Artificial light and quantum order in systems of screened dipoles

Xiao-Gang Wen*

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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The origin of light is an unsolved mystery in nature. Recently, it was suggested that light may originate from a new kind of order, quantum order. To test this idea in experiments, we study systems of screened magnetic/electric dipoles in two-dimensional (2D) and 3D lattices. We show that our models contain an artificial light–a photonlike collective excitation. We discuss how to design realistic devices that realize our models. We show that the ‘‘speed of light’’ and the ‘‘fine-structure constant’’ of the artificial light can be tuned in our models. The properties of artificial atoms (bound states of pairs of artificial charges) are also discussed. The existence of artificial light (as well as artificial electron) in condensed-matter systems suggests that elementary particles, such as light and electron, may not be elementary. They may be collective excitations of quantum order in our vacuum. In our model, light is realized as a fluctuation of string-nets and charges as the ends of open strings (or nodes of string nets).

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I. INTRODUCTION

What is light? Where light comes from? Why light exists? Every one probably agrees that these are fundamental questions. But one may wonder if they are scientific questions, philosophical questions, or even religious question. Before answering these questions and the questions about the questions, we would like to ask three more questions: What is phonon? Where phonon comes from? Why phonon exists?[46] We know that these are scientific questions and we know their answers. Phonon is a vibration of a crystal. Phonon comes from a spontaneous translation symmetry breaking. Phonon exists because the translation-symmetry-breaking phase actually exists in nature.

It is quite interesting to see that our understanding of a gapless excitation phonon is rooted in our understanding of phases of matter. According to Landau's theory,[1] phases of matter are different because they have different broken symmetries. The symmetry description of phases is very powerful. It allows us to classify all possible crystals. It also provides the origin for gapless phonons and many other gapless excitations.[2,3]

However, light, as a U(1) gauge boson, cannot be a Nambu-Goldstone mode from a broken symmetry. Therefore, unlike phonon, light cannot originate from a symmetry-breaking state. This may be the reason why we treat light differently than phonon. We regard light as an elementary particle and phonon as a collective mode.

However, if we believe in the equality between phonon and light and if we believe that light is also a collective mode of a particular ‘‘order’’ in our vacuum, then the very existence of light implies an order not found earlier in our vacuum. Thus, to understand the origin of light, we need to deepen and expand our understanding of phases of matter. We need to discover a new kind of order that can produce and protect light.

After the discovery of fractional quantum Hall (FQH) effect,[4,5] it became clear that Landau's symmetry-breaking theory cannot describe different FQH states, since those states all have the same symmetry. It was proposed that FQH states contain a new kind of order, a topological order.[6] Recently, we find that even gapless quantum states in two, three, or other dimensions can contain an order that is beyond Landau's symmetry-breaking theory.[7,8] We call this order quantum order. A preliminary theory of quantum order is developed. We find some quantum orders can be characterized by projective symmetry group (PSG) just like symmetry-breaking orders can be characterized by symmetry group. Using quantum orders and PSG, we classified over 100 different two-dimensional (2D) spin liquids that have the same symmetry.[7] Intuitively, we can view quantum/topological order as a description of pattern of quantum entanglements in a quantum state.[8] The pattern of quantum entanglements is much richer than pattern of classical configurations.

We know that the fluctuations of pattern of classical configurations (such as lattices) lead to low-energy collective excitations (such as phonons). Similarly, the fluctuations of pattern of quantum entanglement also lead to low-energy collective excitations. However, we find that the collective excitations from quantum order can be gapless gauge bosons[9–15] and/or gapless fermions. The fermions can even appear from pure bosonic models on lattice.[7,11,12,14,16–18]

If we believe in quantum order, then the three questions about light will be scientific questions. Their answers will be (A) light is a fluctuation of quantum entanglement, (B) light comes from the quantum order in our vacuum, and (C) light exists because our vacuum contains a particular entanglement (i.e., a quantum order) that supports U(1) gauge fluctuations.

According to the picture of quantum order, elementary particles (such as photon and electron) may not be elementary after all. They may be collective excitations of a bosonic system. Without experiments at Planck scale, it is hard to prove or disprove if photon and electron are elementary particles or not. However, one can make a point by showing that photon and electron can emerge as collective excitations in certain lattice bosonic models. So photon and electron do not *have* to be elementary particles.

The emergent gauge fluctuations from *local bosonic mod-*

els (also called dynamically generated gauge fields) has a long history. Emergent U(1) gauge field has been introduced in quantum disordered phase of $1+1D$ CP^N model.[19,20] The U(1) gauge fields have also been found in the slave-boson approach to spin liquid states of SU(2) and SU(N) spin models on 2D square lattice.[9,10] The slave-boson approach not only introduces a U(1) gauge field, it also introduces gapless fermion fields. However, due to the confinement of the U(1) gauge field in $1+1$D and $1+2$D, none of the above gauge field and gapless fermion fields leads to gapless gauge bosons and gapless fermions that appear as low-energy physical quasiparticles. This led to an opinion that the U(1) gauge field and the gapless fermion fields are not real and are just an unphysical artifact of the ''unreliable'' slave-boson approach. Thus, the key to finding emergent gauge boson is not to write down a Lagrangian that contains *gauge fields*, but to show that *gauge bosons* actually appear in the physical low-energy spectrum. Only when the dynamics of gauge field is such that the gauge field is in the deconfined phase can the gauge boson appear as a low-energy quasiparticle. Thus many researches, after the initial finding of Refs. 9,10 have been concentrating on finding the deconfined phase of the gauge field.

One way to obtain deconfined phase is to go to higher dimensions. The CP^N model at three and higher dimensions will have a deconfined phase with emergent U(1) gauge bosons. Following Ref. 10, a gapless deconfined U(1) gauge boson can also be found in a SU(N) spin model on 3D cubic lattice.[14] The model also contains gapless fermions. In $1+3D$, the two kinds of excitations can be separated since they interact weakly at low energies. We will call these $1+3D$ excitations, artificial light, and artificial electron. The gapless properties of these collective excitations are protected by the quantum order in the spin ground state.[7,14,21] Recently, a simpler and more realistic 3D interacting boson model was found to contain an artificial light (but not massless fermions).[15] Exact soluble models and realistic Josephson junction arrays that realize $1+2$D Z_2 gauge excitations and related topological order[12] can be found in Refs. 17,18,22–25. We see that gauge bosons appear naturally and commonly in quantum ordered states of *local bosonic models*. We do not need to introduce them by hand as elementary particles.

Motivated by lattice gauge theory[26] and projection by energy gap introduced in Refs. 27,28, in this paper, we will construct realistic 2D and 3D spin models with screened dipole interaction. Our models contain an artificial light as their low-energy excitation. Concrete devices that realize our models are also designed. Building these devices and observing artificial light in these devices will show for the first time, to the best of our knowledge, that elementary particles, such as light, can be created artificially with designed properties (such as designed ''speed of light'' and designed value of ''fine-structure constant'').

Our models also demonstrate the well-known connection between gauge theory and theory of loops:[26,29–36] a U(1) gauge theory is actually a dynamical theory of nets of closed strings. In other words, gauge theory and string-net theory are dual to each other. This duality is directly connected to the duality between statistical U(1) lattice gauge models and statistical membrane models.[31,34,36] According to the string-net picture, a gapless gauge boson is a fluctuation of large string nets and charge is the end of open strings.

However, we would like to stress that this paper is not about the equivalence between gauge theory and string-net theory. It is about how to construct realistic local bosonic models that have emergent gauge bosons. We find that to have low-energy gauge bosons, we do not need to introduce a gauge theory (or a string-net theory) at high energies. At high energies we may have a local bosonic model. To have emergent gauge bosons we simply need to choose a Hamiltonian such that the model contains strong string-net fluctuations at low energies. These string-net fluctuations naturally lead to gauge bosons. From the point of view of condensed matter physics, such a phase with strong string-net fluctuations represents a state of matter that cannot be characterized by Landau's symmetry-breaking theory. We will briefly discuss how to use PSG to characterize these orders in our models.

In the following few sections, we will discuss in detail 2D and 3D spin models and derive their low-energy effective theory. We will show that these models contain strong string-net fluctuations and emergent gauge bosons. For persons who are interested in experimental realization of the spin models and experimental probe of artificial light, they can go directly to Sec. X and XI.

II. A 2D MODEL

To construct a realistic model that contains artificial light as its low-energy collective excitation, we consider systems formed by integral spins. We will consider two cases. In the first case spins $\boldsymbol{S}$ carry magnetic dipole moment $\boldsymbol{m}\propto\boldsymbol{S}$. In the second case, we want the spins to carry electric dipole moment $\boldsymbol{d}\propto\boldsymbol{S}$. However, due to the time-reversal symmetry in real molecule, it is impossible for a molecules with a finite spin to carry an electric dipole moment proportional to the spin. But it is possible to have a molecule whose ground states are formed by *two* spin-S multiples: $|m,\sigma^z\rangle$, with $m=-S,\dots,+S$ and $\sigma^z=\pm 1$. We will call σ^z the z-component of isospin. Such a molecule can be viewed as carrying spin-S and isospin-$\frac{1}{2}$. This kind of molecules can carry a finite electric dipole moment $\boldsymbol{d}\propto\sigma^z\boldsymbol{S}$ and have a time-reversal symmetry since under time reversal, $(\sigma^z,\boldsymbol{S})\to(-\sigma^z,-\boldsymbol{S})$. We will use the above magnetic dipoles $\boldsymbol{m}\propto\boldsymbol{S}$ or electric dipoles $\boldsymbol{d}\propto\sigma^z\boldsymbol{S}$ to build our systems.

