

## SU( $N$ ) evolution of a frustrated spin ladder

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Recent studies indicate that the weakly coupled,  $J_{\perp} \ll J$ , spin-1/2 Heisenberg antiferromagnet with next-nearest-neighbor frustration,  $J_{\times}$ , supports massive spinons for  $J_{\times} = J_{\perp}/2$ . The straightforward SU( $N$ ) generalization of the low-energy ladder Hamiltonian yields two independent SU( $N$ ) Thirring models with  $N-1$  multiplets of massive “spinon” excitations. We study the evolution of the complete set of low energy dynamical structure factors using form factors. Those corresponding to the smooth (staggered) magnetizations are qualitatively different (the same) in the  $N=2$  and  $N>2$  cases. The absence of single-particle peaks preserves the notion of spinons stabilized by frustration. In contrast to the ladder, we note that the  $N \rightarrow \infty$  limit of the four chain model is not a trivial free theory.

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### I. INTRODUCTION

Frustrated quantum antiferromagnets are a source of considerable theoretical and experimental attention—see, for example, Ref. 1. Their characteristics include enhanced classical ground state degeneracies and the suppression of long-range Néel order. In addition to their intrinsic interest, their prominence is fueled by the high- $T_c$  superconducting cuprates, where hole doping frustrates, and ultimately destroys the long-range Néel order of the parent compounds—see, for example, Ref. 2. This motivates the quest for simple models of frustrated quantum magnets, and a detailed understanding of their properties.

Important examples include nearest-neighbor antiferromagnets on frustrated lattices, such as the triangular,<sup>3</sup> pyrochlore, and Kagomé<sup>4</sup> lattices, and further neighbor models on regular lattices. The second variety embraces frustrated chains<sup>5</sup> and ladders,<sup>6–8</sup> the planar pyrochlore,<sup>9–11</sup> and the square lattice antiferromagnet with next-nearest-neighbor interactions. Indeed, the latter model was suggested by Anderson in his influential work<sup>12</sup> on  $\text{La}_2\text{CuO}_4$ , as a means to realize his “resonating-valence-bond” or “quantum spinliquid” state. With isotropic nearest-neighbor exchange,  $J_1$ , this is often referred to as the  $J_1$ - $J_2$  model—for an introduction to spin liquids see Chap. 6 of the book by Fradkin.<sup>13</sup> Other examples include multispin exchange models, and those of dimers.<sup>14</sup> Although enormous progress continues to be made, frustrated quantum magnetism remains theoretically challenging. In general, one must resort to  $1/S$  or  $1/N$  expansions, numerical simulations, or other approximation schemes—see, for example, Ref. 15.

Building on the work of Ref. 8, Nersisyan and Tsvelik have made considerable advances in the so-called confederate flag model.<sup>16</sup> This is an anisotropic version of the much studied  $J_1$ - $J_2$  model, in which the nearest-neighbor exchange has a strongly preferred chain direction—see, Fig. 1. The limit  $J_{\times} \ll J_{\perp} \ll J$  may be viewed as a collection of weakly coupled, but nevertheless interacting chains, and field theory methods may be employed. In general, the massless spinons of the spin-1/2 chain<sup>17</sup> are confined by the interchain interactions. However, along the line,  $J_{\times} = J_{\perp}/2$ , massive spinons emerge in pairs, as the elementary spin excitations of the

coupled system.<sup>8,16</sup> In general, they are neither bosons nor fermions, but have momentum dependent scattering. There have been many speculations about the existence of such excitations in two-dimensional frustrated antiferromagnets, and their possible rôle in high- $T_c$ . The developments of Ref. 16 deserve further investigation.

In this paper we return to an SU( $N$ ) generalization of the ladder introduced in Ref. 8. Our motivation is twofold: first, the large- $N$  approach is known to miss qualitative features in this case,<sup>8</sup> and we wish to track its evolution in detail. Large- $N$  results will be important in two dimensions, and we hope to gain expertise in all the solvable cases. Second, we calculate the dynamical structure factors of the staggered magnetizations. These involve correlation functions of *interacting* Wess-Zumino-Novikov-Witten (WZNW) fields, and their evaluation beyond the ladder is a highly challenging and open problem.<sup>16,18</sup>

The layout of this paper is as follows: in Sec. II we reacquaint the reader with the spin-1/2 model, and its mapping on to two different “parity” sectors.<sup>8,16</sup> We introduce the SU( $N$ ) variant of the low energy action and comment on this choice of generalization. We emphasize that this treatment is not the same as replacing lattice spins by SU( $N$ ) generators;<sup>19</sup> we expand on this in Appendix A where we comment on the connection between filling and SU( $N$ ) “spin” representations. In Sec. III we calculate the dynamical

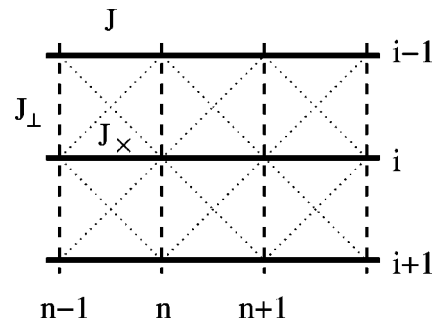


FIG. 1. 2D Heisenberg antiferromagnet with next-nearest-neighbor frustration  $J_{\times} \leq J_{\perp} \ll J$ . The strongly relevant interchain interaction between staggered magnetizations vanishes for  $J_{\times} = J_{\perp}/2$  and renders deconfined spinons.

cal susceptibilities corresponding to the uniform and staggered magnetizations. We conclude in Sec. IV with results for the four chain model. In Appendix B we discuss in detail, the excitations, scattering, and form factors of the  $SU(N)$  Thirring model. We hope this may be of some assistance to the unfamiliar reader.

## II. MODEL

In this section we reacquaint the reader with the spin-1/2 confederate flag model, and its mapping on to two different “parity” sectors;<sup>8,16</sup> we shall specialize to the ladder in due course. Consider a Heisenberg antiferromagnet on a two-dimensional square lattice (of spacing  $a_0$ ) with next-nearest-neighbor exchange interaction  $0 < J_{\times} \leq J_{\perp} \leq J$  as depicted in Fig. 1:

$$H = \sum_{i=1}^L \sum_n [J S_{i,n} \cdot S_{i,n+1} + J_{\perp} S_{i,n} \cdot S_{i+1,n} + J_{\times} (S_{i,n} \cdot S_{i+1,n+1} + S_{i,n+1} \cdot S_{i+1,n})]. \quad (1)$$

It is well established that the low-energy dynamics of a single spin-1/2 (isotropic) Heisenberg chain,

$$H_i^{1D} = \sum_n J S_{i,n} \cdot S_{i,n+1}, \quad (2)$$

are described by the  $\widehat{su}(2)_1$  WZNW model;<sup>20,21</sup> for a review see Refs. 22 and 23. This WZNW model has conserved currents  $\mathbf{J} = \mathcal{L}_{\alpha}^{\dagger} \mathbf{t}_{\alpha\beta} \mathcal{L}_{\beta}$  and  $\bar{\mathbf{J}} = \mathcal{R}_{\alpha}^{\dagger} \mathbf{t}_{\alpha\beta} \mathcal{R}_{\beta}$ , which generate the  $\widehat{su}(2)_1$  Kac-Moody current algebra;  $\mathcal{L}_{\alpha}$  ( $\mathcal{R}_{\alpha}$ ) are left (right) moving fermions with  $\alpha = 1, 2$ , and  $\mathbf{t}_{\alpha\beta}$  are fundamental generators of  $SU(2)$ . The Hamiltonian density,  $H = \int dx \mathcal{H}$ , may be written in the following (Sugawara) form:

$$\mathcal{H}_i^{1D} = \mathcal{N} \hbar v (\mathbf{J}_i \cdot \mathbf{J}_i + \mathbf{\bar{J}}_i \cdot \mathbf{\bar{J}}_i) + \dots \quad (3)$$

