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Number theory implications on the physical properties of elementary cubic networks of Josephson junctions

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Number theory concepts are used to investigate the periodicity properties of the voltage vs applied flux curves of elementary cubic networks of Josephson junctions. It is found that equatorial gaps appearing on the unitary sphere, on which points representing the directions in space for which these curves show periodicity are collected, can be understood by means of Gauss condition on the sum of the squares of three integers.

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After the discovery of high-T$_c$ superconductivity[1] the study of the electrodynamic properties of Josephson junction networks (JJN's) has been undertaken by a growing number of researchers.[2–11] JJN's, indeed, can be adopted as circuital models of these novel superconductors whenever the final product of the fabrication technique adopted is a granular sample.[4] However, the interest in JJN's is not only connected to fundamental aspects of research, but it is also linked to technological applications. In fact, there have been recent proposals for so-called three-dimensional (3D) superconducting quantum interference devices (SQUID's)[12,13] whose basic circuital model consists of a current-biased cubic network of twelve resistively shunted Josephson Junctions (JJ's) in the overdamped limit immersed in a uniform magnetic field. This model system can lead to construction of an ultrasensitive vectorial magnetic field sensor in the same way as the two-junction interferometer has allowed realization of dc SQUID's.[14,15] In the present work we study, by means of known theorems in number theory, the physical implications of the mathematical form of observable periodicities in the time-averaged voltage $\langle \mathrm{v} \rangle$ vs normalized applied flux ψ_{ex} curves of tridimensional cubic JJN's.

In order to investigate the very rich dynamical properties of the 3D SQUID model, some of the authors[16] have adopted an analytical approach based on matrix notation and linear algebra concepts. In this way, the periodicities of the time-averaged voltage $\langle \mathrm{v} \rangle$ with respect to the normalized applied flux ψ_{ex} can be derived for a given direction in space of the external magnetic field. The electrodynamic response of the system is studied through the invariance properties of the vectorial dynamical equation[16]

$$\frac{d}{d\tau}\varphi + \sin\varphi + \frac{1}{2\pi\beta}\mathbf{A}\varphi = f \tag{1}$$

written for the twelve-component gauge-invariant superconducting phase difference vector φ. In Eq. (1) the variable $\tau = (2\pi R I_J/\Phi_0)t$ is a normalized time, R and I_J being the resistive parameter and the maximum Josephson current of the JJ's in the network, respectively, and the parameter β is defined as $\beta = l I_J/\Phi_0$, l being the self-inductance of a single network branch and Φ_0 the elementary flux quantum. The matrix $\mathbf{A}$ in Eq. (1) contains information on the self- and mutual inductance coefficients and the vector f is the external forcing term, written in terms of the bias current I_B and of the normalized applied magnetic flux $\psi_{\mathrm{ex}} = \mu_0 H a^2/\Phi_0$, where μ_0 is the permeability of vacuum, H is the magnetic field amplitude, and a is the length of the side of the cubic network. We may here notice that the dynamical equations for the two gauge-invariant superconducting phases in a dc SQUID can be formally written[16] in the same way as in Eq. (1). Analogously to the case of a dc SQUID, then, the instantaneous voltage $\mathrm{v}(\tau) = (\mathbf{1} - \mathbf{A})(d/d\tau)\varphi$, where $\mathbf{1}$ is the 12×12 unitary matrix, is found by solving the above set of nonlinear ordinary differential equations, given in Eq. (1), by standard numerical routines. However, it can be analytically proven[16] that the $\langle \mathrm{v} \rangle$ vs ψ_{ex} curves are periodic with respect to external flux only for directions $\hat{H}$ of the applied magnetic field $\vec{H}$ which can be expressed as

$$\hat{H} = \frac{(i\hat{x} + j\hat{y} + k\hat{z})}{\sqrt{i^2 + j^2 + k^2}}, \tag{2}$$

where i, j, and k are integers and $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unitary vectors along the three Cartesian axes, which correspond to the directions of the orthogonal sides of the cubic network. Moreover, the possible periods in the $\langle \mathrm{v} \rangle$ vs ψ_{ex} graphs are found to be of the following form:

$$\Delta\psi_{\mathrm{ex}} = \sqrt{i^2 + j^2 + k^2}. \tag{3}$$

Therefore, we shall here study the mathematical implications of Eqs. (2) and (3) on the physical properties of the system. From Lagrange's theorem[17] every positive integer can be expressed as the sum of four squares. However, $\Delta\psi_{\mathrm{ex}}^2$ is the sum of only three squares, so that there might exist nonnegative integers which do not correspond to any value of $\Delta\psi_{\mathrm{ex}}^2$, whatever values of the indices i, j, k we might choose. Indeed, according to Gauss condition,[18] the equation

$$n = i^2 + j^2 + k^2, \tag{4}$$

where n, i, j, k are integers (n being, in particular, nonnegative) is solvable if and only if $n \neq 4^a(8b+7)$, with a and b nonnegative integers. Therefore, it follows that the periodicities of the type

$$\Delta\psi_{\mathrm{ex}}^* = 2^a\sqrt{8b+7}, \tag{5}$$

where a and b are nonnegative integers, are not observable in the $\langle \mathrm{v} \rangle$ vs ψ_{ex} curves for any direction $\hat{H}$ of the external