We start with a honeycomb lattice that will be called the H lattice. To form a magnetic dipole system, we place an integral spin-S on every link of the H lattice. For an electric dipole system, we place an integral spin-S and an isospin-$\frac{1}{2}$ on every link. We note that the spins form a Kagome lattice which will be called the K lattice. There are two ways to label a spin. We can use a site index $\boldsymbol{I}$ of the K lattice or we can use a pair of site indices $\langle\boldsymbol{ij}\rangle$ that labels a link in the H lattice (see Fig. 1). Using these two labels, our model Hamiltonian can be written as

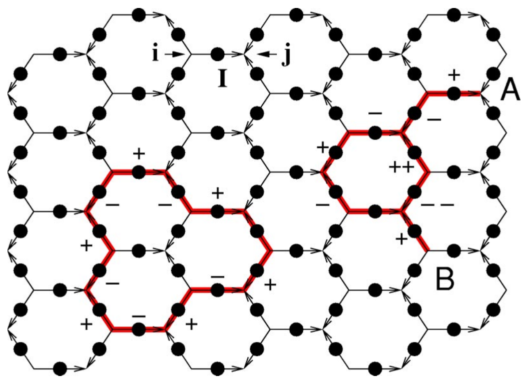

FIG. 1. The K lattice formed by filled dots and the H lattice formed by vertices. The spins in our model are on the filled dots. The lattice contains a closed string created by alternatively increasing and decreasing the $\sigma^z S^z$ of the spins, along a closed path. The lattice also contains a string net with nodes and ends. The string net is created by alternatively increasing and decreasing the $\sigma^z S^z$ by 1 or 2 along the string net. Artificial light corresponds to fluctuations of the string net. A pair of artificial charges, A and B, corresponds to the ends of an open string. Note that the artificial charges live on the H lattice.

$$H = U\sum_{\boldsymbol{i}} \left(\sum_{\boldsymbol{\alpha}} \sigma^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle} S^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle}\right)^2 + J\sum_{\boldsymbol{I}} (S^z_{\boldsymbol{I}})^2 - \frac{1}{S^2}\sum_{\langle \boldsymbol{IJ}\rangle} \sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}(tS^+_{\boldsymbol{I}}S^-_{\boldsymbol{J}} + t' S^+_{\boldsymbol{I}}S^+_{\boldsymbol{J}} e^{i2\phi_{\boldsymbol{IJ}}} + \text{H.c.}). \quad (1)$$

Here $\boldsymbol{\alpha}$ is one of the three vectors that connect a H lattice site $\boldsymbol{i}$ to its three nearest neighbors. The U term enforces a constraint that the total S^z of the three spins around a site in the H lattice is zero. Also, $\phi_{\boldsymbol{IJ}}$ is the angle of link $\boldsymbol{IJ}$ in the xy-plane and $S^{\pm} = S^x \pm iS^y$. Summation $\Sigma_{\langle \boldsymbol{IJ}\rangle}$ is over all nearest neighbors in the K lattice. The above Hamiltonian applies to both magnetic dipole systems and electric dipole systems. For magnetic dipole systems, we regard σ^z as a number $\sigma^z = 1$. For electric dipole systems, we regard σ^z as the z-component of the Pauli matrices.

For the time being, we will treat $\sigma^z_{\boldsymbol{I}}$ classically and assume each $\sigma^z_{\boldsymbol{I}}$ to take a fixed but random value of $+1$ or -1. (For magnetic dipole systems, we will set all $\sigma^z_{\boldsymbol{I}} = 1$.) Let us first assume $J = t = t' = 0$ and $U > 0$. In this case the Hamiltonian is formed by commuting terms that perform local projections. The ground states are highly degenerate and form a projected space. One of the ground states is the state with $\sigma^z_{\boldsymbol{I}} S^z_{\boldsymbol{I}} = 0$ for every spin. Other ground states can be constructed from the first ground state by drawing a loop in the H lattice and then alternatively increasing or decreasing the $\sigma^z_{\boldsymbol{I}} S^z_{\boldsymbol{I}}$ for the spins on the loop by the same amount. Such a process can be repeated to construct all the degenerate ground states. We see that the projected space has some nonlocal characters despite that it is obtained via a local projection.

Let us introduce a string operator that is formed by the product of $S^{\pm}_{\langle \boldsymbol{ij}\rangle}$ operators

$$U(C) = \prod_{\langle \boldsymbol{ij}\rangle} S^{\eta_{\langle \boldsymbol{ij}\rangle}\sigma^z_{\langle \boldsymbol{ij}\rangle}}_{\langle \boldsymbol{ij}\rangle}, \quad (2)$$

where C is a string connecting nearest-neighbor sites in the H lattice and product $\Pi_{\langle \boldsymbol{ij}\rangle}$ is over all the nearest-neighbor links of the H lattice that form the string. $\eta_{\langle \boldsymbol{ij}\rangle} = +1$ if the arrow of link $\langle \boldsymbol{ij}\rangle$ points from $\boldsymbol{i}$ to $\boldsymbol{j}$ and $\eta_{\langle \boldsymbol{ij}\rangle} = -1$ if the arrow of link $\langle \boldsymbol{ij}\rangle$ points from $\boldsymbol{j}$ to $\boldsymbol{i}$. We note that the string operator alternatively increase or decrease $\sigma^z_{\boldsymbol{I}} S^z_{\boldsymbol{I}}$ along the string. If all $\sigma^z_{\langle \boldsymbol{ij}\rangle} = 1$, the string operator has the following simple form:

$$\text{U}(C) = S^+_{\langle \boldsymbol{i}_1\boldsymbol{i}_2\rangle} S^-_{\langle \boldsymbol{i}_2\boldsymbol{i}_3\rangle} S^+_{\langle \boldsymbol{i}_3\boldsymbol{i}_4\rangle} \cdots, \quad (3)$$

where string C is formed by the H lattice sites $\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots$. Using the string operator, we can create all the degenerate ground states by repeatedly applying closed-string operators to one of the ground states.

We note that the above string operator U(C) can be defined even when loop C intersects or overlaps with itself. In fact, those self-intersecting/overlapping loops are more typical configurations of loops. Such kinds of loops look like nets of closed strings and we will call them closed string nets. Nets with open strings will be called open string nets. String operators U(C) will be called string-net operator. The degenerate ground states are formed by closed string nets.

If $t, J \neq 0$, then the ground-state degeneracy will be lifted. The t term will make string nets fluctuate and the J term will give strings in string net a string tension. As we will see later, the closed string net fluctuations become U(1) gauge fluctuations.

The degenerate ground states are invariant under local symmetry transformations generated by

$$U(\phi_{\boldsymbol{i}}) = \exp\left[\boldsymbol{i}\sum_{\boldsymbol{i}} \left(\eta_{\boldsymbol{i}}\phi_{\boldsymbol{i}} \sum_{\boldsymbol{\alpha}} \sigma^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle} S^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle} \right)\right], \quad (4)$$

where $\eta_{\boldsymbol{i}} = +1$ if the arrows of links $\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle$ all point to $\boldsymbol{i}$ and $\eta_{\boldsymbol{i}} = -1$ if the arrows of links $\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle$ all point away from $\boldsymbol{i}$ (see Fig. 1). The above transformation is called the gauge transformation. Thus we can also say that the degenerate ground states are gauge invariant.

III. A FOUR-SPIN SYSTEM

In this Section, we will start to derive the low-energy effective theory of our model for case $t, J \neq 0$ and $t' = 0$. We assume t and J to satisfy $t, J \ll U$ and $J > 0$. The ground state will no longer be degenerate. The low-energy excitations are mainly in the projected space. To understand the low-energy dynamics, we assume $S \gg 1$ and use a semiclassical approach.

To understand the dynamics of our model, let us consider a model of four spins described by $S^z_{\langle 12\rangle}$, $S^z_{\langle 23\rangle}$, $S^z_{\langle 34\rangle}$, and $S^z_{\langle 41\rangle}$ [see Fig. 2(a)]:

$$H = \sum_{\boldsymbol{i}} [\text{U}(S^z_{\langle \boldsymbol{i}-1,\boldsymbol{i}\rangle} - S^z_{\langle \boldsymbol{i},\boldsymbol{i}+1\rangle})^2 + J(S^z_{\langle \boldsymbol{i},\boldsymbol{i}+1\rangle})^2], \quad (5)$$

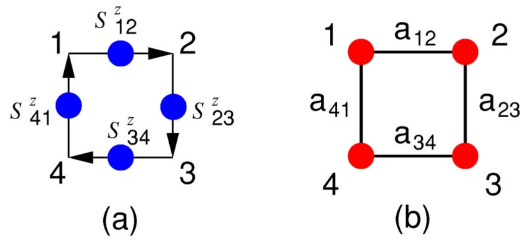

FIG. 2. (a) A four-spin system and (b) a simple lattice gauge theory is described by lattice gauge field $a_{i,i+1}$, and $a_{0,i}$, $i=1,2,3,4$.

where we have assumed $4+1\sim 1$ and $1-1\sim 4$. The Hilbert space is spanned by $|n_{\langle 12\rangle}n_{\langle 23\rangle}n_{\langle 34\rangle}n_{\langle 41\rangle}\rangle$, where integer $n_{\langle i,i+1\rangle}$ is the eigenvalue of $S^z_{\langle i,i+1\rangle}$. If $U\gg J$, then the low-energy excitations are described by $|nnnn\rangle$ states with energy $E=4Jn^2$. All other excitations have energy of order U. As we will see in Sec. IV, these low-energy excitations happen to be identical to the excitations of a U(1) lattice gauge theory on the same square [see Fig. 2(b)]. Thus our four-spin model describe a gauge theory at low energies.