Here  $v$  is the spin velocity,  $\mathcal{N}$  is a normalization constant, and the ellipsis stands for less relevant operators. We replace the perturbing lattice spin operators by their continuous, slowly varying, uniform, and staggered components:

$$\mathbf{S}_{i,n}/a_0 \rightarrow \mathbf{S}_i(x) = \mathbf{M}_i(x) + (-1)^n \mathbf{N}_i(x), \quad (4)$$

where  $x \equiv n a_0$  measures the distance along chain  $i$ . Neglecting oscillatory and derivative terms, Hamiltonian (1) becomes  $H = \int dx \mathcal{H}$ , where

$$\mathcal{H} = \sum_{i=1}^L \mathcal{H}_i^{1D} + J_{\perp} a_0 (\mathbf{M}_i \cdot \mathbf{M}_{i+1} + \mathbf{N}_i \cdot \mathbf{N}_{i+1}) + 2J_{\times} a_0 (\mathbf{M}_i \cdot \mathbf{M}_{i+1} - \mathbf{N}_i \cdot \mathbf{N}_{i+1}). \quad (5)$$

In terms of the currents,  $\mathbf{M}_i \equiv \mathbf{J}_i + \bar{\mathbf{J}}_i$ , the Hamiltonian density (5) may be written as<sup>8,16</sup>

$$\mathcal{H} = \sum_{i=1}^L \mathcal{H}_i^{1D} + \lambda_1 (\mathbf{J} + \bar{\mathbf{J}})_i \cdot (\mathbf{J} + \bar{\mathbf{J}})_{i+1} + \lambda_2 \mathbf{N}_i \cdot \mathbf{N}_{i+1} + \dots, \quad (6)$$

where

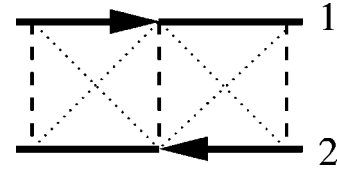


FIG. 2. The ladder Hamiltonian is the sum of two independent  $SU(N)$  Thirring models:  $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-$ . In the sector of positive “parity” the even (odd) chains carry left (right) moving fields. The sector of negative parity is obtained by reversing the arrows. Excitations of the ladder carry this  $\pm$  index and may be produced in both sectors.

$$\lambda_1 = (J_{\perp} + 2J_{\times}) a_0, \quad \lambda_2 = (J_{\perp} - 2J_{\times}) a_0. \quad (7)$$

In particular, for  $J_{\times} = J_{\perp}/2$ , the strongly relevant interchain coupling  $\lambda_2$  between the staggered magnetizations vanishes.<sup>8,16</sup> Setting  $J_{\times} = J_{\perp}/2$ , and neglecting velocity renormalizing terms, the Hamiltonian splits into two independent pieces, or “parity” sectors:<sup>8,16</sup>

$$\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-, \quad (8)$$

where

$$\mathcal{H}_+ = \sum_i \mathcal{N} \hbar v (\mathbf{J}_{2i} \cdot \mathbf{J}_{2i} + \bar{\mathbf{J}}_{2i+1} \cdot \bar{\mathbf{J}}_{2i+1}) + \lambda_1 \mathbf{J}_{2i} \cdot \bar{\mathbf{J}}_{2i+1}, \quad (9)$$

and  $\mathcal{H}_-$  is obtained from  $\mathcal{H}_+$  by the (parity) transformation  $\mathbf{J} \leftrightarrow \bar{\mathbf{J}}$ . In the sector of positive parity, the even (odd) chains carry left (right) moving fields; in the sector of negative parity the situation is reversed (see Fig. 2). Equivalently,  $\mathcal{H}_+$  and  $\mathcal{H}_-$  are interchanged under a shift by  $a_0$  transverse to the chains. Specializing to the ladder,

$$\mathcal{H}_+ = \mathcal{N} \hbar v (\bar{\mathbf{J}}_I \cdot \bar{\mathbf{J}}_I + \mathbf{J}_{II} \cdot \mathbf{J}_{II}) + \lambda_1 (\bar{\mathbf{J}}_I \cdot \mathbf{J}_{II}), \quad (10)$$

where we label the chains by roman numerals to avoid subsequent confusion with space-time indices. Hamiltonian Eq. (10) may be brought into a more familiar form by introducing a spinor, the left component of which resides on one chain and the right resides on the other,

$$\psi_+ = \begin{pmatrix} \mathcal{R}_I \\ \mathcal{L}_{II} \end{pmatrix}. \quad (11)$$

In terms of this spinor, Hamiltonian (10) becomes

$$\mathcal{H}_+ = \mathcal{N} \hbar v (\mathbf{J}_+ \cdot \mathbf{J}_+ + \bar{\mathbf{J}}_+ \cdot \bar{\mathbf{J}}_+) + \lambda_1 (\mathbf{J}_+ \cdot \bar{\mathbf{J}}_+) \quad (12)$$

and similarly for  $\mathcal{H}_-$ . (Equivalently one may perform the chiral interchange  $\mathbf{J}_I \leftrightarrow \mathbf{J}_{II}$  on the original Hamiltonians.) We see that  $\mathcal{H}_+$  is nothing but an  $SU(2)$  Thirring model. That is to say, *the frustrated ladder may be reformulated as the sum of two decoupled  $SU(2)$  Thirring models*, labeled by their parity.<sup>8,16</sup> We emphasize that each of these decoupled models captures the behavior of the coupled ladder, as highlighted in Eq. (11), and not just a single chain. In particular, the elementary excitations of the ladder are those of the  $SU(2)$

Thirring model, namely, massive spinons. These correspond to domain walls separating regions of different spontaneous dimerization.<sup>8</sup>

In this paper we straightforwardly replace the SU(2) currents by SU(N) currents, as suggested in Ref. 8. In each parity sector, the Hamiltonian becomes that of the SU(N) Thirring model with  $N-1$  multiplets of massive spinons (see Appendix B). This is the simplest generalization which retains spinon excitations and parity sectors. We note that the alternative strategy of replacing lattice spins by SU(N) generators leads to problems at the outset.<sup>19</sup> As we discuss in Appendix A, the representation of the generators translates into the filling of the corresponding electronic model. For the critical SU(N) Heisenberg model, with spins in the lowest fundamental representation, the corresponding Hubbard model has one electron per site.<sup>24</sup> The corresponding ‘‘spin’’ density (4) has harmonics at multiples of  $2k_F = 2\pi/Na_0$  due to all the fundamental primaries of the  $\widehat{\text{su}}(N)_1$  WZNW model. In this case, the simple finetuning condition,  $J_\times = J_\perp/2$ , does not remove all relevant perturbations.<sup>19</sup> The absence of such terms is crucial for spinons in the confederate flag model, and such a generalization would be inappropriate. Attempts to reinstate the condition of half-filling with Hubbard chains or the alternating  $N \otimes \bar{N}$  magnet ( $q=N^2$  quantum Potts model) also lead to difficulties; for  $N>2$  they are massive and dimerized<sup>26,27,25,28</sup> and have little in common with the UV limit of decoupled spin-1/2 chains. Since our interest in these generalized models stems from the spinon physics of the confederate flag model, we confine ourselves to the simple minded extension of the low-energy action. We study the SU(N) evolution of the original operators, and retain the terms smooth and staggered magnetizations for these fields.

In the following section, we shall compute the dynamical structure factors of the generalized model. These are a direct probe of the elementary excitations.

