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Meissner response of anisotropic superconductors

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The response field of a half-space anisotropic superconductor is evaluated for an arbitrary weak external field source. Example sources of a point magnetic moment and a circular current are considered in detail. For the penetration depth $\lambda \ll L$ with L being any other relevant distance (the source size or the distance between the source and superconductor), the major contribution to the response is the λ-independent field of the source image. It is shown that the absolute value of λ cannot be extracted from the response field with a better accuracy than that for the source position. Similar problems are considered for thin films.

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I. INTRODUCTION

An experimental technique, scanning superconducting quantum interference device (SQUID) microscopy (SSM), has recently been developed for measuring magnetic fields due to vortices exiting superconducting samples.[1] Knowing the field distributions one can, in principle, extract the London penetration depth λ (either isotropic or anisotropic) and its temperature dependence.[2] In other implementation of this method, one measures the Meissner response of a superconductor to a weak external source of static magnetic field. Again, the response may provide information about the λ temperature dependence and its anisotropy. Also, one can use tunable field sources to study elementary forces acting upon vortices exiting the surface.[3] Similar problems are encountered in the magnetic force microscopy applied to surfaces of anisotropic superconductors.

There are quite a few publications dealing with these problems for isotropic superconductors.[4–8] For anisotropic materials, however, the response field is asymmetric even when the source has certain symmetries, and one cannot use methods developed for isotropic materials. To deal with the problem, one can utilize the two-dimensional (2D) Fourier transform with respect to coordinates x,y of the interface, provided the equations for the field distributions inside and outside the superconducting half-space are *linear*. This is the case if the London description for the field inside the material is adopted. Then, one solves the remaining system of ordinary differential equations in the variable z, normal to the interface. This approach has been developed in Ref. 9 for vortices crossing the superconductor surface.

Let us consider a source with known field distribution $\boldsymbol{h}^s$ in the absence of superconductor. In the presence of the superconductor occupying the half-space $z<0$, the total field in vacuum $z>0$ can be written as

$$\boldsymbol{h}=\boldsymbol{h}^s+\boldsymbol{h}^r, \tag{1}$$

where $\boldsymbol{h}^r$ is the response field which satisfies $\operatorname{div}\boldsymbol{h}^r=$ curl $\boldsymbol{h}^r=0$ in vacuum. One can look for this field as $\nabla\varphi^r$ with the potential φ^r obeying the Laplace equation and the zero boundary condition far from the surface. The general form of such a potential is

$$\varphi^r(\boldsymbol{r},z)=\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\varphi^r(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}-kz}. \tag{2}$$

Here, $\boldsymbol{r}=(x,y)$ and z is directed normal to the superconducting flat surface at $z=0$; $\varphi^r(\boldsymbol{k})e^{-kz}$ is the 2D Fourier transform with respect to variables x,y at any fixed $z>0$. The potential (2) is defined only in the upper half-space; hence, there is no problem of uniqueness which is in general associated with the description of the magnetic field by a potential.

The field inside the superconductor satisfies London equations which read in the general anisotropic case as[10]

$$h_i-\lambda^2 m_{lk}e_{lst}e_{kni}h_{t,ns}=0,\quad i=x,y,z. \tag{3}$$

Here e_{ikl} is the unit antisymmetric tensor, m_{ij} is the mass tensor, and $h_{t,ns}$ abbreviates $\partial^2 h_t/\partial x_n\partial x_s$. The average penetration depth $\lambda=(\lambda_a\lambda_b\lambda_c)^{1/3}$ is related to the actual penetration depth for the currents, e.g., along the crystal direction a: $\lambda_a=\lambda\sqrt{m_a}$. The masses are normalized so that $m_am_bm_c=1$.

Hence, the problem is to match the solutions for the field inside and outside the superconductor with boundary conditions of the field continuity at the interface.