To obtain an effective lattice gauge theory from our spin model, we would like to write down the Lagrangian of our four-spin model. Since the spins are mainly in the xy plane, we have $S^{\pm}_{\langle ij\rangle}=Se^{\pm i\theta_{\langle ij\rangle}}$. In this case $S^z_{\langle ij\rangle}$ is the corresponding momentum $-i\partial/\partial\theta_{\langle ij\rangle}$ of variable $\theta_{\langle ij\rangle}$. If we write the Hamiltonian in the form $H=\frac{1}{2}P^TVP$, where $P^T=(S_{\langle 12\rangle}z,S^z_{\langle 23\rangle},S^z_{\langle 34\rangle},S^z_{\langle 41\rangle})$ and

$$V=\begin{pmatrix} 4U+2J & -2U & 0 & -2U\\ -2U & 4U+2J & -2U & 0\\ 0 & -2U & 4U+2J & -2U\\ -2U & 0 & -2U & 4U+2J\end{pmatrix}, \quad (6)$$

then the Lagrangian will be

$$L=\frac{1}{2}\dot{\Theta}^TM\dot{\Theta}, \quad (7)$$

where $\Theta^T=(\theta_{\langle 12\rangle},\theta_{\langle 23\rangle},\theta_{\langle 34\rangle},\theta_{\langle 41\rangle})$ and $M=V^{-1}$. Obviously, we do not see any sign of gauge theory in the above Lagrangian. To obtain a gauge theory, we need to derive the Lagrangian in another way. Using the path-integral representation of H, we find

$$\begin{aligned}Z&=\int D(p)D(\theta)\exp\left[i\int dt\left(\sum_i S^z_{\langle i,i+1\rangle}\dot{\theta}_{\langle i,i+1\rangle}-H\right)\right]\\&=\int D(p)D(\theta)D(a_0)\exp\left[i\int dt\left(\sum_i S^z_{\langle i,i+1\rangle}\dot{\theta}_{\langle i,i+1\rangle}\right.\right.\\&\quad\left.\left.-\tilde{H}(p,a_0)\right)\right],\end{aligned} \quad (8)$$

where $\tilde{H}=\Sigma_i[J(S^z_{\langle i,i+1\rangle})^2+a_{0,i}(S^z_{\langle i-1,i\rangle}-S^z_{\langle i,i+1\rangle}-(a^2_{0,i}/4U)$. After integrating out $S^z_{\langle i,i+1\rangle}$, we obtain

$$Z=\int D(\theta)D(a_0)\exp\left[i\int dtL(\theta,\dot{\theta},a_0)\right], \quad (9)$$

where Lagrangian

$$L=\frac{1}{4J}\sum_i\left((\dot{\theta}_{\langle i,i+1\rangle}+a_{0,i}-a_{0,i+1})^2+\frac{a^2_{0,i}}{4U}\right). \quad (10)$$

In the large-U limit, we can drop the $a^2_{0,i}/4U$ term and obtain

$$L=\frac{1}{4J}\sum_i(\dot{a}_{i,i+1}+a_{0,i}-a_{0,i+1})^2 \quad (11)$$

which is just the Lagrangian of a U(1) lattice gauge theory on a single square with

$$a_{i,i+1}=\theta_{\langle i,i+1\rangle},\quad a_{i+1,i}=-\theta_{\langle i,i+1\rangle} \quad (12)$$

as the lattice gauge fields [see Fig. 2(b)]. One can check that the above Lagrangian is invariant under the following transformation:

$$a_{ij}(t)\to a_{ij}(t)+\phi_j(t)-\phi_i(t),\quad a_{0,i}(t)\to a_{0,i}(t)+\dot{\phi}_i(t) \quad (13)$$

which is called the gauge transformation.

We note that low-energy wave function $\Psi(a_{12},a_{23},a_{34},a_{41})$ is a superposition of $|nnnn\rangle$ states. All the low-energy states are gauge invariant, i.e., invariant under gauge transformation $a_{ij}\to a_{ij}+\phi_j-\phi_i$.

The electric field of a continuum U(1) gauge theory is given by $\boldsymbol{e}=\dot{\boldsymbol{a}}-\boldsymbol{\partial}a_0$. In a lattice gauge theory, the electric field becomes a quantity defined on the links

$$e_{ij}=\dot{a}_{ij}-(a_{0,j}-a_{0,i}). \quad (14)$$

We see that our lattice gauge Lagrangian can be written as $L=(1/4J)\Sigma_ie^2_{i,i+1}$. Comparing with continuum U(1) gauge theory $\mathcal{L}\propto\boldsymbol{e}^2-\boldsymbol{b}^2$, we see that our Lagrangian contains only the kinetic energy corresponding to $\boldsymbol{e}^2$. A more general lattice gauge theory also contains a potential-energy term corresponding to $\boldsymbol{b}^2$.

To obtain a potential-energy term, we generalized our spin model to

$$\begin{aligned}H=\sum_i&[U(S^z_{\langle i-1,i\rangle}-S^z_{\langle i,i+1\rangle})^2+J(S^z_{\langle i,i+1\rangle})^2\\&+t(e^{i\theta_{\langle i-1,i\rangle}}e^{i\theta_{\langle i,i+1\rangle}}+\text{H.c.})].\end{aligned} \quad (15)$$

We note that $\langle nnn|e^{i\theta_{\langle i-1,i\rangle}}e^{-i\theta_{\langle i,i+1\rangle}}|nnn\rangle=0$. Thus at the first order of t, the extra term has no effect at low energies. The low-energy effect of extra term only appears at the second order of t.

We can repeat the above calculation to obtain the following Lagrangian:

$$\begin{aligned}L=\frac{1}{4J}\sum_i&\left((\dot{a}_{i,i+1}+a_{0,i}-a_{0,i+1})^2\right.\\&\left.-t(e^{i(a_{i-1,i}+a_{i,i+1})}+\text{H.c.})+\frac{a^2_{0,i}}{4U}\right).\end{aligned} \quad (16)$$

It is a little more difficult to see in Lagrangian why the extra term has no low energy effect at the first order of t. Let us concentrate on the fluctuations of the following form:

$$a_{i-1,i}=\phi_i-\phi_{i-1}. \tag{17}$$

In the lattice gauge theory, such types of fluctuations are called the pure gauge fluctuations. After integrating out $a_{0,i}$, the Lagrangian for the above type of fluctuations has a form $L=\frac{1}{2}\dot{\phi}_i m_{ij}\dot{\phi}_j-\Sigma_i t(e^{i(\phi_{i-1}-\phi_{i+1})}+\text{H.c.})$ with $m_{ij}=\text{O}(U^{-1})$. We see that, in the large-U limit, the above form of fluctuations are fast fluctuations. Since ϕ_i live on a compact space (i.e., ϕ_i and $\phi_i+2\pi$ represent the same point), these fast fluctuations all have large energy gap of order U. Now we see that the t term $te^{i(\phi_{i-1}-\phi_{i+1})}$ averages to zero for fast fluctuations and has no effect at the first order in t. However, at second order in t there is a term $t^2\Pi_{i=1,3}e^{i(\theta_{\langle i-1,i\rangle}+\theta_{\langle i,i+1\rangle})}=t^2e^{i\Sigma_i\theta_{\langle i-1,i\rangle}}$. Such a term does not depend on ϕ_i and does not average to zero. Thus we expect the low-energy effective Lagrangian to have a form

$$L=\frac{1}{4J}\sum_i\left((\dot{a}_{i,i+1}+a_{0,i}-a_{0,i+1})^2+\frac{a_{0,i}^2}{4U}\right)+g\cos\Phi, \tag{18}$$

where $g=\text{O}(t^2/U)$ and $\Phi=\Sigma_i a_{i,i+1}$ is the flux of the U(1) gauge field through the square.

To calculate g quantitatively, we would like to first derive the low-energy effective Hamiltonian. If we treat the t term as a perturbation and treat the low-energy states as degenerate states, then at second order in t, we have

$$\begin{aligned}\langle n',n',n',n'|H_{eff}|nnnn\rangle &=-\frac{t^2}{2U}\langle n',n',n',n')|e^{i(\theta_{\langle 34\rangle}+\theta_{\langle 41\rangle})}|n',n',n,n\rangle\\ &\quad\times\langle n',n',n,n|e^{i(\theta_{\langle 12\rangle}+\theta_{\langle 23\rangle})}|nnnn\rangle\\ &\quad+\text{three other similar terms}=-\frac{2t^2}{U},\end{aligned} \tag{19}$$

where $n'=n+1$. Thus the low-energy effective Hamiltonian is

$$\sum_i [U(S^z_{\langle i-1,i\rangle}-S^z_{\langle i,i+1\rangle})^2+J(S^z_{\langle i,i+1\rangle})^2]-\frac{4t^2}{U}\cos(\Phi). \tag{20}$$

The corresponding Lagrangian is given by Eq. (18) with

$$g=\frac{4t^2}{U}. \tag{21}$$

As discussed before, the pure gauge fluctuations has a large energy gap of order U. The low-energy effective theory below U can be obtained by letting $U\to\infty$ and we get

$$L=\frac{1}{4J}\sum_i(\dot{a}_{i,i+1}+a_{0,i}-a_{0,i+1})^2+g\cos\Phi \tag{22}$$

which contains both electric energy and magnetic energy.

