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Self-affine-like magnetoconductance fluctuations: Quasiperiodicity with a Weierstrass-like spectrum

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Using Landauer-Büttiker formula, conductance fluctuations are approximated semiclassically as a summation over periodic orbits that are well coupled to the entrance opening. This formula is valid for open quantum dots with soft-walled boundary everywhere. Conductance fluctuations are therefore in general quasiperiodic whose frequencies correspond to a few shortest periodic orbits that are well coupled to the entrance opening. Applying this formula to open billiard with Barbanis type potential, we will show that the recently observed exact self-affine-like magnetoconductance fluctuations in soft-walled billiards are special cases of quasiperiodicity, namely quasiperiodicity with Weierstrass-like spectrum. We will show that they are the fingerprints of self-similar periodic orbits which cluster in the vicinity of the entrance opening.

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The phenomena of fractal-like magnetoconductance fluctuations in soft-walled quantum dots have attracted many scientific efforts both theoretically and experimentally. It was first predicted by Ketzmerick[1] in 1996 that using semiclassical Landauer-Büttiker formula, the self-similar structure in classical phase space should reveal fractal-like conductance fluctuations as a function of systems parameters (magnetic field, Fermi energy, etc) in soft-walled quantum dots. He argued that the power law for the dwelling time probability in the corresponding classical open systems will cause the fluctuations of the conductance to behave as fractional Brownian fluctuations, the dimension of which can be calculated using the information of the exponent of the power law. A couple of years later, laboratory experiments and numerical simulations confirmed the predictions in the sense that fractal-like magnetoconductance does really exist.[2,3] Yet, there is no evidence on fractal-like fluctuations against other parameters so far. The difficulty to measure the exponent of dwelling time power law in laboratory experiment makes it difficult to check the relation between the dimension of the fractal-like fluctuations and the exponent. In addition, since the theory is based on statistical analysis, it is difficult to explain the interesting phenomena of exact self-affine-like magnetoconductance fluctuations observed in open Sinai quantum dots[3–5] and in dots array.[6] To the latter issue, the authors of the present paper, using semiclassical Kubo formula for conductivity,[7] have suggested a picture that relates the exact self-affine-like fluctuations to self-similar (in configuration space) periodic orbits in the corresponding classical systems.[8]

Landauer-Büttiker formula has been the common way to calculate the conductance of quantum dots. This is because the Landauer-Büttiker formalism provides a clear picture for the treatment of the openings that connect the quantum cavity to the large electronic reservoirs. In contrast to the experiments which are done in small number of the lowest modes, the semiclassical Landauer-Büttiker formula used in Ref.1 is valid in the limit of large mode number.

In this present Rapid Communication, we consider a chaotic quantum dot connected to large reservoirs of source and drain via quantum point contacts (QPC's) with effective longitudinal length L. In real experiment QPC's are soft walled. We therefore assume that the transversal profile of the lower part of the QPC's can be approximated as harmonic potential: $V(y)=V_0+\frac{1}{2}m\omega^2y^2$, where m is the effective mass of the electron.[9] We realize that experiments are done in large potential curvature ω where the ground state is a very narrow Gaussian wave packet: $\phi_0(y)=[1/(\lambda\sqrt{\pi})^{1/2}]e^{-y^2/2\lambda^2}$, whose width $\lambda=\sqrt{\hbar/m\omega}$ is typically much smaller than the width of the QPC's.[9] Noting this experimental fact, we calculate the lowest mode reflection coefficient R_{00} of the open system semiclassically. Conductance is then related to the reflection coefficient via the Landauer-Büttiker formula. Assuming that only the lowest mode contributes to the transport, we will show that the leading fluctuating part can be approximated semiclassically as a summation over periodic orbits that are well coupled to the entrance opening. Applying this formula to open cavity with Barbanis type potential, we will explain that the recently observed self-affine-like conductance fluctuations are the fingerprints of self-similar periodic orbits that are well coupled to the entrance opening. As will be clear later, this approach is essentially different from the one using semiclassical Kubo formula, since it emphasizes the important role played by the nature of the couplings between the cavity and the reservoirs.