II. RESPONSE FIELD

Usually in situations of interest, the sample surface is normal to one of the principal crystal directions. We call this direction c and choose the frame x,y,z as coinciding with a,b,c. In this situation, the mass tensor is diagonal ($m_{xx}=m_a$, $m_{yy}=m_b$, $m_{zz}=m_c$), and Eqs. (3) reduce to

$$h_x+\lambda_b^2(h_{z,x}-h_{x,z})_{,z}+\lambda_c^2(h_{y,x}-h_{x,y})_{,y}=0,$$

$$h_y+\lambda_c^2(h_{x,y}-h_{y,x})_{,x}+\lambda_a^2(h_{z,y}-h_{y,z})_{,z}=0,$$

$$h_z+\lambda_a^2(h_{y,z}-h_{z,y})_{,y}+\lambda_b^2(h_{x,z}-h_{z,x})_{,x}=0. \tag{4}$$

One can replace any of one of these equations by $\operatorname{div}\boldsymbol{h}=h_{x,x}+h_{y,y}+h_{z,z}=0$. It is convenient to replace the third one and exclude $h_{z,z}$ from the first two:

$$h_x-\lambda_b^2h_{x,xx}-\lambda_c^2h_{x,yy}-\lambda_b^2h_{x,zz}+(\lambda_c^2-\lambda_b^2)h_{y,xy}=0,$$

$$h_y-\lambda_c^2 h_{y,xx}-\lambda_a^2 h_{y,yy}-\lambda_a^2 h_{y,zz}+(\lambda_c^2-\lambda_a^2)h_{x,xy}=0,$$

$$h_{z,z}=-(h_{x,x}+h_{y,y}) \tag{5}$$

(the first two equations are decoupled from the third).

We now apply the x,y Fourier transform to Eqs. (5):

$$(1+\lambda_b^2k_x^2+\lambda_c^2k_y^2)h_x-\lambda_b^2h_x''+(\lambda_b^2-\lambda_c^2)k_xk_yh_y=0,$$

$$(\lambda_a^2-\lambda_c^2)k_xk_yh_x+(1+\lambda_c^2k_x^2+\lambda_a^2k_y^2)h_y-\lambda_a^2h_y''=0,$$

$$i(k_xh_x+k_yh_y)+h_z'=0. \tag{6}$$

Here, h_i are functions of k_x,k_y, and z, and the prime denotes derivatives with respect to z. Hence, we are left with the linear system of ordinary second-order differential equations with respect to the variable z for $h_i(\boldsymbol{k},z)$.

The solutions are linear combinations of simple exponentials:

$$h_i(\boldsymbol{k},z)=\sum_n H_i^{(n)}(\boldsymbol{k})e^{q_nz}. \tag{7}$$

The parameters q_n and their number are to be determined. Substituting each term of Eq. (7) in the system (6) we obtain a linear homogeneous system for $H_i^{(n)}$:

$$H_x(1+\lambda_b^2k_x^2+\lambda_c^2k_y^2-\lambda_b^2q^2)+H_y(\lambda_b^2-\lambda_c^2)k_xk_y=0,$$

$$H_x(\lambda_a^2-\lambda_c^2)k_xk_y+H_y(1+\lambda_c^2k_x^2+\lambda_a^2k_y^2-\lambda_a^2q^2)=0,$$

$$i(k_xH_x+k_yH_y)+qH_z=0, \tag{8}$$

for each n; the superscript n is omitted for brevity. As has been mentioned, the first two equations here are decoupled from the third; they have a nonzero solution provided their determinant is zero. This gives a quadratic equation for q^2 which is readily solved:

$$q_{1,2}^2=\frac{P\pm\sqrt{Q}}{2\lambda_a^2\lambda_b^2},$$

$$P=\lambda_a^2+\lambda_b^2+\lambda_c^2(\lambda_b^2k_x^2+\lambda_a^2k_y^2)+\lambda_a^2\lambda_b^2k^2,$$

$$Q=P^2-4\lambda_a^2\lambda_b^2(1+\lambda_b^2k_x^2+\lambda_a^2k_y^2)(1+\lambda_c^2k^2). \tag{9}$$

Note that $Q<P^2$ and both q_1^2 and q_2^2 are positive; therefore, there are only two positive q's (i.e., $n=1,2$) which satisfy the requirement of vanishing fields at $z=-\infty$.

The quantities q determine how the field attenuates in the superconductor. We now have to find the "amplitudes" H_i from Eqs. (8) for each q. It is worth noting that solving the homogeneous system of linear equations implies, in fact, expressing some unknowns in terms of others. To this end, one has to determine the rank of the matrix of coefficients for the system (8), choose a proper subsystem to solve, etc. The actual procedure might differ depending on the situation in question.