IV. QUANTUM GAUGE THEORY

In this Section, we will reverse the above calculation and start with the classical lattice gauge theory described by the Lagrangian Eq. (22). We would like to quantize it and find its Hamiltonian. This will allow us to calculate the energy levels of the lattice gauge theory and compare them with the energy levels of the four-spin model.

As a gauge theory, path integral

$$\begin{aligned}Z=\int D(a)D(a_0)&\\ \times\exp\bigg[-i\int dt\bigg(&\frac{1}{4J}\sum_i(\dot{a}_{i,i+1}+a_{0,i}-a_{0,i+1})^2\\ &+g\cos\Phi\bigg)\bigg],\end{aligned} \tag{23}$$

should not be regarded as a summation over different functions $[a_{ij}(t),a_{0,i}(t)]$. Here we regard two paths related by the gauge transformation Eq. (13) as the same path. Thus the path integral should be regarded as a summation over gauge equivalent classes of paths. Thus $[a_{ij}(t),a_{0,i}(t)]$ is a many-to-one label of the gauge equivalent classes. We can obtain a one-to-one label by "fixing a gauge." We note that $\Sigma_j a_{ij}$ transforms as $\Sigma_j a_{ij}\to\Sigma_j\tilde{a}_{ij}=\Sigma_j(a_{ij}+\phi_i-\phi_j)$ under gauge transformation. By tuning ϕ_i, we can always make $\Sigma_j\tilde{a}_{ij}=0$. Thus for any path $[a_{ij}(t),a_{0,i}(t)]$, we can always make a gauge transformation to make $\Sigma_j a_{ij}=0$. Therefore, we can fix a gauge by choosing a gauge fixing condition

$$\sum_j a_{ij}=0. \tag{24}$$

Such a gauge is called the Coulomb gauge, which has form $\boldsymbol{\partial}\cdot\boldsymbol{a}=0$ for a continuum theory. In the Coulomb gauge our path integral becomes

$$\begin{aligned}Z=\int D(a)D(a_0)\prod_i\delta\bigg(\sum_j a_{ij}\bigg)&\\ \times\exp\bigg[-i\int dt\bigg(&\frac{1}{4J}\sum_i(\dot{a}_{i,i+1}+a_{0,i}-a_{0,i+1})^2\\ &+g\cos\Phi\bigg)\bigg].\end{aligned} \tag{25}$$

We note that a coupling between $a_{0,i}$ and a_{ij} has form $a_{0,i}\Sigma_j\dot{a}_{ij}$. Thus, for a_{ij} satisfying constraint $\Sigma_j a_{ij}=0$, $a_{0,i}$ and a_{ij} do not couple. Since $a_{0,i}$ has no dynamics (i.e., no $\dot{a}_{0,i}$ terms), we can integrate out $a_{0,i}$. The resulting path integral becomes

$$\begin{aligned}Z=\int D(a)\prod_i\delta\bigg(\sum_j a_{ij}\bigg)&\\ \times\exp\bigg[-i\int dt\bigg(\frac{1}{4J}\sum_i\dot{a}^2_{i,i+1}+g\cos\Phi\bigg)\bigg]&\end{aligned} \tag{26}$$

which is the path integral in the Coulomb gauge.

In general, a path integral in the Coulomb gauge can be obtained by the following two simple steps: (a) inserting the gauge fixing condition $\Pi_{\boldsymbol{i}}\delta(\Sigma_{\boldsymbol{j}}a_{\boldsymbol{ij}})$ and (b) drop the $a_{0,\boldsymbol{i}}$ field.

For our problem, constraint $\Pi_{\boldsymbol{i}}\delta(\Sigma_{\boldsymbol{j}}a_{\boldsymbol{ij}})$ makes $a_{12}=a_{23}=a_{34}=a_{41}\equiv\theta/4$. The path integral takes a simple form

$$Z=\int D(\theta)\exp\left[-\boldsymbol{i}\int dt\left(\frac{1}{16J}\dot{\theta}^2+g\cos\theta\right)\right] \tag{27}$$

we note that configuration $(a_{12},a_{23},a_{34},a_{41})=(\pi/2,\pi/2,\pi/2,\pi/2)$ is gauge equivalent to $(a_{12},a_{23},a_{34},a_{41})=(2\pi,0,0,0)$ [i.e., there is a gauge transformation that transforms $(\pi/2,\pi/2,\pi/2,\pi/2)$ to $(2\pi,0,0,0)$]. Also, $a_{12}=2\pi$ is equivalent to $a_{12}=0$ since $a_{\boldsymbol{i},\boldsymbol{i}+1}=\theta_{\langle \boldsymbol{i},\boldsymbol{i}+1\rangle}$ live on a circle. Thus $\theta=2\pi$ and $\theta=0$ correspond to the same physical point. The path integral Eq. (27) describes a particle of mass $(8J)^{-1}$ on a unit circle. Flux energy $-g\cos\theta$ is the potential experienced by the particle. When $g=0$, the energy levels are given by $E_n=4Jn^2$ which agrees exactly with the energy levels of Eq. (5) at low energies. Hence Eq. (5) is indeed a gauge theory at low energies.

V. EFFECTIVE GAUGE THEORY OF LATTICE SPIN MODEL

Using a similar calculation, we find that our 2D lattice model Eq. (1) can be described by the following Lagrangian in the large-U limit:

$$L=\frac{1}{4\tilde{J}}\sum_{\langle\boldsymbol{ij}\rangle}[\dot{a}_{\boldsymbol{ij}}+a_0(\boldsymbol{i})-a_0(\boldsymbol{j})]^2+g\sum_{\boldsymbol{p}}\eta_{\boldsymbol{p}}\cos(\Phi_{\boldsymbol{p}})$$
$$+J_1\sum_{\langle\boldsymbol{IJ}\rangle}\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}. \tag{28}$$

Here, $a_{\boldsymbol{ij}}=\theta_{\langle\boldsymbol{ij}\rangle}$ if the arrow of link (see Fig. 1) points from $\boldsymbol{i}$ to $\boldsymbol{j}$ and $a_{\boldsymbol{ij}}=-\theta_{\langle\boldsymbol{ij}\rangle}$ if the arrow points from $\boldsymbol{j}$ to $\boldsymbol{i}$. $\boldsymbol{p}$ labels the plaquettes in the H lattice and $\Phi_{\boldsymbol{p}}=a_{12}+a_{23}+\cdots+a_{61}$, where $1,\ldots,6$ are the six sites around plaquette $\boldsymbol{p}$. $\Sigma_{\langle\boldsymbol{IJ}\rangle}$ sums over all the nearest-neighbor sites $\langle\boldsymbol{IJ}\rangle$ in the K lattice. $\eta_{\boldsymbol{p}}=1$ if all $\sigma^z_{12},\ldots,\sigma^z_{61}$ are equal and $0\leqslant\eta_{\boldsymbol{p}}\lesssim0.5$ otherwise. In small t limit $\tilde{J}=J$.

Let us first explain the potential term $-J_1\Sigma_{\langle\boldsymbol{IJ}\rangle}\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}$. We start with a low-energy state in the projected space $|\Psi\rangle$. The action of the t term $tS^{-2}\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}S^+_{\boldsymbol{I}}S^-_{\boldsymbol{J}}$ on such a state gives us a high-energy state with energy $4U-2\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}U$. The second order perturbation in t gives rise to contribution $-2\times t^2S^{-4}/(4U-2\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}U)$. We see that $\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}=1$ has a lower energy than $\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}=-1$. The energy difference is $2t^2S^{-4}/3U$. We find that $J_1=t^2S^{-4}/3U$. The dynamics of isospin σ^z is described by an Ising model. The ground state is a ferromagnetic state with all $\sigma^z_{\boldsymbol{I}}=1$ (or $\sigma^z_{\boldsymbol{I}}=-1$).

At second order, the t term can also generate the J term in Eq. (1). Thus $\tilde{J}-J\sim t^2/U$.

Second, let us explain potential term $-g\Sigma_{\boldsymbol{p}}\eta_{\boldsymbol{p}}\cos(\Phi_{\boldsymbol{p}})$. We first note that the gauge transformation Eq. (4) generates the following transformation

$$a_{\boldsymbol{ij}}\rightarrow a_{\boldsymbol{ij}}+\phi_{\boldsymbol{i}}-\phi_{\boldsymbol{j}}. \tag{29}$$

The t term in Eq. (1) can be written as $S^2\cos(\sigma^z_{\langle\boldsymbol{ij}\rangle}a_{\boldsymbol{ij}}+\sigma^z_{\langle\boldsymbol{jk}\rangle}a_{\boldsymbol{jk}})$ which is not gauge invariant. Thus the average of the t term in the projected space is zero. Nonzero potential terms can only be generated from the t term via higher-order perturbation, and the resulting potential term must be gauge invariant. The simplest gauge invariant term has form $\cos(\Phi_{\boldsymbol{p}})$ which is generated at the third order in t/U. Hence $g\sim t^3S^{-6}/U^2$. In the small-t limit, the second order J_1 term will make all $\sigma^z_{\boldsymbol{I}}=1$. In the following we will calculate the g term assuming $\sigma^z_{\boldsymbol{I}}=1$.