The Landauer-Büttiker formula relates dimensionless conductance g in linear response regime to lowest mode reflection coefficient R_{00} as follows:[10]

$$g=1-\int dER_{00}(E)\left(-\frac{df_0}{dE}\right), \tag{1}$$

where we have employed current conservation rule. $f_0(E)=1/\{1+\exp[(E-E_F)/k_BT]\}$ is the Fermi distribution at temperature T and Fermi energy E_F. The lowest mode reflection coefficient is equal to the absolute square of the reflection amplitude $R_{00}=|r_{00}|^2$, where r_{00} at energy E is given by the projection of the retarded Green function $G(y';y;E)$ onto the transverse wave function in the entrance QPC, the incoming and outgoing lowest modes ϕ_0:

$$r_{00}=-1+i\hbar v_0\int_C dy'\int_C dy\,\phi_0^*(y')\phi_0(y)G(y',x';y,x;E). \tag{2}$$

Here $v_0=\hbar k_0/m$, where $k_0=(1/\hbar)\sqrt{2m(E-\hbar\omega/2)}$ is the longitudinal wave number of the ground state in the QPC. The integrations in Eq. (2) take place at the transverse cross sections C on the entrance QPC. Next, we have to consider the fact that for a soft-walled open billiard, the transport properties are insensitive to the positions of boundary line between the reservoirs and the cavity, namely C. To this end, we should perform proper averaging of the location of the initial and final points over the characteristic longitudinal width L of entrance QPC. Futhermore, note that for sufficiently small value of λ, the following relation holds:

$$\phi_0(\xi)\phi_0^*(\xi')\approx\sqrt{2}\,\delta(\xi-\xi')e^{-\xi\xi'/\lambda^2}. \tag{3}$$

Keeping these facts in mind, for sufficiently small $\hbar$, the leading part of reflection coefficient can thus be written as follows:

$$R_{00}=2\hbar^2v_0^2\int_C dy\int_C dy'\,e^{-(y^2+y'^2)/\lambda^2}\times\langle G(y',x';y,x;E)G^*(y',x';y,x;E)\rangle, \tag{4}$$

where the ensemble average is defined as

$$\langle\cdots\rangle=\frac{1}{L^2}\int_{-L/2}^{L/2}dx\int_{-L/2}^{L/2}dx'\cdots. \tag{5}$$

The semiclassical approximation to the transmission proceeds by replacing the Green function G by its semiclassical path-integral expression,[11]

$$G_{sc}(\mathbf{r}';\mathbf{r};E)=\frac{2\pi}{(2\pi i\hbar)^{3/2}}\sum_{s(\mathbf{r}';\mathbf{r})}\sqrt{D_s}\,e^{(i/\hbar)S_s(\mathbf{r}';\mathbf{r};E)-i(\pi/2)\mu_s}, \tag{6}$$

given as a sum over classical trajectories s between points $\mathbf{r}$ and $\mathbf{r}'$ of the entrance cross section. S_s is the action integral along the trajectory s.

$$D_s=\frac{1}{|\dot{q}\dot{q}'|}\frac{\partial^2S_s}{dq_\perp dq'_\perp}\Big|_{q_\perp=q'_\perp=0},$$

where $\dot{q}$ and $\dot{q}'$ are the initial and final velocities along the classical trajectory at entrance opening. The phase index μ_s is given by the number of constant-energy conjugate points.

Substituting the semiclassical Green function (6) into Eq. (4) and evaluating the integration using stationary phase integration, the stationary point conditions are satisfied by pairs of trajectories s and t that have the same initial and final momentum at entrance QPC:

$$\frac{\partial S_s}{\partial\mathbf{r}}\bigg|_{\mathbf{r}=\mathbf{r}_p}-\frac{\partial S_t}{\partial\mathbf{r}}\bigg|_{\mathbf{r}=\mathbf{r}_p}=-\mathbf{p}_s+\mathbf{p}_t=\mathbf{0}, \tag{7}$$

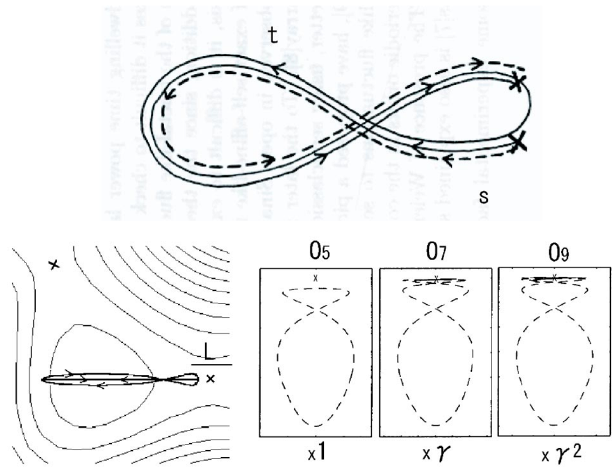

FIG. 1. A pair of nonidentical trajectories $s\neq t$ which satisty the stationary phase conditions (upper); the Barbanis potential (lower left) with two shortest periodic orbits couple well to the entrance lead: a straight-line librating periodic orbit A which oscillates between the wall and the saddle right at the center of entrance QPC, and a rotating periodic orbit O_5; The first three rotating children of A: O_5,O_7,O_9 (lower right). The transversal coordinates of O_7 and O_9 are rescaled by γ and γ^2, respectively. See text for details.