Solving the system (8) we obtain

$$H_x=-i\frac{d-\lambda_a^2q^2}{k_xq(\lambda_a^2-\lambda_b^2)}H_z,\quad H_y=i\frac{d-\lambda_b^2q^2}{k_yq(\lambda_a^2-\lambda_b^2)}H_z,$$

$$d=1+\lambda_b^2k_x^2+\lambda_a^2k_y^2, \tag{10}$$

for each q of Eq. (9).

Let us turn now to the field $\boldsymbol{h}^s$ of the source. As a consequence of $\operatorname{div}\boldsymbol{h}^s=\operatorname{curl}\boldsymbol{h}^s=0$ out of the source, the 2D Fourier components of $\boldsymbol{h}^s$ are not independent. As with the response field, we can look for this field in the from $\boldsymbol{h}^s=\nabla\varphi^s$ such that $\nabla^2\varphi^s=0$. In our situation, the source is situated in the upper half-space $z>0$, and we are interested in the field $\boldsymbol{h}^s$ "under" the source. The general solution of the Laplace equation which vanishes as $z\to-\infty$ is

$$\varphi^s(\boldsymbol{r},z)=\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\varphi^s(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}+kz}. \tag{11}$$

The 2D Fourier components of the source field at the interface are

$$h_\alpha^s=ik_\alpha\varphi^s(\boldsymbol{k})\quad(\alpha=x,y),\quad h_z^s=k\varphi^s(\boldsymbol{k}). \tag{12}$$

Hence, the boundary conditions take the form

$$ik_\alpha(\varphi^s+\varphi^r)=H_\alpha^{(1)}+H_\alpha^{(2)}\quad(\alpha=x,y),$$

$$k(\varphi^s-\varphi^r)=H_z^{(1)}+H_z^{(2)}. \tag{13}$$

Since the components H_α are expressed in terms of H_z's, we can solve the system (13) to obtain

$$\varphi^r=\varphi^s\frac{k(q_1+q_2-k)-q_1q_2(1-1/d)}{k(q_1+q_2+k)+q_1q_2(1-1/d)},$$

$$H_z^{(1)}=\frac{k[\varphi^s(q_2-k)-\varphi^r(q_2+k)]}{q_2-q_1},$$

$$H_z^{(2)}=\frac{k[\varphi^s(k-q_1)+\varphi^r(q_1+k)]}{q_2-q_1}. \tag{14}$$

Thus, the response outside and inside the superconductor is expressed in terms of the source field φ^s. It is worth noting that since $\varphi^s(\boldsymbol{k})$ can be replaced with $h_z^s(\boldsymbol{k})/k$, the response field can be expressed in terms of the z component of the source field at the interface.

One can see, in particular, that the total flux in the z direction "reflected" by the superconductor is equal and opposite in sign to the incident flux of the source crossing the interface. Indeed, this flux is

$$h_z^r|_{k=0}=-k\varphi^r|_{k=0}=-k\varphi^s|_{k=0}=-h_z^s|_{k=0}. \tag{15}$$

It is seen from the general formulas (10) that the case $\lambda_a=\lambda_b$ is singular. The formal reason for this is that the first two equations of the system (8) (which we have used to express H_x and H_y in terms of H_z) are no longer independent. This situation should be treated separately.

A. Small λ

In many situations, the penetration depths are small relative to other relevant lengths in the problem such as the distance z_0 between the source and interface. The characteristic k's then satisfy $k\lambda \ll 1$. The relations between the response and source fields then simplify. In this approximation, Eqs. (9) give $q_1 = 1/\lambda_a$ and $q_2 = 1/\lambda_b$ for $\lambda_a < \lambda_b$. Then, Eqs. (10) yield $d=1$ and

$$H_x^{(1)} = 0, \quad H_z^{(1)} = -ik_y\lambda_a H_y^{(1)}, \tag{16}$$

$$H_y^{(2)} = 0, \quad H_z^{(2)} = -ik_x\lambda_b H_x^{(2)}. \tag{17}$$

Finally, the boundary conditions give

$$\varphi^r = \varphi^s\left(1 - 2\frac{\lambda_b k_x^2 + \lambda_a k_y^2}{k}\right), \tag{18}$$

$$H_y^{(1)} = \frac{k_y}{k_x}H_x^{(2)} = 2ik_y\varphi^s\left(1 - \frac{\lambda_b k_x^2 + \lambda_a k_y^2}{k}\right). \tag{19}$$

Note that in this approximation λ_c does not enter the result; in other words, the currents along z are small and can be disregarded. The results (16)–(19) can also be obtained directly starting with the London equations (4) and taking advantage of $\partial/\partial z \gg \partial/\partial x_\alpha$.