At third order, the effective Hamiltonian in the projected space has the following matrix elements:

$$\langle\Psi_1|H_{eff}|\Psi_2\rangle=\sum_{m,n}{}'\frac{\langle\Psi_1|H_t|m\rangle\langle m|H_t|n\rangle\langle n|H_t|\Psi_2\rangle}{(E_m-E_\Psi)(E_n-E_\Psi)}, \tag{30}$$

where $E_\Psi\sim0$ is the energy of $|\Psi_{1,2}\rangle$, Σ' is a sum over all high-energy states $|m\rangle$ and $|n\rangle$ that are *not* in the projected space, and H_t is the t term $H_t=tS^{-2}\Sigma_{\langle\boldsymbol{IJ}\rangle}\sigma^z_{\boldsymbol{I}}\sigma^z_{\boldsymbol{J}}S^+_{\boldsymbol{I}}S^-_{\boldsymbol{J}}$. When $|\Psi_1\rangle=e^{i\Phi_{\boldsymbol{p}}}|\Psi_{1,2}\rangle$, we find $\langle\Psi_1|H_{eff}|\Psi_2\rangle=6\times2\times t^3S^{-6}/(2U)^2$. Thus $g\eta_{\boldsymbol{p}}=6t^3S^{-6}/U^2$.

In a numerical calculation, we considered our model on a single hexagon, a single cell, of the H lattice and assumed $S=1$. Solving the six-spin model exactly, we found that the low-energy sector and the high-energy sector start to mix when $g\sim0.25U$. In that case, perturbation theory breaks down.

The J_1 term favors a ground state with all $\sigma^z_{\boldsymbol{I}}=1$ or $\sigma^z_{\boldsymbol{I}}=-1$. Such a ground state spontaneously breaks the time-reversal symmetry. The time-reversal-symmetry breaking happens even when we include the quantum fluctuations of $\sigma^z_{\boldsymbol{I}}$ generated by $\delta H=J'\Sigma_{\boldsymbol{I}}\sigma^x_{\boldsymbol{I}}$ as long as $J'\lesssim\max(|g|,t^2/U)$. In the time-reversal symmetry-breaking phase, Eq. (28) describes a U(1) lattice gauge theory.

When $t'\neq0$, more complicated term of form $\cos(\Phi_{\boldsymbol{p}}+\phi)$ can be generated, where ϕ depends on $\sigma^z_{12},\ldots,\sigma^z_{61}$. In this case, $\sigma^z_{\boldsymbol{I}}$ might have a certain pattern in the ground state, which can break translational and/or rotational symmetry. But as long as J' is small, the quantum fluctuations of $\sigma^z_{\boldsymbol{I}}$ can be ignored and the model contains a U(1) gapless gauge boson if we ignore the instanton effect.

However, in $2+1$D, we do have an instanton effect. Due to the instanton effect, a U(1) gauge excitation develops a gap.[37] The instanton effect is associated with a change of the U(1) flux Φ from 0 to 2π on a plaquette. To estimate the importance of the instanton effect, let us consider a model with only a single plaquette (i.e., the single-hexagon model discussed before). Such a model is described by

$$L=\frac{1}{24\tilde{J}}\dot{\theta}^2+g\cos\theta. \tag{31}$$

The instanton effect corresponds to a path $\theta(t)$, where θ goes from $\theta(-\infty)=0$ to $\theta(+\infty)=2\pi$. To estimate the instanton action, we assume

$$\theta(t)=\begin{cases}0 & \text{for } t<0\\ 2\pi t/T & \text{for } 0<t<T\\ 2\pi & \text{for } T<t.\end{cases} \quad (32)$$

The minimal instanton action is found to be

$$S_c=\pi\sqrt{2g/3\tilde{J}} \quad (33)$$

when $T=(\pi/2)\sqrt{3g\tilde{J}/2}$. From the density of the instanton gas, $\sqrt{\tilde{J}g}e^{-S_c}$, we estimate the energy gap of the U(1) gauge boson to be

$$\Delta\sim\sqrt{\tilde{J}g}e^{-\pi\sqrt{2g/3\tilde{J}}}. \quad (34)$$

Thus, to have a nearly gapless gauge boson, we require the above gap to be much less than the bandwidth of gauge field $\sqrt{g\tilde{J}}$. This requires

$$g\lesssim 0.25U, \quad \exp\left(-2.4\sqrt{\frac{g}{\tilde{J}}}\right)\ll 1. \quad (35)$$

If the above condition is satisfied, we can ignore the mass gap of the gauge boson and regard the U(1) gauge theory as in the deconfined phase. Therefore, Eq. (35) is the condition to have an artificial light in our 2D model.

VI. STRING NET THEORY AND STRING NET PICTURE OF ARTIFICIAL LIGHT AND ARTIFICIAL CHARGE

As mentioned before, the low-energy excitations below U are describe by closed string nets of increased/decreased $\sigma^z S^z$. (see Fig. 1). To make this picture more precise, we would like to define a closed-string-net theory on a lattice.

The Hilbert space of the closed-string net theory is a subspace of the Hilbert space of our model Eq. (1) (here we assume all $\sigma^z_{\boldsymbol{I}}=1$). The closed-string net Hilbert space contains a state with all $S^z_{\boldsymbol{I}}=0$. If we apply the closed-string net operator Eq. (2) to the $S^z_{\boldsymbol{I}}=0$ state, we obtain another state in the closed-string-net Hilbert space. Such a state is formed by $S^z_{\boldsymbol{I}}=\pm 1$ along the closed loop, or more generally a closed string net C if we include self-intersection and overlap. Thus, U(C) in Eq. (2) can be viewed as a string-net creation operator. Other states in the closed-string-net Hilbert space correspond to multiple-string-net states and are generated by repeatedly applying the closed-string net operators Eq. (2) to the $S^z_{\boldsymbol{I}}=0$ state.

The Hamiltonian of our closed-string net theory is given by

$$H_{str}=\sum_{\boldsymbol{I}}\tilde{J}(S^z_{\boldsymbol{I}})^2-\sum_{\boldsymbol{p}}\frac{1}{2}(gW_{\boldsymbol{p}}+\text{H.c.}), \quad (36)$$

where $\Sigma_{\boldsymbol{p}}$ sums over all the plaquettes of the H lattice, and $W_{\boldsymbol{p}}$ is the closed-string-net operator for the closed string around plaquette $\boldsymbol{p}$. One can check that the above Hamiltonian acts within the closed-string net Hilbert space. The $\tilde{J}$ term gives strings in string nets a finite string tension, and the g term causes the string nets to fluctuate.

From the construction, it is clear that the closed-string-net Hilbert space is identical to the low-energy Hilbert space of our model Eq. (1), which is formed by states with energy less than U. From our derivation of effective lattice gauge theory Eq. (28), it is also clear that the closed-string-net Hamiltonian Eq. (36) is directly related to the lattice gauge Lagrangian Eq. (28). In fact, the Hamiltonian of the lattice gauge theory is identical to the closed-string-net Hamiltonian Eq. (36). The string tension $\Sigma_{\boldsymbol{I}}\tilde{J}(S^z_{\boldsymbol{I}})^2$ term in the string-net theory corresponds to the $(1/4\tilde{J})\Sigma_{\langle\boldsymbol{ij}\rangle}[\dot{a}_{\boldsymbol{ij}}+a_0(\boldsymbol{i})-a_0(\boldsymbol{j})]^2$ term in the gauge theory, and string hopping $\Sigma_{\boldsymbol{p}}\frac{1}{2}g(W_{\boldsymbol{p}}+\text{H.c.})$ term in the string-net theory corresponds to the $g\Sigma_{\boldsymbol{p}}\eta_{\boldsymbol{p}}\cos(\Phi_{\boldsymbol{p}})$ term in the gauge theory. Since $S^z\sim\dot{\theta}_{\langle\boldsymbol{ij}\rangle}=\dot{a}_{\boldsymbol{ij}}$ corresponds to the electric flux along the link, a closed loop of increased/decreased $\sigma^z S^z$ corresponds to a loop of electric flux tube. A string net corresponds to a "river" network of electric flux.

We see that the U(1) gauge theory Eq. (28) is actually a dynamical theory of nets of closed strings. Typically, one expects a dynamical theory of closed string nets to be written in terms of string nets, as in Eq. (36). However, since we are more familiar with field theory, what we did in the last few sections can be viewed as an attempt in trying to describe a string net theory using a field theory. Through some mathematical trick, we have achieved our goal. We are able to write the string-net theory in the form of gauge-field theory. The gauge-field theory is a special field theory in which the field *does not* correspond to physical degrees of freedom and the physical Hilbert space is nonlocal (in the sense that the total physical Hilbert space cannot be written as a direct product of local Hilbert spaces). The point we try to make here is that gauge theory (at least the one discussed here) is a closed-string-net theory in disguise. Or, in other words, gauge theory and closed-string-net theory are dual to each other. We would like to point out that in Refs. 34,36 various duality relations between lattice gauge theories and theories of extended objects were reviewed. In particular, some statistical lattice gauge models were found to be dual to certain statistical membrane models.[31] This duality relation is directly connected to the relation between gauge theory and closed-string-net theory in our dipole models.

