp(-i\phi_1) = w_{\mathrm{L}\alpha} ,$$

$$W_{\mathrm{L}\beta}\exp(-i\phi_2) = W_{\beta\mathrm{L}}\exp(i\phi_2) = w_{\mathrm{L}\beta} \tag{29}$$

with $w_{\mathrm{L}\rho}$ real. A further modification accounts for the presence of the wire which furnishes a direct link between the two QD's. We represent this link by a real hopping matrix element $V_{\mathrm{LR}} = V_{\mathrm{RL}}$. This element appears in the matrix D which now takes the form

$$D_{\mathrm{PP'}} = \delta_{\mathrm{PP'}}[E - E_{\mathrm{P}}] + i\pi \sum_{\rho} W_{\mathrm{P}\rho} W_{\rho\mathrm{P'}} + (1 - \delta_{\mathrm{PP'}}) V_{\mathrm{RL}} . \tag{30}$$

Except for these modifications, all formulas in Sec. III remain unchanged.

For a discussion of the form of the matrix D in Eq. (30), we distinguish two limiting cases, where $|\Gamma_{\mathrm{LR}}|$ dominates $|V_{\mathrm{RL}}|$ or vice versa. It is obvious that for $|V_{\mathrm{RL}}| \ll |\Gamma_{\mathrm{LR}}|$ we (approximately) retrieve our previous results since $\phi_1 + \phi_2 = \phi$. The distribution of the AB phase over two sets of matrix elements only complicates the notation. Therefore, the interesting limiting case is the one where $|V_{\mathrm{RL}}| \gg |\Gamma_{\mathrm{LR}}|$. We neglect Γ_{LR} in comparison with V_{RL} and consider again the symmetric case with $\Gamma_{\mathrm{L}} = \Gamma_{\mathrm{R}} = \Gamma$. Explicitly, the matrix D is given by

$$D = \begin{pmatrix} E - E_{\mathrm{L}} + (i/2)\Gamma & V_{\mathrm{LR}} \\ V_{\mathrm{RL}} & E - E_{\mathrm{R}} + (i/2)\Gamma \end{pmatrix} . \tag{31}$$

The matrix D does not depend upon the AB phase [which now appears only in the matrix elements W in Eq. (13)]. Moreover, the interaction V_{LR} causes standard level repulsion between the two resonances. The AB phase dependence of the conductance coefficients becomes complicated not because of the matrix D but because the electron may traverse several different paths on its way from the entrance channel to the exit channel. For instance, if the source (sink) is located in part 1 (part 2) of the AB device, there are four possible paths. One enters and leaves QDL, one enters and

leaves QDR, one enters QDL but leaves QDR, and one enters QDR and leaves QDL. The relative weight of the four contributions depends upon the eigenvectors and eigenvalues of the matrix *D*. Again, these can be worked out straightforwardly.

Corrections to these limiting cases can easily be calculated in terms of a power-series expansion in V_{LR}, or in Γ_{LR} and Γ_{RL}. A full diagonalization of the matrix *D* in Eq. (30) is also possible, covers all the intermediary cases, and yields interesting results. The eigenvalues $\varepsilon_{1,2}$ are given by

$$\varepsilon_{1,2}=\frac{1}{2}[E_{\mathrm{L}}+E_{\mathrm{R}}-(i/2)(\Gamma_{\mathrm{L}}+\Gamma_{\mathrm{R}})]\pm\frac{1}{2}\sqrt{[E_{\mathrm{L}}-E_{\mathrm{R}}-(i/2)(\Gamma_{\mathrm{L}}-\Gamma_{\mathrm{R}})]^2+4[V_{\mathrm{LR}}+(i/2)\Gamma_{\mathrm{LR}}][V_{\mathrm{RL}}+(i/2)\Gamma_{\mathrm{RL}}]}. \tag{32}$$

The eigenvalues coincide whenever the argument of the square root vanishes, i.e., whenever

$$[E_{\mathrm{L}}-E_{\mathrm{R}}-(i/2)(\Gamma_{\mathrm{L}}-\Gamma_{\mathrm{R}})]^2=-4[V_{\mathrm{LR}}+(i/2)\Gamma_{\mathrm{LR}}]\times[V_{\mathrm{RL}}+(i/2)\Gamma_{\mathrm{RL}}]. \tag{33}$$

Equation (33) extends the definition of an exceptional point to the figure-eight topology. We note that the right-hand side of Eq. (33) is a periodic function of ϕ.