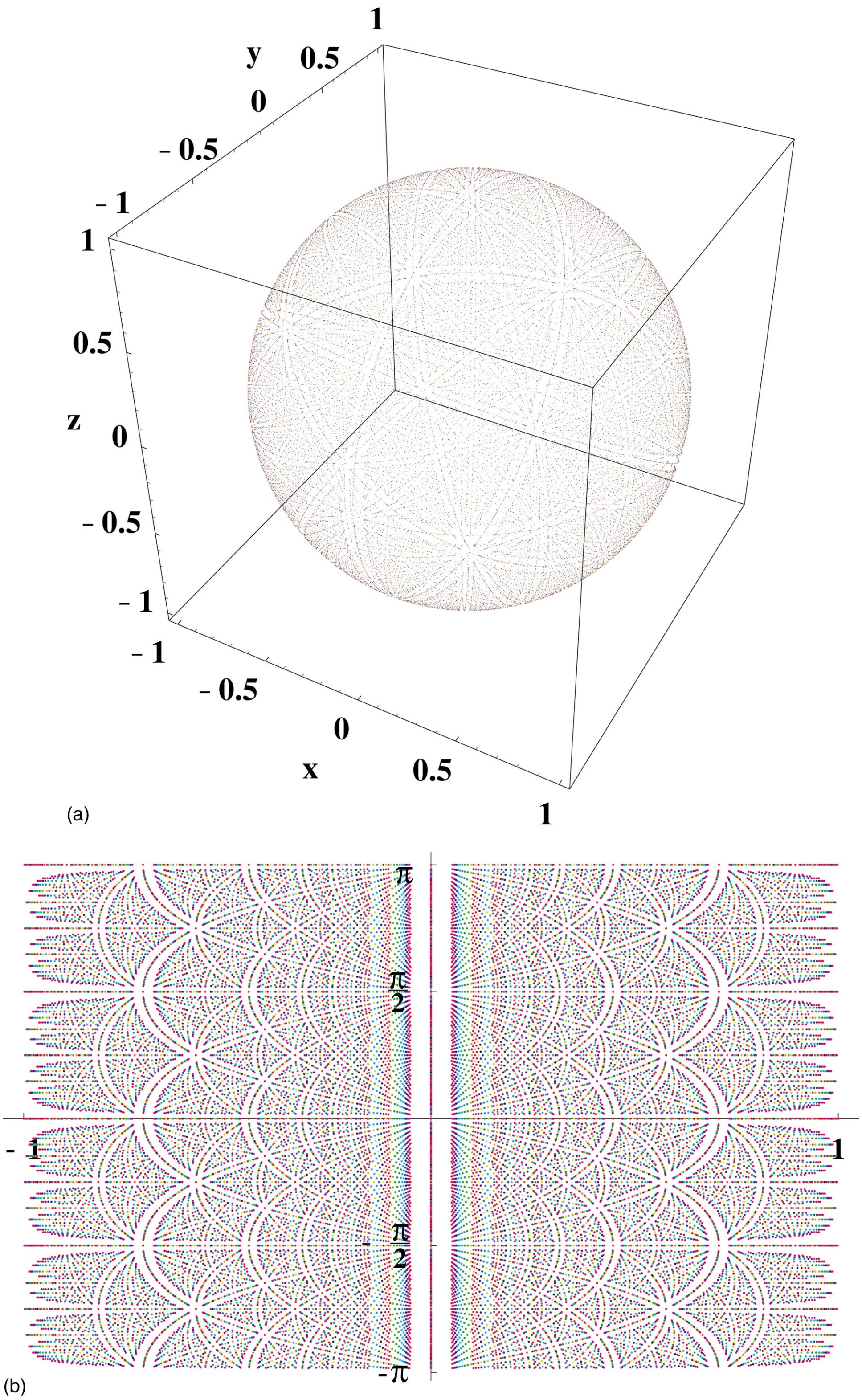

FIG. 1. (Color) (a) Representation on the unitary sphere of the directions in space as specified by Eq. (2) in the text for the value of n, given by Eq. (4), ranging from 1 to 400. (b) Equivalent representation of the directions in space as specified by Eq. (2) in the text for the value of n, given by Eq. (4), ranging from 1 to 1000. The equivalent representation is obtained by wrapping a sheet of length 2π and height 2 around the unitary sphere and by projecting the points on the sphere on the sheet itself, which can now be thought to be a circumscribing cylinder. Finally, the sheet is unwrapped and the points can be visualized on the subset $[-\pi,+\pi]\times[-1,+1]$ of $\Re^2$.