It is also worth observing that in zero order in $k\lambda \ll 1$, $\varphi^r = \varphi^s$. This means that the x and y components of the response field are the same as those for the source, whereas $h_z^r = -h_z^s$; see Eqs. (2) and (11). In other words, in this approximation the response field is the mirror image of the source field.

B. Isotropic materials

It is readily seen that in this case

$$q_1 = q_2 = \sqrt{\lambda^{-2} + k^2}. \tag{20}$$

The first two equations of Eqs. (8) turn identities, whereas the third gives one of H_i's in terms of two others, e.g., $H_z = (k_x H_x + k_y H_y)/iq$. The boundary conditions then give

$$\varphi^r = \frac{q-k}{q+k}\varphi^s, \quad H_\alpha = \frac{2iq}{q+k}k_\alpha\varphi^s. \tag{21}$$

C. $\lambda_a = \lambda_b < \lambda_c$

This is the case, e.g., of a layered crystal with the *ab* plane being the surface. Equations (9) yield

$$q_1 = \sqrt{\lambda_{ab}^{-2} + k^2}, \quad q_2 = \sqrt{\lambda_{ab}^{-2} + \gamma^2 k^2}, \tag{22}$$

where $\gamma = \lambda_c/\lambda_{ab}$ is the anisotropy parameter. Substituting q_1 in the first two equations of the system (8), we obtain two identical results $H_x^{(1)}k_y - H_y^{(1)}k_x = 0$ which together with the third equation yield

$$H_x^{(1)} = \frac{iq_1 k_x}{k^2}H_z^{(1)}, \quad H_y^{(1)} = \frac{iq_1 k_y}{k^2}H_z^{(1)}. \tag{23}$$

Doing the same for q_2, we obtain from the first two equations $H_x^{(2)}k_x + H_y^{(2)}k_y = 0$ which is compatible with the third only if $H_z^{(2)} = 0$. Hence, we have

$$H_z^{(2)} = 0, \quad H_y^{(2)} = -\frac{k_x}{k_y}H_x^{(2)}. \tag{24}$$

The boundary conditions (13) yield

$$\varphi^r = \frac{q_1 - k}{q_1 + k}\varphi^s, \quad H_z^{(1)} = \frac{2k^2}{q_1 + k}\varphi^s, \quad \boldsymbol{H}^{(2)} = 0. \tag{25}$$

Note that for this case, q_2 along with λ_c drops off the result for *any* source. In other words, the response of a uniaxial superconducting half-space with the c axis normal to the interface to an arbitrary weak source is as if the superconductor were isotropic with the penetration depth λ_{ab}. Clearly, for real samples this statement holds provided the size of the flat sample surface is large compared to the characteristic source size along with the distance from the sample surface to the source; also, the sample should be thick relative to the penetration depth.

D. $\lambda_c = \lambda_a < \lambda_b$

This is the case of the screening by the "side surface" of a uniaxial crystal. Our notation, however, differs from that commonly used (for standard notation, we should have replaced in our formulas λ_c and λ_a with λ_{ab} and $\lambda_b \to \lambda_c$). We then obtain using Eq. (9)

$$q_1 = \sqrt{\lambda_a^{-2} + k^2}, \quad q_2 = \sqrt{1 + \lambda_b^2 k_x^2 + \lambda_a^2 k_y^2}/\lambda_b. \tag{26}$$

With $q = q_1$, Eq. (10) gives

$$H_x^{(1)} = i\frac{k_x}{q_1}H_z^{(1)}, \quad H_y^{(1)} = i\frac{1 + \lambda_a^2 k_y^2}{\lambda_a^2 k_y q_1}H_z^{(1)}, \tag{27}$$

and for $q = q_2$

$$H_x^{(2)} = i\frac{q_2}{k_x}H_z^{(2)}, \quad H_y^{(2)} = 0. \tag{28}$$

Finally, Eqs. (14) give φ^r, $H_z^{(1)}$, and $H_z^{(2)}$ in terms of the source field φ^s.