### III. DYNAMICAL STRUCTURE FACTOR

In this section we compute the dynamical structure factor (as may be seen by neutrons) for momentum transfers close to the ‘‘soft modes’’ at 0 and  $\pi$ . This is nothing but a Fourier transform of the spin-spin correlation functions,

$$S(\omega, q, q_\perp) \propto \text{Im} i \int_{-\infty}^{\infty} dx \int_0^{\infty} dt e^{i(\omega+i\delta)t - ivqx} \times \langle [S_I^a(t, x) \pm S_{II}^a(t, x), S_I^a(0, 0) \pm S_{II}^a(0, 0)] \rangle. \quad (13)$$

The plus (minus) sign corresponds to  $q_\perp = 0$  ( $q_\perp = \pi$ ), and  $\delta$  ensures convergence of the temporal integral. The longitudinal momentum transfers in the vicinity of  $q=0$  ( $q=\pi$ ) probe the smooth (staggered) components of the spin operators. The task is to relate the spin operators entering Eq. (13) to the operators of the Thirring models, and to evaluate their matrix elements.

#### A. Smooth components

The smooth component of the sum of the chain spin densities may be expressed in terms of the two Thirring models as follows:

$$\mathbf{S}_I + \mathbf{S}_{II}|_{\text{smooth}} = \mathbf{J}_I + \bar{\mathbf{J}}_I + \mathbf{J}_{II} + \bar{\mathbf{J}}_{II} \equiv \mathbf{J}_{0,+} + \mathbf{J}_{0,-},$$

where  $\mathbf{J}_{0,+} = \bar{\mathbf{J}}_I + \mathbf{J}_{II}$  ( $\mathbf{J}_{0,-} = \mathbf{J}_I + \bar{\mathbf{J}}_{II}$ ) is the temporal component of the Thirring current in the model of positive (negative) parity. Simply put, the structure factor  $S(\omega, q \sim 0, 0)$  of the frustrated ladder may be obtained from the correlators of  $\mathbf{J}_0$  in the SU(N) Thirring model:

$$S(\omega, q \sim 0, 0) \propto \text{Im} \sum_{p=\pm} i \int_{-\infty}^{\infty} dx \int_0^{\infty} dt e^{i(\omega+i\delta)t - ivqx} \times \langle [J_{0,p}^a(t, x), J_{0,p}^a(0, 0)] \rangle, \quad (14)$$

where the summation is over parity sectors. The elementary excitations of the SU(N) Thirring model consist of  $N-1$  multiplets of massive particles, corresponding to the fundamental representations of SU(N). The length of the Young tableau is termed the ‘‘rank’’ of the particle,<sup>29</sup> and their masses are given by Eq. (B1). It is convenient to move to a basis of such particles and to parametrize their energy and momentum in terms of rapidity:

$$E_i = m_i \cosh \theta_i, \quad P_i = m_i \sinh \theta_i. \quad (15)$$

One may now insert a complete set of states between the current operators in Eq. (14),

$$\mathbb{I} = \sum_{n=0}^{\infty} \sum_{\epsilon_i} \int \frac{d\theta_1, \dots, d\theta_n}{(2\pi)^{2n}!} \times |\theta_n, \dots, \theta_1\rangle_{\epsilon_n, \dots, \epsilon_1}^{\epsilon_1, \dots, \epsilon_n} \langle \theta_1, \dots, \theta_n| \quad (16)$$

where the  $\epsilon_i$  are the internal (or isotopic) indices carried by the members of each multiplet. Using

$$\begin{aligned} & \epsilon'_1, \dots, \epsilon'_n \langle \theta'_1, \dots, \theta'_n | \mathcal{O}(t, x) | \theta_n, \dots, \theta_1 \rangle_{\epsilon_n, \dots, \epsilon_1} \\ & \equiv e^{i \sum_j (E'_j - E_j)t - (P'_j - P_j)x} \epsilon'_1, \dots, \epsilon'_n \\ & \times \langle \theta'_1, \dots, \theta'_n | \mathcal{O}(0, 0) | \theta_n, \dots, \theta_1 \rangle_{\epsilon_n, \dots, \epsilon_1} \end{aligned} \quad (17)$$

one obtains

$$\begin{aligned}
S(\omega, q \sim 0, 0) &\propto -2\pi \operatorname{Im} \sum_{n=0}^{\infty} \sum_{\epsilon_i} \int \frac{d\theta_1, \dots, d\theta_n}{(2\pi)^n n!} \\
&\times |F_{J_0^a}(\theta_1, \dots, \theta_n)_{\epsilon_1, \dots, \epsilon_n}|^2 \\
&\times \left[ \frac{\delta\left(vq - \sum_j m_j \sinh \theta_j\right)}{\omega - \sum_j m_j \cosh \theta_j + i\delta} - \frac{\delta\left(vq + \sum_j m_j \sinh \theta_j\right)}{\omega + \sum_j m_j \cosh \theta_j + i\delta} \right], \quad (18)
\end{aligned}$$

where  $F_{J_0^a}(\theta_1, \dots, \theta_n)_{\epsilon_1, \dots, \epsilon_n}$  is a multiparticle form factor of the temporal Thirring current,

$$F_{J_0^a}(\theta_1, \dots, \theta_n)_{\epsilon_1, \dots, \epsilon_n} \equiv \langle 0 | J_0^a(0, 0) | \theta_n, \dots, \theta_1 \rangle_{\epsilon_n, \dots, \epsilon_1}. \quad (19)$$

The dominant contributions to Eq. (18) come from the states with the lowest mass. In the case at hand these are two particle states of the (rank-1) fundamental  $\square$ , and its (rank- $N-1$ ) conjugate  $\bar{\square}$ . The box notation serves as a mnemonic for the lowest fundamental multiplet of  $SU(N)$  spinons; in the subsequent analysis we do not distinguish between its  $N$  members. In particular, the current operator couples to the adjoint representation occurring in the  $SU(N)$  tensor product  $\square \otimes \bar{\square}$ ; for  $N=2$ ,  $\bar{\square}$  is  $\square$ . As we discuss in Appendix B, this form factor is

$$F_{J_0}(\theta_1, \theta_2)_{\square, \bar{\square}} \propto m \sinh\left(\frac{\theta_1 + \theta_2}{2}\right) f_{\text{adj}}^{\square, \bar{\square}}(\theta_{12}), \quad (20)$$

where

$$f_{\text{adj}}^{\square, \bar{\square}}(\theta_{12}) = \exp\left\{ \int_0^\infty dx \frac{2 \exp(x/N) \sinh(x/N) \sin^2(x\hat{\theta}/2\pi)}{x \sinh^2 x} \right\} \quad (21)$$

and  $\hat{\theta} = i\pi - \theta$ ; see Eqs. (B24) and (B27). We have suppressed the isotopic and component information in Eq. (21) and concentrated solely on the rapidity dependence. Inserting this into Eq. (18) and performing the  $\theta$  integrations one obtains

$$S(\omega, q \sim 0, 0) \propto \frac{m^2 v^2 q^2}{s^3 \sqrt{s^2 - 4m^2}} |f_{\text{adj}}^{\square, \bar{\square}}[2\theta(s)]|^2, \quad (22)$$

where  $s^2 = \omega^2 - v^2 q^2$ ,  $\theta(s) = \operatorname{arcosh}(s/2m)$  and

$$4m^2 < s^2 < \begin{cases} 16m^2, & N=2, \\ 9m^2, & N=3, \\ 16m^2 \cos^2(\pi/N), & N>3. \end{cases} \quad (23)$$

This result is plotted in Fig. 3 and is exact, provided Eq. (23) is fulfilled. For larger energy transfers there are small corrections due to higher mass states; the upper thresholds correspond to four rank-1 solitons, three rank-1 (or rank-2) solitons, and a rank-2 bound state and its conjugate, respectively. In particular, there are no single-particle bound states appear-

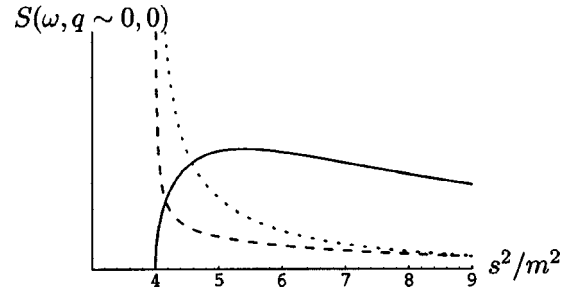


FIG. 3. Exact dynamical structure factor  $S(\omega, q \sim 0, 0)$  for  $N=2$  (solid),  $N=3$  (dashed), and  $N=\infty$  (dotted) with fixed  $q$  and arbitrary normalization. The  $\sqrt{s^2 - 4m^2}$  threshold behavior for the  $N=2$  physical case is replaced by a  $1/\sqrt{s^2 - 4m^2}$  divergence for all  $N>2$ .

ing below the gap; the elementary Thirring excitations correspond to fundamental  $SU(N)$  representations, and do not couple to the current directly, which spans the adjoint.