magnetic field. We might therefore argue that, no matter how large we take the number n in Eq. (4), it is never possible to completely fill the unitary sphere S_1 with points corresponding to field directions giving a periodicity equal to $\sqrt{n}$. In order to illustrate this point, let us show, in Fig. 1(a), all the directions on S_1 giving a period equal to $\sqrt{n}$, for the value of n ranging from 1 to 400. The colors of points in Figs. 1(a) and 1(b) range from red to violet, in the iris, as the value of n increases. We notice the presence of regions on the unitary sphere where points are absent. We call these regions equatorial gaps. We may thus try to increase the number of points plotted on S_1, by letting n range from 1 to 1000, in order to see whether these regions are still present. We do this by opportunely giving an equivalent (area preserving) representation of the points on the subset $[-\pi,+\pi]\times[-1,+1]$ of $\Re^2$ and the results are shown in Fig. 1(b). In Fig. 1(b) we notice that, even though regions between equatorial gaps tend to be filled with more points, the same equatorial gaps do not disappear. This is in accordance with what noted by Duke and Schulze-Pillot[19] who have proven that points on S_1, representing the directions in space as specified by Eq. (2), tend to become equidistributed on S_1 for increasing squarefree integers n given by Eq. (4), with $n\neq 4^a(8m+7)$. We recall that a squarefree integer is a product of distinct prime factors (i.e., all appearing to the first power). Naturally, gaps corresponding to not allowed values of n, according to the Gauss condition, should still appear even for very large values of squarefree integers n. Notice that in stating this theorem we only take a single value of n. For our purposes, this is sufficient, since we do not need to consider just one high enough value $n=n_{sf}$ of this type of integers, but all the contributions on the unitary sphere obtained by considering the "allowed" directions in space for positive integers n, such that $n\leqslant n_{sf}$, as done in Figs. 1(a) and 1(b), where a whole range of values for n is considered.

We might now ask for what field directions $\hat{H}$ we have integral periodicities $\Delta\psi_{\mathrm{ex}}=m$. This is a trivial question for fields directed along the coordinate axes, since the periodicity is always an integer in this case. For fields lying along the coordinate planes, where only one of the indices i, j, and k is equal to zero, integral periodicities are obtained for all directions in which the period and the two nonzero indices form a Pythagorean triple. The question is rather subtle for fields for which all of the above indices are different from zero. In this respect, Hurwitz' theorem[20] states that every square s^2, where s is an integer different from 2^m and $5(2^m)$, m being a nonnegative integer, is the sum of the squares of three integers. Apparently, then, we can have as many integral periods as we want except for $\Delta\psi_{\mathrm{ex}}=2^m$ and $\Delta\psi_{\mathrm{ex}}=5(2^m)$, when the magnetic field lies along the directions $\hat{H}$ having all components different than zero. Naturally, these selection rules are only valid for the specific set of field directions we are considering. In fact, if we consider directions $\hat{H}$ for which it is possible to have one zero in its components, for example, we immediately see that periodicities of the type $\Delta\psi_{\mathrm{ex}}=5(2^m)$ are allowed, while those of the type $\Delta\psi_{\mathrm{ex}}=2^m$ are still excluded. The latter integral periodicities reappear if we consider all field directions, including those lying along the coordinate axes. Finally, notice that, for field orientations not lying along the coordinate axes the lowest observable integral period is $\Delta\psi_{\mathrm{ex}}=3$ (since we have $\Delta\psi_{\mathrm{ex}}=\sqrt{(\pm 2)^2+(\pm 2)^2+(\pm 1)^2}$ and all other possible permutations of the index with unitary absolute value).

In conclusion, we might state that number theory is useful in determining the directions in space in which the external field can induce periodicities in the $\langle \mathrm{v}\rangle$ vs ψ_{ex} curves of a so-called 3D SQUID. We have noticed that equatorial gaps appearing on the unitary sphere, where field directions allowing periodicities in the electrodynamic response of the 3D-SQUID model are represented, can be understood by means of Gauss condition. Finally, when one examines the field directions giving integral periodicities, known theorems in the theory of numbers allow us to establish selection rules whose validity is limited to the particular set of directions considered. These results can be helpful in solving the inverse problem related to the external magnetic field vectorial reconstruction by a hypothetical magnetic field sensor based on three-dimensional Josephson junction network models. Further work is still needed, however, to transfer the mathematical properties of the model system to the actual performance of a so-called 3D SQUID, whose prototype is yet to be constructed.

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