III. EXAMPLES OF SOURCES

To apply the above formulas for the response fields, one needs the Fourier transform $\varphi^s(\boldsymbol{k})$ of the source field at the interface. Below we provide examples for which $\varphi^s(\boldsymbol{k})$ can be calculated analytically.

A. Point magnetic moment

Consider a magnetic moment $\boldsymbol{\mu}$ situated at the height z_0 above the superconductor. The corresponding potential (in the absence of the superconductor) at $z=0$ is

$$\varphi^s = -\frac{\boldsymbol{\mu}\cdot\boldsymbol{R}}{R^3} = \frac{\mu_z z_0 - \boldsymbol{\mu}\cdot\boldsymbol{r}}{(r^2+z_0^2)^{3/2}}. \tag{29}$$

Here, $\boldsymbol{R}=(x,y,z-z_0)$ is the radius vector originating at the source; the first minus sign is due to the definition $\boldsymbol{h}^s = \nabla\varphi^s$. The 2D Fourier transform is[11]

$$\varphi^s(\boldsymbol{k}) = 2\pi e^{-kz_0}\left(\mu_z + i\frac{\boldsymbol{\mu}\cdot\boldsymbol{k}}{k}\right). \tag{30}$$

The response field can now be calculated with the help of Eqs. (14). In general, this can be done numerically; for $\lambda \ll z_0$, an analytic evaluation is possible.

As an example take the moment directed along z above the flat isotropic superconducting surface; for the isotropic films this problem has been considered in Refs. 6,8, and 12. According to Eq. (18) for the isotropic case $\varphi^r = \varphi^s(1-2k\lambda)$. Transforming back to real space we obtain

$$\varphi^r(r,z) = \mu\left(\frac{Z}{R^3} - 2\lambda\frac{2Z^2-r^2}{R^5}\right),$$

$$R=\sqrt{r^2+Z^2}, \quad Z=z+z_0. \tag{31}$$

Here, the first term is the field of a moment $-\boldsymbol{\mu}$ at $z=-z_0$, i.e., of the image source. The second term is the field of a magnetic quadrupole proportional to λ and situated at the same point.

For the magnetic force microscopy, the quantity of interest is the interaction energy which is given by

$$\mathcal{E} = -\boldsymbol{\mu}\cdot\boldsymbol{h}^r(0,0,z_0)/2$$

$$= -\frac{1}{2}\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\varphi^r(\boldsymbol{k})e^{-kz_0}(-\mu_z k + i\mu_\alpha k_\alpha). \tag{32}$$

The factor 1/2 here is due to the field $\boldsymbol{h}^r$ being induced by the moment $\boldsymbol{\mu}$; see, e.g., Ref. 13. Substituting here Eqs. (18) and (30) and integrating we obtain

$$\mathcal{E} = \frac{\mu_z^2}{8z_0^3}\left[1+\frac{3(\lambda_a+\lambda_b)}{4z_0}\right] + \frac{\mu_x^2+\mu_y^2}{16z_0^3} + \frac{3}{64z_0^4}[\mu_x^2(3\lambda_b+\lambda_a)$$

$$+\mu_y^2(3\lambda_a+\lambda_b)]. \tag{33}$$

In addition to a repulsive force $-\partial\mathcal{E}/\partial z_0$, the magnetic moment $\boldsymbol{\mu}$ experiences a torque, because the energy depends on the moment orientation. It is readily seen that if the position and the value of the magnetic moment are fixed, the minimum of $\mathcal{E}$ corresponds to $\boldsymbol{\mu}$ situated in the plane xy and parallel to the direction of largest λ. The z component of the torque is easily evaluated:

$$\tau_z = \frac{3\mu_\perp^2}{32z_0^4}(\lambda_a-\lambda_b)\sin 2\beta, \tag{34}$$

where $\mu_\perp$ is the in-plane part of the magnetic moment, and β is the angle between $\boldsymbol{\mu}_\perp$ and $\hat{x}$. For $\lambda_a=\lambda_b$, the position of $\boldsymbol{\mu}_\perp$ in the xy plane is arbitrary; still, there is a torque which tends to rotate $\boldsymbol{\mu}$ out of the z direction and to place it in the xy plane.