$$\frac{\partial S_s}{\partial\mathbf{r}'}\bigg|_{\mathbf{r}'=\mathbf{r}'_p}-\frac{\partial S_t}{\partial\mathbf{r}'}\bigg|_{\mathbf{r}'=\mathbf{r}'_p}=\mathbf{p}'_s-\mathbf{p}'_t=\mathbf{0}. \tag{8}$$

There are two kinds of pairs of trajectories that satisfy the above conditions. One is pairs of identical trajectories: $s=t$. Contribution from these trajectories leads to the classical part of the reflection coefficient R_{00}^{cl} and after convolution with the derivative of the Fermi distribution gives the classical conductance. Second is pairs of trajectories that are both part of a periodic orbit (see Fig. 1, upper). Then s and t can differ by an integer number n of period of the periodic orbit. These trajectories will give the quantum correction δR_{00} to R_{00}^{cl}. The quantum correction to the transport through the open quantum dots is therefore mediated by periodic orbits that are well coupled to the entrance opening. Performing the stationary integration and substituting the result into Eq. (1) we obtain

$$\delta g(E_F,B)\sim-4\sum_{p(C)}\Phi_p^2\sum_{n=1}^{\infty}\frac{R_n(T_p/\tau_\beta)}{\|(M_p^n-1)\|^{1/2}}\times\cos\left[n\left(S_p(E_F,B)/\hbar-\frac{\pi}{2}\sigma_p\right)\right]. \tag{9}$$

$$\Phi_p=\frac{1}{\tau_0}\int dt\,e^{-y_p^2(t)/\lambda^2}. \tag{10}$$

Here the first summation is over the prime periodic orbits and n denotes its repetition. M_p and σ_p are the stability matrix and the Maslov index of the periodic orbit p. All periodic orbits are assumed to be isolated, otherwise the amplitude will diverge. T_p is the period of the prime periodic orbit, and $R_n(T_p/\tau_\beta)=nT_p/\tau_\beta/\sinh(nT_p/\tau_\beta)$ is the damping factor, where $\tau_\beta \equiv \hbar/\pi k_B T$. The presence of the damping factor guarantees that the dominant contribution to the transport comes from the shortest periodic orbits, and therefore ensures the convergence of the sum. It also suppresses the amplitude of the conductance fluctuations as the temperature of the quantum dot is increased. In Eq. (10), $\tau_0 = L/v_0$ and the integration over time t runs along the part of the periodic orbit which is covered by the entrance QPC, where y_p is the y component of the periodic orbit. Φ_p, which measures the strength of the coupling between each periodic orbit and the entrance opening, is given by the ratio between the Gaussian weighted time consumed by a periodic orbit in the QPC region and the time needed by an electron to pass through the QPC in longitudinal direction. It is then obvious that the nature of coupling between the cavity and reservoir does play an important role in transport. In contrast to this, the semiclassical Kubo formula does not give any explanation on this issue.

For a weak magnetic field B, we can assume that only the phase of the electron is changed, the periodic orbits remain unchanged. Then we can expand S_p up to the first order in B as follows: $S_p(E,B) = S_p(E,B=0) + (e/c)\Theta_p B$. Here Θ_p is the area enclosed by each periodic orbit. Considering the contribution from $\pm\Theta_p$, i.e., from a pair of time-reversal symmetric orbits, we can rewrite the cosine term in Eq. (9) as

$$2\cos\left[n\left(\frac{S_p(E,0)}{\hbar} - \frac{\sigma_p}{2}\pi\right)\right]\cos\left(\frac{e}{\hbar c} n\Theta_p B\right).$$