In the large $\tilde{J}/g$ (hence large Δ_{gauge}) limit, the ground states for both the dipole model and string net model are given by $S^z=0$ for every spin. In this phase, the closed string nets or the electric flux tubes do not fluctuate much and have an energy proportional to their length. This implies that the U(1) gauge theory is in the confining phase. In the small $\tilde{J}/g$ limit, the closed string nets fluctuate strongly and the space is filled with closed string nets of arbitrary sizes. According to the calculation in the preceding section, we note that the small $\tilde{J}/g$ phase can also be viewed as the Coulomb phase with gapless gauge bosons. Combining the two pictures, we see that gapless gauge bosons correspond to fluctuations of large closed string nets.

After relating the closed strings (or closed string nets) to artificial light, we now turn to artificial charges. To create a pair of particles with opposite artificial charges for the artificial U(1) gauge field, we need to draw an open string (or an open string net) and alternatively increase and decrease the $\sigma^z S^z$ of the spins along the string (see Fig. 1). The end points of the open strings, as the end points of electric flux tubes, correspond to particles with opposite artificial charges. We note that charged particles live on the H-lattice. In the confining phase, the string connecting the two artificial charges does not fluctuate much. The energy of the string is proportional to the length of the string. We see that there is a linear confinement between the artificial charges.

In the small $\tilde{J}/g$ limit, the large g causes strong fluctuations of the closed string nets, which leads to gapless U(1) gauge fluctuations. The strong fluctuations of the string connecting the two charges also changes the linear confining potential to the $\ln(r)$ potential between the charges.

To understand the dynamics of particles with artificial charges, let us derive the low-energy effective theory for those charged particles. Let us first assume $J=t=t'=0$. A pair of charged particles with opposite unit artificial charges can be created by applying the open-string operator Eq. (2) to the ground state. We find that each charged particle has energy U and the string costs no energy. Let us first treat charged particles as independent particles. In this case the total Hilbert space of charged particles is formed by state $|\{n_{\boldsymbol{i}}\}\rangle$, where $n_{\boldsymbol{i}}$ is the number of artificial charges on site $\boldsymbol{i}$ of the H lattice. $|\{n_{\boldsymbol{i}}\}\rangle$ is an energy eigenstate with energy $E=U\Sigma_{\boldsymbol{i}} n_{\boldsymbol{i}}^2$. Such a system can be described by the following rotor Lagrangian:

$$L=\sum_{\boldsymbol{i}} \frac{1}{4U}\dot{\varphi}_{\boldsymbol{i}}^2, \tag{37}$$

where $\varphi_{\boldsymbol{i}}$ is an angular variable. The creation operator of the charged particle is given by $e^{i\varphi_{\boldsymbol{i}}}$. Now, let us include the fact that the charged particles are always the ends of open strings (or nodes of string nets). Such a fact can be implemented by including the U(1) gauge field in the above Lagrangian. Using the gauge invariance, we find that the gauged Lagrangian has a form

$$L=\sum_{\boldsymbol{i}} \frac{1}{4U}[\dot{\varphi}_{\boldsymbol{i}}+a_0(\boldsymbol{i})]^2. \tag{38}$$

After including the gauge field, the single charge creation operator $e^{i\varphi_{\boldsymbol{i}}}$ is no longer physical since it is not gauge invariant. The gauge invariant operator

$$e^{-i\varphi_{\boldsymbol{i}_1}}e^{ia_{\boldsymbol{i}_1\boldsymbol{i}_2}}\cdots e^{ia_{\boldsymbol{i}_{N-1}\boldsymbol{i}_N}}e^{i\varphi_{\boldsymbol{i}_N}} \tag{39}$$

always creates a pair of opposite charges. In fact, the above gauge invariant operator is nothing but the open-string net operator Eq. (2). We also see that the string-net operator Eq. (2) is closely related to the Wegner-Wilson loop operator.[30,38,39]

The t term generates a hopping of charged particles to the next-nearest neighbor in the H lattice. Thus, if $t\neq 0$, the charged particles will have a nontrivial dispersion. The corresponding Lagrangian is given by

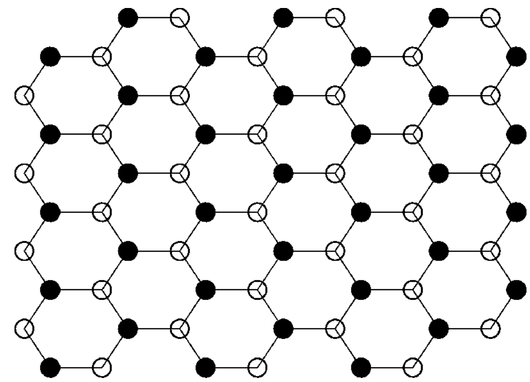

FIG. 3. Two sublattices of a H lattice. In a 3D H lattice, the open dots link to the layer above and the filled dots to the layer below.

$$L=\sum_{\boldsymbol{i}} \frac{[\dot{\varphi}_{\boldsymbol{i}}+a_0(\boldsymbol{i})]^2}{4U}-\sum_{(\boldsymbol{ij})} t(e^{i(\varphi_{\boldsymbol{i}}-\varphi_{\boldsymbol{j}}-a_{\boldsymbol{ik}}-a_{\boldsymbol{kj}})}+\text{H.c.}), \tag{40}$$

where $(\boldsymbol{ij})$ are next-nearest neighbors in the H lattice and $\boldsymbol{k}$ is the site between site $\boldsymbol{i}$ and site $\boldsymbol{j}$. The above Lagrangian also tells us that the charged particles are bosons. We also note that a flipped spin corresponds to two artificial charges. Therefore, each unit of artificial charge corresponds to a half-integer spin.

Using the string net picture, we can give more concrete answers to the three questions about light:

1) What is light? Light is a fluctuation of closed string nets of arbitrary sizes.

2) Where light comes from? Light comes from the collective motions of "things" that our vacuum is made of.[47] In particular, light comes from the large closed string nets that fill the vacuum.

3) Why light exists? Light exists because our vacuum contains strong fluctuations of looplike objects (the closed string nets) of arbitrary size.

We would like to stress that the above string net picture of the actual light in nature is just a proposal. There may be other theories that explain what is light and where light comes from. In this paper, we try to argue that the string-net picture is at least self-consistent, since there are actual models that realize the string-net picture of light. We also try to argue that the string-net picture of light is more natural than the current theory of light where light is regarded as a vector gauge field that is introduced by hand.

VII. A 3D MODEL

Our 2D model and the related calculations can be easily generalized to three dimensions. To construct our 3D model, we first construct a 3D H lattice that is formed by layers of H lattices stacked on top of each other. Note that a H lattice can be divided into two triangular sublattices (see Fig. 3). We link the sites in one sublattice to the corresponding sites in the layer above and link the sites in the other sublattice to the layer below. The spins are place on the links of the 3D H lattice. The lattice formed by the spins is called 3D K lattice. Actually, the 3D K lattice is nothing but the conner-sharing tetrahedron lattice or the pyrochlore lattice. The 3D Hamiltonian still has the form in Eq. (1). But now $\boldsymbol{i}$ label the sites in the 3D H lattice and $\boldsymbol{I}$ the sites in the 3D K lattice. $\boldsymbol{\alpha}$ connects site $\boldsymbol{i}$ to its four linked neighbors in the 3D H lattice. The low-energy effective theory still has the form in Eq. (28) and the conditions to observe artificial light are still given by Eq. (35). The main difference between the $1+2$D model and $1+3$D model is that the artificial light, if it exists, is exactly gapless in $1+3$D. The effective Lagrangian for the charged particles still has the form in Eq. (40).

VIII. EMERGENT QUANTUM ORDERS

Our 3D model contains two $T=0$ quantum phases with the same symmetry. One phase (phase A) appears in $\tilde{J}\gg|g|$ limit and is gapped [see Eq. (35))]. The other phase (phase B) appears in $|g|\gg\tilde{J}$ limit. Phase B contains a nontrivial quantum order which is closely related to the artificial light in it.

By including the t term between spins beyond nearest neighbors, our model can even support different kinds of nontrivial quantum orders. For example, by adjusting the different t terms, we can independently tune the value and the sign of g in $g\cos(\Phi_{\boldsymbol{p}})$ for different kind of plaquettes. If all g are positive then we get phase B discussed above, where there is zero gauge flux through all the plaquettes: $e^{i\Phi_p}\sim 1$. If we tune g to be negative for the plaquettes in the layers of the 3D H lattice and positive for the plaquettes between the layers, then we get a phase (phase C) with a quantum order not found earlier. In phase C there is π flux through the plaquettes in the layers and zero flux through the other plaquettes. Phase C has the same symmetry as phase A and B, and contains a gapless artificial light. The phase B and phase C are separated by phase A that appears in small g limit.