B. Source as a current loop

Let the source be a circular current of a radius a situated in the plane $z=z_0$. If a and z_0 are of the same order of magnitude (practically, they are both of a few-micron size), modeling of the source by a point-size magnetic moment does not suffice. The scalar potential of the field created by a loop in the plane $z=0$ reads[14]

$$\varphi^s(\boldsymbol{r}) = -\frac{I}{c}\int\frac{d\boldsymbol{S}\cdot\boldsymbol{R}}{R^3}. \tag{35}$$

The source current I flows counterclockwise relative to the z axis so that an area element $d\boldsymbol{S}=dS\hat{\boldsymbol{z}}$. $\boldsymbol{R}$ is the radius vector from this element to the interface point $(\boldsymbol{r},0)$, and the integral is over the area of the current contour. The position of the element $d\boldsymbol{S}=d^2\boldsymbol{r}'\hat{\boldsymbol{z}}$ in our situation is $(\boldsymbol{r}',z_0)$ so that $\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'-z_0\hat{z}$ and

$$\varphi^s(\boldsymbol{r},0) = \frac{Iz_0}{c}\int\frac{d^2\boldsymbol{r}'}{[(\boldsymbol{r}-\boldsymbol{r}')^2+z_0^2]^{3/2}}. \tag{36}$$

For a circular loop of radius a, the integral is over the circle area. Compare this with Eq. (29): the source field can be considered as created by magnetic moments distributed uniformly over the loop area with the density $I\hat{\boldsymbol{z}}/c$ so that the total moment of the loop is $\pi a^2 I\hat{\boldsymbol{z}}/c$.

The 2D Fourier transform of $z_0[(\boldsymbol{r}-\boldsymbol{r}')^2+z_0^2]^{-3/2}$ with respect to $\boldsymbol{r}$ is $2\pi e^{-kz_0}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}$ [see Eq. (30)]; therefore,

$$\varphi^s(\boldsymbol{k}) = \frac{I}{c}2\pi e^{-kz_0}\int d^2\boldsymbol{r}'e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} = \frac{4\pi^2 Ia}{ck}e^{-kz_0}J_1(ka). \tag{37}$$

According to Eq. (25) the 2D Fourier transform of the response potential for isotropic superconductor is

$$\varphi^r(\boldsymbol{k}) = \frac{4\pi^2 Ia}{ck}\frac{q-k}{q+k}e^{-kz_0}J_1(ka). \tag{38}$$

The z component of the response field follows:

$$h_z^r(\boldsymbol{r},z) = -\frac{Ia}{c}\int d^2\boldsymbol{k}\frac{q-k}{q+k}e^{-k(z+z_0)}J_1(ka)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$= -\frac{2\pi Ia}{c}\int_0^\infty dkk\frac{q-k}{q+k}e^{-k(z+z_0)}J_1(ka)J_0(kr). \tag{39}$$

Further, one can evaluate the response flux through a flat probe placed above the superconductor. If the probe is a circular loop of a radius a_p with the center at $\boldsymbol{r}=0$ at the height $z_p \leqslant z_0$, the flux is given by

$$\Phi_z^r = -\frac{4\pi^2 I a a_p}{c}\int_0^\infty dk \frac{q-k}{q+k} e^{-k(z_0+z_p)} J_1(ka) J_1(ka_p). \tag{40}$$

All formulas for the isotropic case have been worked out earlier by Clem and Coffey making use of the cylindrical symmetry of the problem.[4]

C. Interaction with vortices

It is of interest to evaluate the force acting on the vortex tip by screening currents in the sample induced by a circular current as the field source; experiments for which this is relevant are described in Ref. 3. A similar problem for isotropic films and the magnetic moment as a source has been considered in Ref. 6. We do this for materials isotropic in the *xy* plane, for which the force depends only on the distance r from the loop center along with the loop height z_0. One can calculate $F_x(x,0;z_0)$ and replace x with r in the result:

$$\begin{aligned} F_x(x,0) &= \frac{\phi_0}{c}\int_{-\infty}^0 dz j_y \\ &= \frac{\phi_0}{4\pi}\int_{-\infty}^0 dz (h_{x,z} - h_{z,x}) \\ &= \frac{\phi_0}{4\pi}\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} e^{ik_x x}\left(H_x^{(1)} - \frac{ik_x}{q_1} H_z^{(1)}\right). \end{aligned} \tag{41}$$