Next, let us suppose that through some kind of bifurcations, we have a sequence of periodic orbits well coupled to the entrance opening, which are self-similar in configuration space and satisfy the following approximate scaling relation:

$$\Theta_p^m = \gamma \Theta_p^{m-1}. \tag{11}$$

Here Θ_p^m denotes the area enclosed by periodic orbit generated at the mth bifurcation. Then we can expect that the fluctuations of the magnetoconductance should be characterized by many scales. Below, we will show a mechanism how the above relation is generated in open cavities with soft-walled boundary. Any introduction of opening in a cavity will naturally create a harmonic saddle right on the QPC, i.e., it gives a maximum to a longitudinal cut and a minimum (which is harmonic) to a transversal cut.[9] To make the picture clear, let us look at the well-known Barbanis potential shown in Fig. 1 (lower left), which is given by the following equation:

$$V(x,y) = \frac{1}{2}(x^2 + y^2) - \epsilon x^2 y. \tag{12}$$

Barbanis potential can be taken as a simple ideal model for a soft-walled billiard with two leads attached to its two corners. The harmonic saddles are obvious ($\times$). Fortunately, in this potential as rigorously evaluated in Refs. 12,13, the shortest periodic orbits that are well coupled to the entrance opening are the straight-line librating periodic orbit A which oscillates between the wall and the saddle, and its self-similar children. For E_F which is smaller and close enough to the saddle energy E_S, the librating orbit A oscillates between stability and instability as E_F is increased, undergoing an infinite number of isochronous bifurcations that cummulate at the saddle energy, where the period T_A becomes infinity. During the bifurcations two types of self-similar periodic orbits are generated. Both types are well coupled to the entrance opening; a sequence of pairs of librating orbits L_{2k}, L'_{2k} with vanishing area, which are born when the orbit A goes from unstable to stable, and a sequence of rotating orbits O_{2k-1} with nonvanishing area, which are born when the orbit A changes from stable to unstable. Here, the lower index denotes the Maslov index of the periodic orbits. We are only interested in the orbits with nonvanishing area, O_{2k-1} (see Fig. 1). The closer they are born to E_S, the smaller is their amplitude in transversal direction. For large enough E_F, the ratio of the amplitude of a child to that of its next brother is the same for all children and is equal to exp $(-\pi/\omega)$, where ω is the transversal curvature at the harmonic saddle mentioned at the beginning of this paper. There is another scaling relation in the longitudinal direction which relates the tip of the periodic orbit to the whole periodic orbit. They make one more oscillation in the transverse direction in each generation. However, its contribution to the area is very small. Neglecting this contribution we have the scaling relation that relates the area of two immediate brothers as in Eq. (11) with $\gamma = \exp(-\pi/\omega)$, which is smaller than 1.

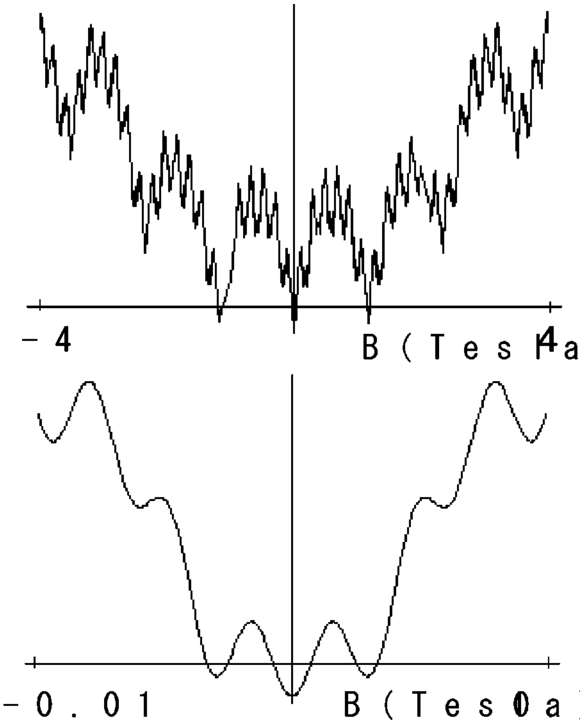

FIG. 2. Succesive magnifications of fractal fluctuations around $B=0$ T, for a sufficiently low temperature. The vertical coordinate is scaled in arbitrary unit. The scale factor of each magnification in the horizontal direction is equal to $1/\gamma^2$.