Quantum orders in phase B and C can be more precisely characterized by the projective symmetry group or PSG.[7,8] In semiclassical limit, phase B is described by an ansatz where all $\langle e^{ia_{\boldsymbol{ij}}}\rangle\sim 1$, while phase C is described by an ansatz where some $\langle e^{ia_{\boldsymbol{ij}}}\rangle\sim 1$ and other $\langle e^{ia_{\boldsymbol{ij}}}\rangle\sim -1$. The PSG for an ansatz is formed by all the combined gauge and symmetry transformations that leave the ansatz invariant.[7,8] We find that the PSG's for the ansatz of phase B and the ansatz of phase C are different. It was shown that PSG is a universal property of a quantum phase that can be changed only through phase transitions.[7,8] The different PSG's for phase B and phase C indicate that phase B and phase C are indeed different quantum phases that cannot be changed into each other at $T=0$ without a phase transition. Using PSG we can also describe more complicated quantum orders (or flux configurations). We can even use PSG to classify all the quantum orders in our model (in semiclassical limit).

The different quantum orders in phase B and phase C can be distinguished in experiments by measuring the dispersion relation of the charged particle. From Eq. (40), we see that the hopping of the charged particles is affected by the flux through the plaquettes.

IX. EMERGENT LOW-ENERGY GAUGE INVARIANCE

After seeing the importance of gauge transformation Eq. (4) in obtaining artificial light and in PSG characterization of quantum orders, we are ready to make a remark about the gauge invariance. We note that after including the higher order t/U terms, formally, the Lagrangian is not invariant under gauge transformation Eq. (29). As a result, the so-called pure gauge fluctuations (which should be unphysical in gauge theory) actually represent physical degrees of freedom. However, those fluctuations all have a large energy gap of order U.[27,28] The low-energy fluctuations (assuming there is a finite energy gap between the low-energy and high-energy excitations) should be gauge invariant, and the effective Lagrangian that describes their dynamics should be gauge invariant.

Due to the finite mixing between the low-energy and high-energy excitations caused by the t term, the low-energy excitations are not invariant under the particular gauge transformation defined in Eq. (4). However, since the mixing is perturbative, we can perturbatively modify the gauge transformations such that the low-energy excitations are invariant under a modified gauge transformation. To obtain the modified gauge transformation, we continuously change t from zero to a small value. This will cause the eigenstates of our model to rotate. The rotation is generated by a unitary matrix W. Then the modified gauge transformation is given by $\tilde{U}(\phi_{\boldsymbol{i}})=W\exp[i\Sigma_{\boldsymbol{i}}(\eta_{\boldsymbol{i}}\phi_{\boldsymbol{i}}\Sigma_{\boldsymbol{\alpha}}\sigma^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle}S^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle})]W^{\dagger}$. By definition, the modified gauge transformation will leave the low-energy excitations invariant. The nontrivial point here is that the modified gauge generator $W\Sigma_{\boldsymbol{i}}(\eta_{\boldsymbol{i}}\phi_{\boldsymbol{i}}\Sigma_{\boldsymbol{\alpha}}\sigma^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle}S^z_{\langle \boldsymbol{i},\boldsymbol{i}+\boldsymbol{\alpha}\rangle})W^{\dagger}$ is still a local operator. This is likely to be the case if t is not too large to destroy the energy gap between the low- and high-energy excitations. We see that both the U(1) gauge structure and the PSG are emergent properties in our model.

We would like to remark that the key to obtaining a low-energy effective gauge theory is not to formally derive an effective Lagrangian that have a gauge invariance, but to show all the pure gauge fluctuations to have a large energy gap. In this limit, as we have seen for the t term, all the gauge noninvariant terms will drop out from the low-energy effective theory. Only gauge invariant combinations can appear in the effective theory.[27,28]

X. REALISTIC DEVICES

In the following, we will discuss how to design realistic devices that realize our 2D and 3D models. First we note that our 2D model Hamiltonian Eq. (1) can be realized by magnetic or electric dipoles which form a Kagome lattice (assuming only dipolar interactions between the dipoles). For such a system $t=S^2U/2$, $t'=3S^2U/2$, and $J=-2U$. So the coupling constants do not have the right values to support an artificial light. Thus the key to design a working device is to find a way to reduce the couping between $S^{\pm}$. We need to reduce the t term and t' term by a factor $\sim 4S^2$. We also need to introduce an anisotropic spin term $(S^z)^2$ to bring J close to zero.

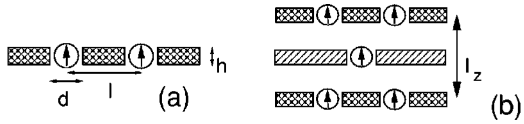

FIG. 4. (a) A 2D device and (b) a 3D device made of electric dipoles screened by superconducting films.

We can use molecules with a finite electric dipole moment $\boldsymbol{d}$ as our spins. For a fixed $\boldsymbol{d}$, the molecule should have two degenerate ground states with angular momentum $\pm S$ in the $\boldsymbol{d}$ direction. If we allow the molecule to rotate, the ground states of the molecule will contain $2(2S+1)$ states $|S^z,\sigma^z\rangle$, $S^z=-S,\ldots,S$ and $\sigma^z=\pm 1$. S^z corresponds to the spin degree of freedom and σ^z the isospin degree of freedom. The tunneling between the $\pm S$ states generates term $\delta H = J'\sigma^x$, which leads to quantum fluctuation of σ^z. We also need to put the molecule, say, in a C_{80} buckyball so the dipole can rotate freely. We note that endohedral $S_cA_2N@C_{80}$ (Ref. 40) is commercially available from Luna Nanomaterials (http://www.lunananomaterials.com), where A is a rare-earth atom such as Y,Er,Gd, . . . If endohedral $S_cABN@C_{80}$ can be made with A and B being different rare-earth atoms, such an endohedral may have the properties discussed above.

One way to reduce t,t' is to embed dipoles in a fully gapped superconductor. A particular design for our 2D model is given in Fig. 4(a). The sample is formed by a superconducting film. Circular holes of diameter d are drilled through the film to form a Kagome lattice. The dipoles are placed in the holes. A large h will reduce t. The screening of the superconducting film also makes the dipoles tend to point horizontally (i.e., $S^z=0$). In this case J can be tuned by changing d/l. If we choose $l=10$ nm, $S=2$, and dipole moment $0.1e$ nm, we find $U\sim 40$ mK. The operating temperature to observe artificial light is about 1 mK, which is achievable.

The 3D model can be realized by the device in Fig. 4(b). We note that the 3D K lattice is formed by alternatively stacking K lattices and triangular lattices together. The top and the bottom layers in Fig. 4(b) are screened K lattices just like Fig. 4(a), while the middle layer is a screened triangular lattice. The distance between layers and d/l needs to be tuned to reproduce the U term. The t term and J term can be adjusted similarly, as in the 2D device.

XI. PHYSICAL PROPERTIES OF 2D AND 3D DEVICES

The 2D and 3D devices are described by model Hamiltonian Eq. (1) with coupling constants U, J, t, and t'. The low-energy effective theory Eq. (28) contains only two coupling constants $\tilde{J}$ and g in large-U limit. $\tilde{J}$ and g are determined by U, J, t, and t'. If $U=40$ mK, we can tune t to make $g=6$ mK. We can tune J to make $\tilde{J}=g/2=3$ mK.

The phase diagrams of the 2D and 3D devices are sketched in Fig. 5. Both 2D and 3D electric dipole systems have a phase transition at $T_c\sim t^2S^{-4}/U$, which breaks the time-reversal symmetry. The 3D system also has a quantum phase transition at $g/\tilde{J}\sim 1$. The quantum phase transition separates the confined phase where the artificial light has an energy gap and the Coulomb phase where the artificial light is gapless. In principle, the quantum phase transition can be a continuous phase transition although it does not change any symmetry. We know that the Coulomb phase corresponds to a phase with strong fluctuations of large closed strings. There are two ways in which the Coulomb phase can change into the confined phase. In the first method, the large closed string nets break up into small open string nets. This corresponds to condensation of charged bosons and produces a Anderson-Higgs phase. (Note that a confined phase is the same as a Anderson-Higgs phase.) Such a transition is expected to be of first order.[41–43] In the second method, the large closed string nets start to cost too much energy and the ground state, after transition, contains only dilute small closed string nets. Such a transition was believed to be continuous.[44]

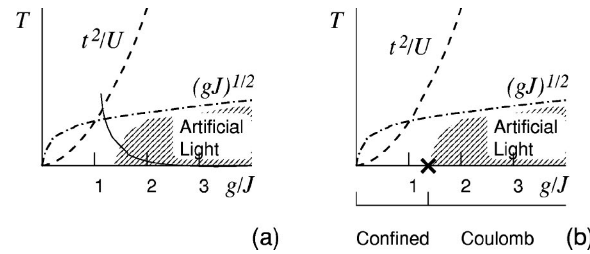

FIG. 5. The phase diagram of (a) 2D and (b) 3D electric dipole systems. The dash line represents the time-reversal symmetry-breaking transition. The transition happens at $T_c\sim t^2S^{-4}/U$. The dash-dot line marks the bandwidth of artificial light, which is of order $\sqrt{\tilde{J}g}$. The thin solid line in (a) marks energy gap Δ of the artificial light. The artificial light exists in the shaded region where $T,\Delta<\sqrt{\tilde{J}g}$. In (b) the cross marks the position of a zero-temperature phase transition between the confined phase and the Coulomb phase. The artificial light is exactly gapless in the Coulomb phase.