The result (22) interpolates between two known limits. For  $N=2$ , it coincides with Eq. (34) of Ref. 8, and in the limit  $N=\infty$ , where Eq. (21) tends to unity, we recover the result for free massive fermions.<sup>6,8</sup> In particular, the ( $\theta=0$ ) threshold behavior of Eq. (21) is quite instructive: for  $N=2$  it vanishes like  $\sinh(\theta/2)$ , as may be seen from Eq. (B32), whereas it is finite and nonvanishing for any  $N>2$ . As a result, the structure factor Eq. (22) vanishes as  $\sqrt{s^2 - 4m^2}$  in the physical case of  $N=2$ , but diverges as  $1/\sqrt{s^2 - 4m^2}$  for any  $N>2$ —see Fig. 3. Solely on the basis of the  $N=2$  and  $N=\infty$  limits,<sup>8,6</sup> one might have expected the threshold to get steeper and narrower with increasing  $N$ , but to remain qualitatively correct for  $N<\infty$ . The actual evolution, and the departure even for  $N=3$ , is a sobering example of how  $SU(N)$  treatments may miss simple features over the entire range of  $N$ .

Likewise, the smooth component of the difference of the chain spin densities may be expressed in terms of the two Thirring models as follows:

$$\mathbf{S}_I - \mathbf{S}_{II}|_{\text{smooth}} = \mathbf{J}_I + \bar{\mathbf{J}}_I - \mathbf{J}_{II} - \bar{\mathbf{J}}_{II} \equiv \mathbf{J}_{I,+} - \mathbf{J}_{I,-}$$

where  $\mathbf{J}_{I,+} = \bar{\mathbf{J}}_I - \mathbf{J}_{II}$  ( $\mathbf{J}_{I,-} = \mathbf{J}_I - \bar{\mathbf{J}}_{II}$ ) is the spatial component of the Thirring current in the model of positive (negative) parity. Simply put, the structure factor  $S(\omega, q \sim 0, \pi)$  of the frustrated ladder may be obtained from the correlators of  $\mathbf{J}_I$  in the  $SU(N)$  Thirring model. The corresponding form factor is given by Eq. (B25),

$$F_{J_1}(\theta_1, \theta_2)_{\square, \bar{\square}} \propto m \cosh\left(\frac{\theta_1 + \theta_2}{2}\right) f_{\text{adj}}^{\square, \bar{\square}}(\theta_{12}). \quad (24)$$

We obtain

$$S(\omega, q \sim 0, \pi) \propto \frac{m^2 \omega^2}{s^3 \sqrt{s^2 - 4m^2}} |f_{\text{adj}}^{\square, \bar{\square}}[2\theta(s)]|^2. \quad (25)$$

Once again, this result interpolates between the known  $N=2$  and  $N=\infty$  results,<sup>8,6</sup> and the SU( $N$ ) approach leads to qualitatively incorrect results over the entire range of  $N > 2$ .

### B. Staggered components

We denote the staggered component of the spin on chain I,  $\mathbf{S}_I(t, x)|_{\text{stagg}}$ , by  $\mathbf{N}_I(t, x)$ . In the UV limit (corresponding to decoupled chains and  $m=0$ )  $\mathbf{N}(t, x)$  is a spinless  $\widehat{\text{su}}(N)_1$  primary field with (full) scaling dimension  $\Delta=1-1/N$ . For the ladder we propose the following formula for the long-distance asymptotics of the real space correlation functions:

$$\begin{aligned} & \langle [\mathbf{N}_I(t, x) \pm \mathbf{N}_{II}(t, x)] \cdot [\mathbf{N}_I(0, 0) \pm \mathbf{N}_{II}(0, 0)] \rangle \\ & \propto \langle \mathbf{N}_I(t, x) \cdot \mathbf{N}_I(0, 0) \rangle \pm \langle \mathbf{N}_I(t, x) \cdot \mathbf{N}_{II}(0, 0) \rangle \\ & \propto m^{2\Delta} [K_\Delta^2(mr) \pm K_0^2(mr)] + \dots, \end{aligned} \quad (26)$$

where  $r \equiv \sqrt{z\bar{z}} = \sqrt{x^2 - t^2}$  ( $v=1$ ) and  $K_\nu(x)$  is Macdonald's function,<sup>30</sup> also known as the modified Bessel function of the third kind.<sup>31</sup> The dots stand for more rapidly decaying terms. In order to get a feel for this result we begin by studying a few limits. In the limit  $N \rightarrow \infty$ ,  $\Delta \rightarrow 1$ , each parity sector reduces to noninteracting massive fermions. More specifically,  $\mathbf{N}_i$  may be replaced by the fermion bilinear  $\mathcal{L}_{i,\alpha}^\dagger \mathbf{t}_{\alpha\beta} \mathcal{R}_{i,\beta} + \mathcal{R}_{i,\alpha}^\dagger \mathbf{t}_{\alpha\beta} \mathcal{L}_{i,\beta}$  and one obtains

$$\langle \mathbf{N}_I \cdot \mathbf{N}_I \rangle \propto \langle \mathcal{L}_I^\dagger \mathcal{L}_I \rangle \langle \mathcal{R}_I \mathcal{R}_I^\dagger \rangle, \quad (27)$$

$$\langle \mathbf{N}_I \cdot \mathbf{N}_{II} \rangle \propto \langle \mathcal{L}_I^\dagger \mathcal{R}_{II} \rangle \langle \mathcal{R}_I \mathcal{L}_{II}^\dagger \rangle \quad (28)$$

with the usual massive Dirac fermion correlators:

$$\langle \mathcal{L}^\dagger \mathcal{L} \rangle = 2m \sqrt{\frac{\bar{z}}{z}} K_1(mr), \quad (29)$$

$$\langle \mathcal{L}^\dagger \mathcal{R} \rangle = 2m K_0(mr), \quad (30)$$

see, for example, Chap. 13 of Ref. 32. In Eqs. (27) and (28) we see quite clearly that the correlators of staggered magnetizations are *products* of correlators from the sectors of *different* parity,<sup>8,16</sup> by definition the left and right moving fields on a given chain belong to different sectors. In coupling to the staggered magnetizations, the solitons are still created in pairs, but belong to different sectors.<sup>8,16</sup> In a given sector (i.e., Thirring model) we thus require the matrix elements of *single-soliton* creation operators. The matrix elements of such operators have only recently become available.<sup>33-35</sup> The free fermions appearing in Eqs. (27) and (28) for  $N \rightarrow \infty$  are replaced by chiral fields  $L_s, R_s$ , which are nonlocal single-soliton creation operators and carry the Lorentz spin,  $\pm \Delta/2$ , of a Thirring soliton;<sup>36,37</sup> we take the plus (minus) sign for left (right) movers. These chiral fields are the components of an (interacting)  $\widehat{\text{su}}(N)_1$  primary field, and the Lorentz spin is nothing but the UV conformal dimension. The single-soliton form factors of such operators are governed (upto normalization) solely by their Lorentz transformation properties:

$$\langle 0 | \mathcal{L}_s | \theta \rangle = m^{\Delta/2} e^{\Delta\theta/2}, \quad \langle 0 | \mathcal{R}_s | \theta \rangle = m^{\Delta/2} e^{-\Delta\theta/2}, \quad (31)$$

and their two-point functions are now readily computed:

$$\langle \mathcal{L}_s^\dagger \mathcal{L}_s \rangle = m^\Delta \int d\theta e^{\Delta\theta} e^{-\tau m \text{ch } \theta + ix m \text{sh } \theta} \quad (32)$$

$$= m^\Delta \left( \frac{\bar{z}}{z} \right)^{\Delta/2} 2K_\Delta(m\sqrt{z\bar{z}}), \quad (33)$$

$$\langle \mathcal{R}_s^\dagger \mathcal{R}_s \rangle = m^\Delta \int d\theta e^{-\tau m \text{ch } \theta + ix m \text{sh } \theta} \quad (34)$$

$$= m^\Delta 2K_0(m\sqrt{z\bar{z}}), \quad (35)$$

where  $z = \tau - ix$  and  $\tau = it$ . The results for  $\langle \mathcal{R}_s^\dagger \mathcal{R}_s \rangle$  and  $\langle \mathcal{L}_s^\dagger \mathcal{L}_s \rangle$  follow by interchanging  $z$  and  $\bar{z}$ . In particular, Eq. (33) first appeared in the study of weakly coupled one-dimensional Mott insulators.<sup>34</sup> Replacing the correlators in Eqs. (27) and (28) with these more general expressions, the result (26) follows immediately.

Further, the Macdonald function has the asymptotic expansion given by Eq. (9.7.2) of Ref. 38:

$$\begin{aligned} K_\Delta(mr) &= \sqrt{\frac{\pi}{2mr}} e^{-mr} \\ &\times \left[ 1 + \frac{\mu-1}{8mr} + \frac{(\mu-1)(\mu-9)}{2!(8mr)^2} + \dots \right], \end{aligned} \quad (36)$$

where  $\mu = 4\Delta^2$ . The leading term in Eq. (36) is independent of  $\Delta$ , and at separations  $r \gg 1/m$ , the interchain and intrachain correlations (amusingly) coincide:

$$\langle \mathbf{N}_a(t, x) \cdot \mathbf{N}_b(0, 0) \rangle \sim \frac{m^{2\Delta-1}}{r} e^{-2mr}. \quad (37)$$

Coupling the chains together not only generates exponentially decaying interchain correlations, but also modifies the  $1/r^{2\Delta}$  behavior within the chains.

Substituting Eq. (26) into definition (13) and effecting the Fourier transforms we obtain the following structure factors:

$$S(\omega, q \sim \pi, 0) \propto \frac{[s + \sqrt{s^2 - 4m^2}]^{2\Delta} + (2m)^{2\Delta}}{s \sqrt{s^2 - 4m^2}}, \quad (38)$$

$$S(\omega, q \sim \pi, \pi) \propto \frac{[s + \sqrt{s^2 - 4m^2}]^{2\Delta} - (2m)^{2\Delta}}{s \sqrt{s^2 - 4m^2}}, \quad (39)$$

where  $s^2 = \omega^2 - (q - \pi)^2$ . In deriving these expressions the reader may find the integral representations (32) and (34) more convenient. At threshold,  $S(\omega, q \sim \pi, 0)$  diverges as  $1/\sqrt{s^2 - 4m^2}$  for *all*  $N$ , and we plot this behavior in Fig. 4; the large  $s$  behavior is  $s^{-2/N}$ . Similarly, at threshold,  $S(\omega, q \sim \pi, \pi)$  tends to a constant for *all*  $N$ . In contrast to the magnetization correlators, we obtain qualitatively similar results over the entire range of  $N$ .

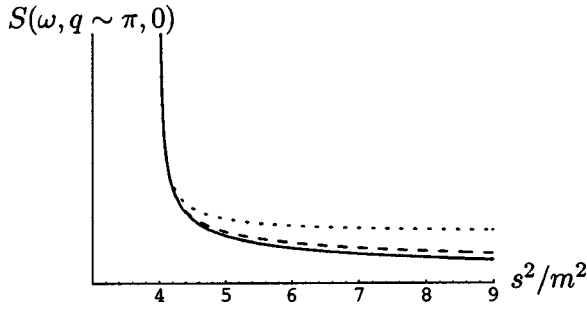


FIG. 4. Exact dynamical structure factor  $S(\omega, q \sim \pi, 0)$  for  $N=2$  (solid),  $N=3$  (dashed), and  $N=\infty$  (dotted). The threshold behavior is  $1/\sqrt{s^2-m^2}$  for all  $N$ , and we have normalized accordingly. The  $SU(N)$  approach leads to qualitatively similar results over the entire range of  $N$ .

#### IV. CONCLUSIONS

In this paper we have studied the  $SU(N)$  evolution of a frustrated spin ladder.<sup>8,16</sup> The dynamical structure factors corresponding to the smooth (staggered) magnetizations are shown to be qualitatively different (the same) in the  $N=2$  and  $N>2$  cases. A robust feature which survives, however, is the absence of coherent single-particle excitations at low energies. This is in stark contrast to the unfrustrated ladder, and reinforces the notion of spinons stabilized by frustration.<sup>8,16</sup> The success or failure of the large  $N$  limit is not only model and approach specific, but also depends on the physical quantity probed. We are thus unable to extract a wider guiding principle other than one of caution. In the case at hand, the Lorentz invariant contribution to the uniform susceptibility has a smooth and unalarming dependence on  $N$ . Yet for  $N=2$ , it vanishes rapidly enough to remove the threshold singularity that persists for all other  $N$ .

In closing we note that the  $N=\infty$  limit of the two chain model is a free theory, but this is not so in general. In particular, each parity sector of the four chain model may be viewed as two (nonchiral)  $\widehat{su}(N)_1$  WZNW models coupled by currents:

$$\mathcal{H}_+ = \widehat{su}(N)_1 + \widehat{su}(N)_1 + \lambda J_+ \bar{J}_+.$$

With transverse periodic boundary conditions the currents  $\bar{J}_+ = \bar{J}_I + \bar{J}_{III}$ ,  $J_+ = J_{II} + J_{IV}$  generate  $\widehat{su}(N)_2$  Kac-Moody algebras, and it is convenient to write:<sup>16,39,40</sup>

$$\mathcal{H}_+ = Z_N + [\widehat{su}(N)_2 + \lambda J_+ \bar{J}_+].$$

The  $Z_N$  parafermions<sup>40</sup> describe gapless nonmagnetic excitations with central charge  $c = 2(N-1)/(N+2)$ . They are unaffected by the interaction (due to the boundary conditions) and for  $N \rightarrow \infty$  they reduce to two Gaussian models. The  $\widehat{su}(N)_2$  model, on the other hand, is rendered massive by the interaction, and is in fact integrable. The mass spectrum coincides with that of the (two chain)  $SU(N)$  Thirring model (B1), but the scattering is notably different. The  $S$  matrix can be extracted from a straightforward generalization of the thermodynamic Bethe ansatz equations derived in Ref. 41.