In Fig. 2, using the first 10 self-similar children of periodic orbit A, i.e., O_{2k-1}, $k = 3, \ldots, 12$, and their first repetitions, conductance fluctuations are plotted against B for E_F slightly larger than E_S. The exact self-affinity is obvious. Since for $E_F > E_S$, the phase space of the corresponding classical system is more than 95% filled with chaotic sea, we conclude that the self-affine-like magnetoconductance fluctuations are not the finger-prints of the mixed phase space. This supports recent numerical evidence reported in Ref. 14. In mathematical point of view, Eq. (11) guarantees the existence of Weierstrass-like spectrum, and together with the increasing progression of the amplitude in Eq. (10), they comprise a Weierstrass-like function, a well-known self-affine function. The increasing progression of the amplitude is guaranteed since the later children enjoy longer time being trapped in the entrance QPC. However unlike the Weierstrass function, the function we obtained using Eqs. (9)–(11) has smallest bound scale corresponding to O_5 or its repetitions which have the largest area among other self-similar periodic orbits. By decreasing the temperature, more repetitions of O_5 will contribute to the fluctuations to generate much finer scale than shown in Fig. 2. Experimentalists usually say that the lowest scale obtained in experiments is due to the limitation of the measuring devices. We therefore believe that our results should give a new insight to perform a realistic measurement.

Some notes should be mentioned here before finishing with the conclusion. Our semiclassical picture is a one-mode calculation, whereas fractal-like conductance fluctuations can also be observed in experiments with more than 1, yet small number of modes. However for a sufficiently low temperature, we believe that our one-mode approximation is valid to explain the interesting phenomena.

The main ingredient of our paper is the occurrence of the isochronous pitchfork bifurcations of a straight-line librating orbit oscillating towards a harmonic saddle (entrance QPC). This is not specific to the model in Fig. 1. In fact, as shown by Brack,[12,13] besides in the above Barbanis potential, pitchfork bifurcations also occur in other kind of potentials of the same type. The periodic orbits born from the straight-line librating orbit oscillate towards a harmonic saddle or between two harmonic saddles (as in 4-quartic Henon-Heiles potential), they are also self-similar with different value of scaling constant, γ. The latter corresponds to quantum dot with its entrance and exit opening placed aligned opposite to each other. Since our derivation is based on local information on the geometrical feature of the QPC, we find that the dimension of the fractal-like magnetoconductance fluctuations is independent of the detailed geometrical shapes of the cavity. From Eq. (9), it is obvious, however, that there is a systematic dependence of the fractal structure on temperature and inelastic scattering rate τ_β. Nevertheless, the important finding reported in Ref. 15, that the fractal dimension depends only on the resolution of the energy level spacings of the system is yet to be confirmed. We leave it for future work.

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[1] R. Ketzmerick, Phys. Rev. B **54**, 10 841 (1996).
[2] A.S. Sachrajda, R. Ketzmerick, C. Gould, Y. Feng, P.J. Kelly, A. Delage, and Z. Wasilewski, Phys. Rev. Lett. **80**, 1948 (1998).
[3] C.R. Tench *et al.*, Physica E (Amsterdam) **7**, 726 (2000).
[4] R.P. Taylor, R. Newburg, A.S. Sachrajda, Y. Feng, P.T. Coleridge, C. Dettmann, Ningjia Zhu, Hang Gua, A. Delage, P.J. Kelly, and Z. Wasilewski, Phys. Rev. Lett. **78**, 1952 (1997).
[5] A.P. Micolich *et al.*, Europhys. Lett. **49**, 417 (2000).
[6] Y. Ochiai *et al.*, Physica E (Amsterdam) **18**, 147 (2003).
[7] K. Richter, Europhys. Lett. **29**, 7 (1995); G. Hackenbroich and F. von Oppen, *ibid.* **29**, 151 (1995).
[8] A. Budiyono and K. Nakamura, J. Phys. Soc. Jpn. **71**, 2090 (2002); A. Budiyono and K. Nakamura, Chaos, Solitons Fractals **17/1**, 89 (2003).
[9] D.K. Ferry and S.M. Goodnick, *Transport in Nanostructures* (Cambridge University Press, Cambridge, 1997).
[10] D.S. Fisher and P.A. Lee, Phys. Rev. B **23**, 6851 (1981).
[11] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer Verlag, New York, 1991).
[12] M. Brack, nlin.CD/0006034 (unpublished).
[13] M. Brack, M. Mehta, and K. Tanaka, nlin.CD/0105048 (unpublished).
[14] Y. Takagaki and K.H. Ploog, Phys. Rev. B **61**, 4457 (2000).
[15] A.P. Micolich *et al.*, Phys. Rev. Lett. **87**, 036802 (2001).