Both types of transitions are between the same pair of phases, the Coulomb and the confined phases. Both types of transition can appear in our 3D dipole model. However, in the large-U limit, we expect the quantum phase transition from the Coulomb phase to the confined phase to be of the second type and continuous. The continuous phase transition will become a smooth cross over at finite temperatures [see Fig. 5(b)]. If U is not large enough, the quantum phase transition can be of the first type, which is a first-order phase transition. Such a first-order phase transition will extend to finite temperatures.

In the 2D electric-dipole system, there is no zero-temperature quantum phase transition and the artificial light always has a finite energy gap Δ [see Eq. (34)]. The thin solid line in Fig. 5(a) marks the scale of the energy gap. When the energy gap is much less than the bandwidth $\sqrt{\tilde{J}g}$ of the artificial light, we say the artificial light exists.

Our 2D and 3D dipole systems have boundaries. Some interesting questions arise. To the artificial light, what is the properties of the boundary? If we shine artificial light onto the boundary, does artificial light get reflected or absorbed? If we place an artificial charge near the boundary, whether the charge is attracted or repelled by the boundary? These questions can be answered by our string net picture of artificial light and artificial charges. We note that the closed strings are always confined in the sample. The ends of open strings always cost an energy of order U even when the open string is ended on the boundary. This means that the closed strings do not break up near the boundary. Since the closed strings represent electric flux tube, we find that the electric flux of the artificial light can never leave the sample, neither can it end at the boundary of the sample. Therefore, to the artificial light, the outside of the sample behaves like a perfect dia-electric media that repels all the artificial electric flux. If we place an artificial charge near the boundary, the charge will be repelled by the boundary.

To understand the physical properties of the artificial light in the 2D model, we can take the continuum limit by writing

$$a_{\boldsymbol{ij}}=\delta\boldsymbol{x}_{\langle \boldsymbol{ij}\rangle}\cdot\boldsymbol{a}(\boldsymbol{x}),$$

$$a_{0,\boldsymbol{i}}=a_0(\boldsymbol{x}), \tag{41}$$

where $\boldsymbol{a}=(a_x,a_y)$ is a 2D vector field (the vector gauge potential in 2D), a_0 corresponds to the potential field, $\boldsymbol{x}$ is near site $\boldsymbol{i}$, $\delta\boldsymbol{x}_{\langle \boldsymbol{ij}\rangle}$ is the vector that connects the $\boldsymbol{i}$ and $\boldsymbol{j}$ sites in the H lattice, and l is the distance between the neighboring sites in the H lattice. In the continuum limit, the Lagrangian Eq. (28) becomes

$$L=\int d^2\boldsymbol{x}\left(\frac{1}{4\tilde{J}\sqrt{3}}\,\boldsymbol{e}^2-\frac{3\sqrt{3}gl^2}{4}\boldsymbol{b}^2\right), \tag{42}$$

where $\boldsymbol{e}=\partial_t\boldsymbol{a}-\partial_{\boldsymbol{x}}a_0$ and $\boldsymbol{b}=\partial_x a_y-\partial_y a_x$ are the corresponding artificial electric field and artificial magnetic field. We see that the velocity of our artificial light is $c_a=\sqrt{9g\tilde{J}l^2/\hbar^2}\sim(\sqrt{t\tilde{J}}/U)tl$. If we take $\tilde{J}=3$ mK, $g=6$ mK, and $l=10$ nm, we find that the speed of the artificial light is about $c_a=20$ m/s. The bandwidth of the artificial light is about $E_a=\pi c_a\hbar/2l=20$ mK. The gap of the artificial light is about $\Delta\sim0.03E_a$.

From Eq. (40), we find the continuum Lagrangian that describes the charged particles in the 2D model (in the $U\gg t$ limit) to be

$$L=\int d^2\boldsymbol{x}\sum_{I=1,2}\Bigg(\phi_I^\dagger(i\partial_t-a_0-U)\phi_I-\frac{9tl^2}{2}|(\partial_{\boldsymbol{i}}+ia_{\boldsymbol{i}})\phi_I|^2$$
$$+\bar{\phi}_I^\dagger(i\partial_t+a_0-U)\bar{\phi}_I-\frac{9tl^2}{2}|(\partial_{\boldsymbol{i}}-ia_{\boldsymbol{i}})\bar{\phi}_I|^2\Bigg), \tag{43}$$

where ϕ_I describe the positively charged bosons, $\bar{\phi}_I$ describe the negatively charged bosons, $\psi_1,\bar{\psi}_1$ describe the charged bosons on the even sites of the H lattice, and $\psi_2,\bar{\psi}_2$ describe the charged bosons on the odd sites of the H lattice. It costs $2U$ energy to create a pair of charged bosons. The mass of the bosons is $m=(9tl^2)^{-1}$ and $mc_a^2=2.3$ mK. We would like to note that the boson velocity can be larger than the speed of artificial light. The potential energy between a positive and a negative charge is $V(r)=(\sqrt{3}\tilde{J}/\pi)\ln r$. A bound state of a positive charge and a negative charge (an artificial atom) has a size of order $l\sqrt{3\sqrt{3}\pi t/\tilde{J}}=6.6l$. For each angular momentum $m\hbar$, the lowest-energy level of the artificial atom is of order $[\ln(m)\sqrt{3}\tilde{J}/\pi]=1.7\ln(m)$ mK.

For the 3D model, if the layer separation is l_z, we find that the Lagrangian in the continuum limit is given by

$$L=\int d^3\boldsymbol{x}\frac{1}{4\tilde{J}\sqrt{3}l_z}\left(\boldsymbol{e}_x^2+\boldsymbol{e}_y^2+\frac{l_z^2}{l^2}\boldsymbol{e}_z^2\right)$$
$$-\int d^3\boldsymbol{x}\frac{3\sqrt{3}gl^2}{4l_z}\left(\frac{2l_z^2\boldsymbol{b}_x^2}{3l^2}+\frac{2l_z^2\boldsymbol{b}_y^2}{3l^2}+\boldsymbol{b}_z^2\right), \tag{44}$$

where $\boldsymbol{e}$ and $\boldsymbol{b}$ are the artificial electric field and artificial magnetic field in 3D, respectively. We see that, in general, the speed of artificial light is different in different directions. For simplicity, we choose $l_z=l$ and ignore the anisotropy in the speed of artificial light. That is, we work with the following simplified Lagrangian:

$$L=\int d^3\boldsymbol{x}\left(\frac{1}{4\tilde{J}\sqrt{3}l}\boldsymbol{e}^2-\frac{3\sqrt{3}gl}{4}\boldsymbol{b}^2\right). \tag{45}$$

The speed of artificial light is $c_a=\sqrt{9g\tilde{J}l^2/\hbar^2}$. If we take $\tilde{J}=3$ mK, $g=6$ mK, and $l=10$ nm, we find $c_a=20$ m/s. The bandwidth of the artificial light is about $E_a=\pi c_a\hbar/2l=20$ mK. The above 3D Lagrangian can be rewritten in a more standard form

$$L=\int d^3\boldsymbol{x}\frac{1}{8\pi\alpha}\left(\frac{1}{c_a}\boldsymbol{e}^2-c_a\boldsymbol{b}^2\right), \tag{46}$$

where $\alpha=(1/2\pi)\sqrt{\tilde{J}/3g}=1/15$ is the artificial fine structure constant. The mass of the charged boson m is of order $(9tl^2)^{-1}$ and $mc_a^2\sim2.3$ mK. The artificial atom has an energy-level spacing $\frac{1}{2}mc_a^2\alpha^2\sim0.01$ mK and a size of order $1/\alpha mc_a\sim(6\pi\sqrt{3}t/\tilde{J})l=87l$.

In the following, we will discuss one experiment that can detect some of the above properties in the 2D system (Note that it is easier to create a 2D device.) If we place the tip of a scanning tunneling microscope near an electric dipole, we can induce coupling $\delta H=E(t)S_{\boldsymbol{I}}^{+}+\text{H.c.}$ to the electric dipole. $S_{\boldsymbol{I}}^{\pm}$ flips a spin on a link, which creates a pair of bosons on the two ends of the link. The two bosons carry positive and negative artificial charges. If we measure the high-frequency capacitance of the tip,[45] we can see peaks at the energy levels of the artificial atom $\omega=2U+[\ln(m)\sqrt{3}\tilde{J}/\pi]=[80+1.7\ln(m)]\,\text{mK}=[1668+35\ln(m)]$ MHz. We also note that an ac voltage on the tip, at lower frequencies, can generate artificial light. However, the tip of scanning tunneling microscope is not an efficient antenna to generate artificial light.

From the above discussion, it is clear that the electric dipole systems, if they can be created, really provide a model for artificial light, artificial charge, and artificial electromagnetic interaction in both two and three dimensions. We know that the SU(N)-spin model that realizes 3D artificial light, artificial electron, and artificial proton[14] is not realistic. The dipole systems discussed here contain only artificial light. It would be very interesting to design a realistic device that has artificial light, artificial electron, and artificial proton. In that case, we can have an artificial world sitting on our palm.

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*URL: http://dao.mit.edu/wen

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[46] Note that the term phonon always refers to gapless phonon in solid in this paper.
[47] Note that our vacuum is not empty. It is filled with ''things'' that form the space time.