At low temperatures ( $T \ll M_1$ ) the free energy of the perturbed  $\widehat{su}(N)_k$  WZNW model is given by

$$F/L = -T \sum_{j=1}^{N-1} M_j \int \frac{d\theta}{2\pi} \text{ch } \theta \ln[1 + e^{-\varepsilon_k^{(j)}(\theta)/T}],$$

where in this case  $k=2$ . The excitation energies  $\varepsilon_n^{(j)}$  ( $j=1, \dots, N-1$ ,  $n=1, 2, \dots$ ) satisfy

$$\begin{aligned} T \ln(1 + e^{\varepsilon_n^{(i)}(\lambda)/T}) - T \mathcal{A}_{ij}^* C_{nm}^* \ln(1 + e^{-\varepsilon_m^{(j)}(\lambda)/T}) \\ = \delta_{n,k} M_i \text{ch}(2\pi\lambda/N), \end{aligned}$$

where  $*$  denotes convolution, the kernels  $C_{nm}(\lambda)$  and  $\mathcal{A}_{ij}(\lambda)$  are given in Ref. 41 and  $\lambda = N\theta/2\pi$ . We extract the Bethe equations  $E = \sum_{a=1}^n m \text{ch } \theta_a$  and

$$\begin{aligned} \exp(imL \text{sh } \theta_a) = \prod_{b \neq a} S_0(\theta_a - \theta_b) \\ \times \prod_{\alpha} e_1(\theta_a - \lambda_{\alpha}) \prod_{\beta} \mathcal{E}(\theta_a - \mu_{\beta}), \end{aligned}$$

where

$$\begin{aligned} S_0(\theta) = \exp \left\{ - \int_{-\infty}^{\infty} \frac{d\omega}{\omega} e^{-i\theta\omega} \left[ -1 + \frac{1}{(1 + e^{-2|\omega|\pi/N})^2} \right. \right. \\ \left. \left. \times \left( 1 + \frac{\text{sh } \pi \left( 1 - \frac{2}{N} \right) \omega}{\text{sh } \pi\omega} \right) \right] \right\} \end{aligned}$$

and

$$e_n(x) = \frac{x - i\pi n/N}{x + i\pi n/N}, \quad \mathcal{E}(x) = \frac{e^{Nx/2 - i}}{e^{Nx/2 + i}}.$$

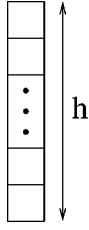
The rapidities  $\lambda_{\alpha}$  and  $\mu_{\beta}$  are distributed according to the  $A^{N-1}$  hierarchy, the details of which do not concern us here. Similar equations occur for the  $SU(N)$  invariant Thirring model ( $k=1$ ) but, without the  $\mu$  rapidities and with a different  $S_0(\theta)$ . In the limit  $N \rightarrow \infty$  one obtains

$$\exp(imL \text{sh } \theta_a) = \prod_{b \neq a} S_0^{\infty}(\theta_{ab}),$$

where  $S_0^{\infty}(\theta) = \exp[-i(\pi/2)\text{sgn}\theta]$ . This is to be contrasted with the  $SU(N)$  invariant Thirring model where  $S_0^{\infty}(\theta) = -1$ . The absence of a simple  $N \rightarrow \infty$  limit will be crucial for multiparticle form factors, and renders excitations with non-trivial statistics. In future publications we hope to study these pertinent issues in more detail. Recent progress on spinon propagation in the four chain model may be found in the work of Smirnov and Tselik.<sup>18</sup>

#### ACKNOWLEDGMENTS

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 FIG. 5. Antisymmetric  $\text{su}(N)$  Young tableau of height  $h$ .

able comments. We also acknowledge support from the U.S. DOE under Contract No. DE-AC02-98 CH 10886.

### APPENDIX A: SPIN OPERATORS

In this appendix we comment on the connection between  $\text{SU}(N)$  spin representations and filling. At each lattice site (labeled by  $n$ ) one may introduce the fermionic spin operators

$$S_n^a = \sum_{\alpha, \beta=1}^N c_{n,\alpha}^\dagger t_{\alpha\beta}^a c_{n,\beta}, \quad (\text{A1})$$

where  $c$  and  $c^\dagger$  obey the canonical fermionic anticommutation relations

$$\{c_{n,\alpha}^\dagger, c_{m,\beta}\} = \delta_{n,m} \delta_{\alpha,\beta}, \quad \{c, c\} = 0, \quad \{c^\dagger, c^\dagger\} = 0 \quad (\text{A2})$$

and the generators  $t^a$  span the algebra  $\text{su}(N)$ :  $[t^a, t^b] = i f^{abc} t^c$ . It is readily verified that spins on different sites commute, whereas those on the same site satisfy the  $\text{su}(N)$  algebra:  $[S_n^a, S_m^b] = i \delta_{n,m} f^{abc} S_n^c$ . In the fundamental representation, the generators are chosen to satisfy  $\text{tr}(t^a t^b) = C \delta^{ab}$ ,  $t^a t^a = C_2 \mathbb{1}$ , with  $C = 1/2$  and  $C_2 = (N^2 - 1)/2N$ ; see Appendix A.3 of Ref. 42.

One may specify the  $\text{su}(N)$  representation on which spin operators  $S_n$  act by the relevant Young tableau—see, for example, Ref. 43. In particular, this fixes the value of the quadratic Casimir  $S_n^2$ , and thus by Eq. (A1), constrains fermion occupation numbers. As we shall demonstrate, the constraint

$$\sum_{\alpha=1}^N c_{n,\alpha}^\dagger c_{n,\alpha} = h, \quad \forall n. \quad (\text{A3})$$

corresponds to the vertical (i.e., antisymmetric) Young tableau of height  $h$ , as depicted in Fig. 5. The constraint (A3) fixes  $h$  electrons per site, and the permissible states to be of the form

$$\psi_{\alpha_1, \alpha_2, \dots, \alpha_h} = c_{n,\alpha_1}^\dagger c_{n,\alpha_2}^\dagger \cdots c_{n,\alpha_h}^\dagger |0\rangle, \quad (\text{A4})$$

where  $\alpha_i \in [1, \dots, N]$ . By virtue of the fermion anticommutation relations (A2), this may be viewed as a tensor of rank  $h$ , antisymmetric under the interchange of any pair of labels  $\alpha$ ; by the standard conventions for Young tableau<sup>43</sup> this corresponds to a vertical diagram of  $h$  boxes. Moreover, it also follows from the anticommutation relations (A2) that there are  $N(N-1) \cdots (N-h+1)/h!$  independent states of form (A4); this coincides with the dimension of the representation

corresponding to the Young tableau of Fig. 5; see Sec. 8.4 of Ref. 43. Further, squaring Eq. (A1) and enforcing constraint (A3), one obtains<sup>51</sup>

$$S_n^2 = \frac{h(N^2 - h)}{2N} + \frac{h(1-h)}{2}. \quad (\text{A5})$$

This coincides with the quadratic Casimir of the  $\text{su}(N)$  Young tableau depicted in Fig. 5;<sup>44</sup> see Eq. (2.19) of Ref. 45, e.g., for the fundamental  $\square$  of  $\text{su}(2)$  ( $h=1, N=2$ ) one obtains  $S_n^2 = 3/4$ , as appropriate for spin 1/2.

Thus, Eq. (A1) supplemented by the constraint (A3) leads to spin operators  $S_n$  described by the Young tableau of Fig. 5.

### APPENDIX B: SU(N) THIRRING MODEL

In this appendix we discuss the excitations, scattering matrices, and form factors of the  $\text{SU}(N)$  Thirring (chiral Gross-Neveu) model. More details may be found in appendix A of Smirnov<sup>29</sup> and the literature.<sup>37,46,36,47,48</sup>

#### 1. Excitations

The excitations of the  $\text{SU}(N)$  invariant Thirring (chiral Gross-Neveu) model are  $N-1$  multiplets of fundamental particles, corresponding to the  $N-1$  fundamental representations of  $\text{SU}(N)$ . Their masses are given by

$$M_a = m \frac{\sin \pi a/N}{\sin \pi/N}; \quad a = 1, 2, \dots, N-1, \quad (\text{B1})$$

and following Smirnov, we shall refer to the label  $a$  as the ‘‘rank’’ of the particle.