With the help of Eqs. (22), (23), and (37) we obtain

$$F_r(r) = -\frac{\phi_0 I a}{c\lambda_{ab}^2}\int_0^\infty \frac{dk k e^{-kz_0}}{q_1(q_1+k)} J_1(ka) J_1(kr). \tag{42}$$

Using $1/\lambda^2 = q_1^2 - k^2$ and the substitution $t = ka$, we write

$$F_r = -\frac{\phi_0 I}{2ca}\int_0^\infty dt G\left(\frac{\lambda}{a}t\right) t e^{-z't} J_1(t) J_1(r't),$$

$$G = 1 - \frac{\lambda}{a} t\left(1 + \frac{\lambda^2}{a^2} t^2\right)^{-1/2}, \tag{43}$$

where $z' = z_0/a$ and $r' = r/a$. Usually, the parameter $\lambda/a \ll 1$. Besides, due to the factor $e^{-z't} J_1(t)$, the region contributing to the integral is $0 < t < \min(1/z', 1)$ because of the oscillating $J_1(t)$ at large t (unless $r' \approx 1$). Then, one can expand G in powers of $\lambda t/a$ and keep only the linear term:

$$F_r = -\frac{\phi_0 I}{2ca}\left(f_0 - \frac{\lambda}{a} f_1\right),$$

$$f_0 = \int_0^\infty dt t e^{-z't} J_1(t) J_1(r't),$$

$$f_1 = \int_0^\infty dt t^2 e^{-z't} J_1(t) J_1(r't), \tag{44}$$

The integrals here can be expressed in terms of the hypergeometric functions [see Ref. 11, (6.612.3)] or alternatively of the complete elliptic integrals convenient for numerical evaluation.[12]

One can define a potential $U(r)$ so that $F_r = -dU/dr$:

$$U = -\frac{\phi_0 I}{2c}\left(u_0 - \frac{\lambda}{a} u_1\right),$$

$$u_0 = \int_0^\infty dt e^{-z't} J_1(t) J_0(r't),$$

$$u_1 = \int_0^\infty dt t e^{-z't} J_1(t) J_0(r't), \tag{45}$$

The energy U is defined so that $U(\infty) = 0$; its value at the origin is

$$U(0) = -\frac{\phi_0 I}{2c}\left[1 - \frac{z_0}{(z_0^2 + a^2)^{1/2}} - \frac{\lambda a^2}{(z_0^2 + a^2)^{3/2}}\right]. \tag{46}$$

Thus, the source loop creates a potential well of depth $|U(0)|$ or a barrier of height $|U(0)|$ for vortices underneath depending on the current and vortex directions. This opens an interesting possibility for studying the behavior of vortices (or antivortices) in tunable potentials.[3] In a similar manner, one can evaluate interactions of vortices with other types of sources.

D. Accuracy of the SSM determination of λ

Magnetic fluxes of the response field, in particular $\Phi_z = \int d^2\boldsymbol{r} h_z^r(\boldsymbol{r}, z_p)$ (the integral is over the area of a pickup coil placed at the height z_p above the superconducting surface), can be measured with high accuracy for a given source, given geometry of the coil and a known height z_p. In principle, this leads to a possibility to measure the penetration depth. However, the accuracy of this determination in bound by the accuracy with which the heights z_0 and z_p are known. To demonstrate this consider the response field of an isotropic material for small λ's:

$$\begin{aligned} h_z^r(\boldsymbol{r}, z_p) &= -\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} k\varphi^r(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r} - kz_p} \\ &= -\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} k\varphi^s(\boldsymbol{k})(1 + 2\lambda k) e^{i\boldsymbol{k}\cdot\boldsymbol{r} - kz_p}. \end{aligned} \tag{47}$$

If z_p varies by δz_p, the response field variation is

$$\delta h_z^r = \delta z_p \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} k^2 \varphi^s(\boldsymbol{k})(1 + 2\lambda k) e^{i\boldsymbol{k}\cdot\boldsymbol{r} - kz_p}. \tag{48}$$

If only λ varies, we have

$$\delta h_z^r = -2\delta\lambda \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} k^2 \varphi^s(\boldsymbol{k})(1 + 2\lambda k) e^{i\boldsymbol{k}\cdot\boldsymbol{r} - kz_p}; \tag{49}$$

in other words,

$$\frac{\delta h_z^r}{\delta\lambda} = -2\frac{\delta h_z^r}{\delta z_p}. \tag{50}$$

Therefore, the accuracy of extracting λ from the data on h_z^r cannot be much better than knowledge of z_p (the same is true about the source position z_0). The latter is usually known within a fraction of a micron. This is a severe restriction upon the accuracy of the absolute determination of λ. Still, in principle, SSM allows one to determine accurately the temperature dependence of λ (for fixed z_p and z_0).