VI. SUMMARY AND DISCUSSION

We have presented a very general approach to the transport properties of an AB device containing two QD's. Our main assumptions are the following.

(i) The electrons do not interact. Then, we can use the Landauer-Büttiker approach and express every observable in terms of the conductance coefficients *G*. The latter are given as squares of the elements of the scattering matrix *S*.

(ii) For the description of the the two Coulomb blockade resonances, we use the single-level approximation.

(iii) The only relevant energy dependence of *S* is due to the two Coulomb blockade resonances, one in either QD. Then, scattering in parts 1 and 2 of the AB device is independent of energy, and the scattering matrix *S* attains the form of Eq. (8), with $U^{(1)}$ and $U^{(2)}$ independent of energy and AB phase.

Under these assumptions, we have presented a comprehensive description of an AB device with the topology of a ring or of a figure eight. In particular, we have displayed explicitly the dependence of the *S* matrix upon energy and AB phase. We have shown that an unfamiliar situation arises in the case of a ring topology. Here the two resonances (with complex energies) are coupled via the channels in part 1 and part 2 of the AB device. This coupling is given by a Hermitian matrix multiplied by the imaginary unit *i*. This case differs fundamentally from the standard coupling of two bound states by a Hermitian interaction. The latter case leads to level repulsion, the former may lead to coalescence of levels. It seems that this phenomenon has been observed in Ref. 1.

We now address the approximations we have made. Perhaps most importantly, we have neglected the Coulomb interaction between the two electrons populating the two QD's, and that between each of these and the electrons on either QD. Inclusion of the Coulomb interaction would make it impossible to use the Landauer-Büttiker approach as we have done. Alternatives are discussed in a recent review.[15] The standard procedure employs rate equations for the occupation probabilities of the single-particle levels. However, this approach is manifestly unsuited to deal with phase correlations between scattering amplitudes. The latter are of central importance for an AB device. A more elaborate approach[15] uses a description in terms of an effective Hamiltonian. This approach assumes that the single-particle states in the QD are described by random-matrix theory. The effective Hamiltonian for the isolated QD is obtained as the leading term in a systematic expansion in inverse powers of *g*, the dimensionless conductance. The two-terminal conductance is then obtained from the Kubo formula and another effective Hamiltonian that includes the coupling to the leads. The latter is determined via a nontrivial theoretical derivation which in turn involves approximations. To the best of our knowledge, this approach has never been used for a multiterminal device involving an AB ring. Therefore, it is not known whether the approach is able to account for the phases that are relevant for the present system. At sufficiently low temperatures, the Coulomb interaction leads to Kondo-like effects in QD's. Remarkably, the calculation of the phase of a QD embedded in an AB ring has recently been worked out in the Kondo regime,[16] in spite of the difficulties just mentioned to deal with the Coulomb interaction outside this regime.

In view of this situation, we can only offer a few qualitative remarks in support of the present approach. First, the Coulomb interaction has likewise been neglected in Refs. 5,4 and 6, which addressed the AB phase for a single QD embedded in an AB ring. The results offered what seems a realistic and useful description of the overall phase dependence of experimental observables. Second, our use of the single-level approximation for each QD lends greater plausibility to the inclusion of the charging energy in the definition of the energies labeled E_{P}. We admit, however, that the Coulomb energy between the two electrons (one on each QD) is not covered by this argument. Our neglect of the Coulomb interaction is not restricted to the neglect of the charging energy. We have likewise neglected the spin-dependent interaction between electrons. The latter is induced via the exchange term and lifts the degeneracy between singlet and triplet states.[3,15] This spin-dependent interaction plays a prominent role in Kondo-type effects. We expect that this is

likewise the case in the present situation, especially when the two resonances overlap. Therefore, our approach can only be expected to work above the Kondo temperature.

We believe that our other approximations are less severe. The single-level approximation should work at and near an isolated Coulomb blockade resonance whenever resonance width and temperature are small compared to the charging energy. The neglect of all other energy dependence but that due to the resonances in the scattering matrix should be excellent, barring very special circumstances.

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