#### 2. S matrices

The S-matrix describing the scattering of two (rank-1) fundamental particles in the  $\text{SU}(N)$  invariant Thirring (chiral Gross-Neveu) model is given by<sup>37,29</sup>

$$S_{\epsilon_1, \epsilon_2}^{\epsilon'_1, \epsilon'_2}(\theta) \equiv_{\epsilon_1 \epsilon_2} \langle \theta_2 \theta_1 | S^{\square \square}(\theta) | \theta_1 \theta_2 \rangle_{\epsilon_1 \epsilon_2}, \quad (\text{B2})$$

where  $\theta = \theta_1 - \theta_2$ ,  $\epsilon \in [1, \dots, N]$ , and the S-matrix operator acts on the two body Hilbert space  $\square \otimes \square$ ,

$$S^{\square \square}(\theta) = S_0(\theta) \begin{pmatrix} \theta \mathcal{I} - \frac{2\pi i}{N} \mathcal{P}_{12} \\ \theta - \frac{2\pi i}{N} \end{pmatrix}. \quad (\text{B3})$$

$\mathcal{I}$  and  $\mathcal{P}_{12}$  are the identity and permutation operator, respectively, with matrix elements

$$\epsilon'_2 \epsilon'_1 \langle \theta_2 \theta_1 | \mathcal{I} | \theta_1 \theta_2 \rangle_{\epsilon_1 \epsilon_2} = \delta_{\epsilon'_1}^{\epsilon_1} \delta_{\epsilon'_2}^{\epsilon_2}, \quad (\text{B4})$$

$$\epsilon'_2 \epsilon'_1 \langle \theta_2 \theta_1 | \mathcal{P}_{12} | \theta_1 \theta_2 \rangle_{\epsilon_1 \epsilon_2} = \delta_{\epsilon'_1}^{\epsilon_2} \delta_{\epsilon'_2}^{\epsilon_1} \quad (\text{B5})$$

and

$$S_0(\theta) = \frac{\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right) \Gamma\left(-\frac{\theta}{2\pi i}\right)}{\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right) \Gamma\left(\frac{\theta}{2\pi i}\right)}. \quad (\text{B6})$$

See Eq. (11a) of Ref. 37 or Appendix A (p. 182) of Ref. 29, e.g., for  $N=2$  this reduces to Eq. (6) of Ref. 29. Using the decompositions  $\mathcal{I} = \mathcal{P}^{(+)} + \mathcal{P}^{(-)}$ , and  $\mathcal{P}_{12} = \mathcal{P}^{(+)} - \mathcal{P}^{(-)}$ , one may also write (B3) in the form

$$\begin{aligned} S^{\square\square}(\theta) &\equiv \sum_r S_r^{\square\square}(\theta) \mathcal{P}^{(r)} \\ &= S_0(\theta) \left( \mathcal{P}^{(+)} + \frac{\theta + \frac{2\pi i}{N}}{\theta - \frac{2\pi i}{N}} \mathcal{P}^{(-)} \right), \end{aligned} \quad (\text{B7})$$

where  $\mathcal{P}^{(+)}$  and  $\mathcal{P}^{(-)}$  act on the symmetric and antisymmetric representations occurring in the tensor product  $\square \otimes \square$ ; e.g.,  $3 \otimes 3 = 6 + \bar{3}$  in  $SU(3)$ . Bound states correspond to poles of the  $S$  matrix, with masses

$$m_b = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cosh(\theta_{12})}. \quad (\text{B8})$$

Since  $\Gamma(z)$  is free of zeros, and exhibits simple poles at  $z = 0, -1, -2, \dots$ ,<sup>30</sup> it follows that Eq. (B7) has a single simple pole at  $\theta = 2\pi i/N$  occurring within the physical strip,  $0 < \theta < \pi i$ . This yields the bound state mass of the second fundamental particle,  $M_2 = m \sin(2\pi/N)/\sin(\pi/N)$ , as given by Eq. (B1).

The  $S$  matrix describing the scattering of a (rank-1) fundamental particle off its conjugate (rank- $N-1$ ) may be obtained from Eq. (B7) by the crossing transformation:

$$S^{\square\square}(\theta) = C_{\square} S^{\square\square}(i\pi - \theta) C_{\square}, \quad (\text{B9})$$

where  $C_{\square}$  is the conjugation operator on  $\square$ . Utilizing  $\mathcal{P}^{(0)} = C_{\square} \mathcal{P}_{12} C_{\square} / N$ , and  $\mathcal{I} = \mathcal{P}^{(\text{adj})} + \mathcal{P}^{(0)}$ , one obtains

$$\begin{aligned} S^{\square\square}(\theta) &\equiv \sum_r S_r^{\square\square}(\theta) \mathcal{P}^{(r)} \\ &= -S_1(\theta) \left( \mathcal{P}^{(\text{adj})} + \frac{\theta + \pi i}{\theta - \pi i} \mathcal{P}^{(0)} \right), \end{aligned} \quad (\text{B10})$$

where

$$S_1(\theta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right) \Gamma\left(\frac{1}{2} - \frac{1}{N} - \frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2} - \frac{\theta}{2\pi i}\right) \Gamma\left(\frac{1}{2} - \frac{1}{N} + \frac{\theta}{2\pi i}\right)}, \quad (\text{B11})$$

and  $\mathcal{P}^{(\text{adj})}$  and  $\mathcal{P}^{(0)}$  act on the adjoint and singlet representations occurring in the tensor product  $\square \otimes \bar{\square}$ ; e.g.  $3 \otimes \bar{3} = 8 + 1$  in  $SU(3)$ . In particular, for  $N=2$ , Eqs. (B7) and (B10) coincide (up to sign) as expected from the identification of  $\square$  and  $\bar{\square}$  in  $SU(2)$ . Moreover, Eq. (B7) is also in agreement with Eq. (1.6a) of Ref. 49. The  $S$  matrix (B10) has

a pole at  $\theta = \pi i - 2\pi i/N$  occurring within the physical strip,  $0 < \theta < \pi i$ . This is a cross channel pole.

For the purpose of calculating form factors in the following Sec. , it proves useful to have the  $S$  matrices in an integral form. Taking the logarithm of Eqs. (B6) and (B11) and employing

$$\ln \Gamma(z) = \int_0^{\infty} \frac{dt}{t} \left[ (z-1)e^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}} \right], \quad (\text{B12})$$

one obtains

$$S_a(\theta) = \exp \left\{ \int_0^{\infty} dx f_a(x) \sinh \left( \frac{x\theta}{\pi i} \right) \right\}, \quad (\text{B13})$$

where

$$f_0(x) = \frac{2 \exp(x/N) \sinh[x(1 - 1/N)]}{x \sinh x}, \quad (\text{B14})$$

$$f_1(x) = \frac{2 \exp(x/N) \sinh(x/N)}{x \sinh x}. \quad (\text{B15})$$

In particular, the  $su(2)$  Thirring  $S$  matrix coincides with the sine-Gordon  $S$  matrix with  $\beta^2 = 8\pi$ ,<sup>29</sup>

$$S_a(\theta) \xrightarrow{N=2} -\exp \left\{ i \int_0^{\infty} d\kappa \frac{\exp(-\pi\kappa/2)}{\kappa \cosh \pi\kappa/2} \sin \kappa \theta \right\}. \quad (\text{B16})$$