IV. THIN FILMS

The Meissner response of superconducting thin films can be probed by the SSM method in yet greater detail than that of bulk samples. Some films have a large Pearl length $\Lambda = 2\lambda^2/d$ (d is the film thickness) exceeding substantially the size of the SSM sensing loop, making the SSM measurement to a local probe. Formally, the problem of a thin film in a field of an external source is simpler than that of the superconducting half-space, because in the film case there is no ''internal problem'' to solve; instead, the film provides a boundary condition for the outside field distribution.

Consider a film in the x,y plane made of a uniaxial material with the c axis at an angle θ to the film normal z. We write the London equation (3) for $i=z$, $h_z - 4\pi\lambda^2(m_{xl}j_{l,y} - m_{yl}j_{l,x})/c = 0$ and integrate it over the film thickness:

$$h_z - \frac{2\pi}{c}\Lambda(m_{xx}g_{x,y} - m_a g_{y,x}) = 0, \quad \Lambda = \frac{2\lambda^2}{d}, \tag{51}$$

where $\boldsymbol{g}$ is the sheet current. Note that only two components of the mass tensor, $m_{xx} = m_a\cos^2\theta + m_c\sin^2\theta$ and $m_{yy} = m_a$, determine the film anisotropy.

Using the relation of sheet currents $\boldsymbol{g}$ to the tangential components of the response field,

$$\frac{2\pi}{c}g_x = -h_y^r(+0), \quad \frac{2\pi}{c}g_y = h_x^r(+0) \tag{52}$$

[$+0$ denotes the upper film face; the tangential components satisfy $h_t^r(+0) = -h_t^r(-0)$], we obtain for $z=+0$

$$h_z^s + h_z^r + \Lambda(m_{xx}h_{y,y}^r + m_a h_{x,x}^r) = 0; \tag{53}$$

for more detail, see Ref. 9. Further, since $h_z^s + h_z^r = k(\varphi^s - \varphi^r)$, we obtain

$$\varphi^r = \frac{k\varphi^s}{k + \Lambda(m_{xx}k_y^2 + m_a k_x^2)}, \quad z=+0. \tag{54}$$

In particular, for $\theta=0$ we have

$$\varphi^r = \frac{\varphi^s}{1+\Lambda_a k}, \quad \Lambda_a = m_a\Lambda = \frac{2\lambda_{ab}^2}{d}. \tag{55}$$

This result holds also for isotropic materials where $m_a = 1$. In this case, one can readily obtain the field h_z for the circular current (37) at the height z_0 above the film; its 2D Fourier transform for $0<z<z_0$ is given by

$$h_z^r(k,z) = -k\varphi^r e^{-kz} = -\frac{4\pi^2 Ia}{c}\frac{J_1(ka)}{1+k\Lambda}e^{-k(z+z_0)}. \tag{56}$$

This field can be measured by SSM.[15]

For completeness, we write down the field on the opposite film side, i.e., for $z<0$, where the response potential is given by

$$\varphi^r(\boldsymbol{r}, z<0) = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\varphi^r(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}+kz}. \tag{57}$$

The London boundary condition (53) should now be written in terms of the field components at $z=-0$:

$$h_z^s + h_z^r - \Lambda(m_{xx}h_{y,y}^r + m_a h_{x,x}^r) = 0, \tag{58}$$

which yields the 2D Fourier transform of the response potential (54) with the minus sign. Proceeding as above, we obtain for the isotropic case

$$h_z^r(k, z<0) = -\frac{4\pi^2 Ia}{c}\frac{J_1(ka)}{1+k\Lambda}e^{k(z-z_0)}. \tag{59}$$

The force acting upon a Pearl vortex situated in the film at a radial distance r from the current ring center is readily evaluated:

$$F_r(r) = -\frac{\phi_0 Ia}{c}\int_0^\infty \frac{ke^{-kz_0}}{1+k\Lambda}J_1(ka)J_1(kr)dk. \tag{60}$$

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