### 3. Form factors

In the previous paragraphs, we have discussed the elementary excitations of the  $SU(N)$  Thirring model. They are massive particles labeled by their rapidities,  $\theta_i$ , and carrying quantum numbers or isotopic indices,  $\epsilon_i$ . In order to compute correlation functions and dynamical susceptibilities, we will need the matrix elements of various physical operators,  $\mathcal{O}$ , between the vacuum and the (lowest) multiparticle excited states. Such matrix elements are termed *form factors*, and their computation is an important enterprise; see, for example, Ref. 29. As is discussed in Chap. 1 of Smirnov's book,<sup>29</sup> the two-particle form factors

$$F_{\mathcal{O}}(\theta_1, \theta_2)_{\epsilon_1, \epsilon_2} \equiv \langle 0 | \mathcal{O}(0,0) | \theta_2 \theta_1 \rangle_{\epsilon_2, \epsilon_1} \quad (\text{B17})$$

satisfy a matrix (Riemann-Hilbert) problem, also known as Watson's equations,

$$F(\theta_1, \theta_2 + 2\pi i)_{\epsilon_1, \epsilon_2} = F(\theta_1, \theta_2)_{\epsilon'_1, \epsilon'_2} \mathcal{S}_{\epsilon_1, \epsilon_2}^{\epsilon'_1, \epsilon'_2}(\theta_{12}). \quad (\text{B18})$$

This equation may be diagonalized to yield the simpler scalar problem(s)

$$F(\theta_1, \theta_2 + 2\pi i) = F(\theta_1, \theta_2) \mathcal{S}(\theta_{12}), \quad (\text{B19})$$

where  $\mathcal{S}(\theta)$  are the  $S$  matrix eigenvalues. In particular, the Thirring current operator  $\mathbf{J}_{\mu}$  (with  $N^2 - 1$  components)



couples to the adjoint representation occurring in the tensor product  $\square \otimes \bar{\square}$ ; the relevant eigenvalue is

$$S(\theta) \equiv S_{\text{adj}}^{\square \bar{\square}}(\theta) = -S_1(\theta). \quad (\text{B20})$$

Another constraint on the form factors comes from Lorentz invariance. Under a Lorentz boost, corresponding to a simultaneous shift of all rapidities by  $\Lambda$ , the two-particle form factor of an operator  $\mathcal{O}$  of spin  $s$  satisfies

$$F_{\mathcal{O}}(\theta_1 + \Lambda, \theta_2 + \Lambda) = e^{s\Lambda} F_{\mathcal{O}}(\theta_1, \theta_2). \quad (\text{B21})$$

In particular, the left (right) component of the Thirring current has spin  $s = +1$  ( $s = -1$ ) and one obtains

$$F_{j_L^a}(\theta_1, \theta_2) \propto e^{+(\theta_1 + \theta_2)/2} f_{\text{adj}}^{\square \bar{\square}}(\theta_{12}), \quad (\text{B22})$$

$$F_{j_R^a}(\theta_1, \theta_2) \propto e^{-(\theta_1 + \theta_2)/2} f_{\text{adj}}^{\square \bar{\square}}(\theta_{12}). \quad (\text{B23})$$

Note that  $f_{\text{adj}}^{\square \bar{\square}}(\theta_{12})$  is a function of  $\theta_1 - \theta_2$ , and is thus Lorentz invariant. The form factors corresponding to the temporal and spatial components of the current may be written as

$$F_{j_0^a}(\theta_1, \theta_2) \propto m \sinh\left(\frac{\theta_1 + \theta_2}{2}\right) f_{\text{adj}}^{\square \bar{\square}}(\theta_{12}), \quad (\text{B24})$$

$$F_{j_1^a}(\theta_1, \theta_2) \propto m \cosh\left(\frac{\theta_1 + \theta_2}{2}\right) f_{\text{adj}}^{\square \bar{\square}}(\theta_{12}). \quad (\text{B25})$$

Substituting Eq. (B20) and either of Eqs. (B24) and (B25) into Eq. (B19), one obtains a constraint on  $f_{\text{adj}}^{\square \bar{\square}}(\theta)$ ,

$$f_{\text{adj}}^{\square \bar{\square}}(\theta - 2\pi i) = f_{\text{adj}}^{\square \bar{\square}}(\theta) S_1(\theta). \quad (\text{B26})$$

Following the general arguments of Karowski and Weisz<sup>50</sup> [Eqs. (2.18) and (2.19)] Eq. (B26) may be solved by

$$f_{\text{adj}}^{\square \bar{\square}}(\theta) = \exp\left\{\int_0^\infty dx f_1(x) \frac{\sin^2(x\hat{\theta}/2\pi)}{\sinh x}\right\}, \quad (\text{B27})$$

where  $\hat{\theta} = i\pi - \theta$ .<sup>52</sup> Expanding the denominator factors in powers of  $e^{-2x}$ , and employing the identity

$$\exp\int_0^\infty \frac{dx}{x} 2e^{-\beta x} \sinh \gamma x = \frac{\beta + \gamma}{\beta - \gamma}, \quad (\text{B28})$$

one obtains the equivalent representation

$$f_{\text{adj}}^{\square \bar{\square}}(\theta) = \prod_{l,m=0}^{\infty} \left[ \frac{1+l+m}{1-\frac{1}{N}+l+m} \right]^2 \left[ \frac{\frac{1}{2}-\frac{1}{N}+l+m+\frac{\theta}{2\pi i}}{\frac{1}{2}+l+m+\frac{\theta}{2\pi i}} \right] \times \left[ \frac{\frac{3}{2}-\frac{1}{N}+l+m-\frac{\theta}{2\pi i}}{\frac{3}{2}+l+m-\frac{\theta}{2\pi i}} \right]. \quad (\text{B29})$$

Application of Euler's Formula yields

$$f_{\text{adj}}^{\square \bar{\square}}(\theta) = \prod_{l=0}^{\infty} \left[ \frac{\Gamma\left(1-\frac{1}{N}+l\right)}{\Gamma(1+l)} \right]^2 \left[ \frac{\Gamma\left(\frac{1}{2}+l+\frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{N}+l+\frac{\theta}{2\pi i}\right)} \right] \times \left[ \frac{\Gamma\left(\frac{3}{2}+l-\frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{N}+l-\frac{\theta}{2\pi i}\right)} \right]. \quad (\text{B30})$$

As may be seen most clearly from Eqs. (B29) and (B30), this form factor is free of poles in the physical strip  $0 < \theta < i\pi$ , and Watson's minimal equations (in Karowski-Weisz form) are explicitly satisfied. It is indeed a *minimal* form factor. Expressions (B27), (B29), and (B30) conform to the Karowski-Weisz normalization  $F(i\pi) = 1$ , and have the asymptotic behavior

$$\lim_{\theta \rightarrow \pm\infty} f_{\text{adj}}^{\square \bar{\square}}(\theta) \sim \exp(\pm\theta/2N). \quad (\text{B31})$$

In the limit  $N=2$ , one may write Eq. (B27) in the form

$$f_{\text{adj}}^{\square \bar{\square}}(\theta) \rightarrow -i \sinh(\theta/2) \times \exp\left\{\int_0^\infty dx \frac{\sin^2(x\hat{\theta}/2\pi)}{x \sinh x} [\tanh(x/2) - 1]\right\} \quad (\text{B32})$$

and expressions (B24) and (B25) coincide with the known results for the SU(2) invariant Thirring (or sine-Gordon) model; see Eq. (33) of Allen *et al.*<sup>8</sup> or let  $\xi \rightarrow \infty$  in the formula for  $f_\mu(\beta_1, \beta_2)$  given on p.46 of Ref. 29 and note the different definition of the physical strip. In the limit  $N \rightarrow \infty$ , the SU(N) Thirring model maps onto a theory of free massive fermions, as reflected in the explicit  $S$ -matrices. In this limit  $f_{\text{adj}}^{\square \bar{\square}}(\theta) \rightarrow 1$ , and Eqs. (B24) and (B25) coincide with the free fermion form factors given in Eq. (108) of Smirnov.<sup>29</sup>

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- <sup>52</sup>More accurately,  $f(\theta)$  as given by Eq. (B27) may be seen to satisfy  $f(\theta - 2\pi i) = f(\theta)\mathcal{S}_1(\theta - 2\pi i)$ , as follows from Eq. (2.13) of Ref. 50; by definition  $\mathcal{S}_1(\theta - 2\pi i) := \mathcal{S}_1(\theta)$ .