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Some consequences of the Luttinger theorem: The Luttinger surfaces in non-Fermi liquids and Mott insulators

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Landau and Luttinger have shown that the properties of so-called Fermi liquids are determined by the Fermi surface of their excitations. The present analysis of mathematics of the Luttinger paper proves that non-Fermi liquids where there are no well-defined excitations and Mott insulators are essentially characterized by the Fermi-surface analog—the Luttinger surface, a feature of particle's propagator.

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I. INTRODUCTION

About five decades ago, Landau[1] formulated a concept of the Fermi liquid (FL) claiming that thermodynamic and kinetic properties of a system of (bare) fermions in FL state are fundamentally the same as those of a weakly interacting gas of Fermi excitations (dressed fermions) notwithstanding the actual strength of interactions between bare fermions. Subsequently, all Landau claims were proven correct (see, e.g., Ref. 2).

The present paper concerns the most fundamental of Landau's claims: that the number of bare articles in FL's (at given chemical potential μ) is equal to the number of Fermi excitations, specifically, what is the relation between the number of bare fermions (actual density) and the system excitations *if* there is a gap in the excitation spectrum, the so-called Mott insulator (MI), or if there are no well-defined Fermi excitations at the chemical potential level [the so-called non-Fermi liquid (NFL)]. The proper tool to use here is the so-called Luttinger theorem. Regretfully, nowadays the contents of the theorem are so densely clouded by folklore that it is sensible to begin with a statement what is meant when I say "the Luttinger theorem."

Back in 1960, Luttinger[3] gave an alternative proof of the Landau conjecture concerning density of particles and excitations in FL's. However, the method he designed reveals a deep relation between density and some properties of renormalized fermion propagator $G(\epsilon,p)$ which transcends FL limitations and can be applied to Mott insulators and NFL's. The relation runs,[3] see also Ref. 2,

$$\frac{N}{V}=2\int_{G(0,p)>0}\frac{d^3p}{(2\pi)^3}. \tag{1}$$

The integral in momentum space is taken over the region where the propagator at the chemical-potential level $\epsilon=0$, $G(\epsilon=0,p)$ is positive. Picking the FL propagator

$$G(\epsilon,p)\approx\frac{Z}{\epsilon-\epsilon_{ex}(p)}, \tag{2}$$

one immediately comes to the Landau statement: density equals number of excitations below Fermi surface $|p|=P_F:\epsilon_{ex}(P_F)=0$.

The author applied[4] the Luttinger theorem in its most general form (1) to NFL's where there is no fermionic excitations at $\epsilon=0$, and consequently no pole (2) in the propagator. Now the theorem claims that density is equal to the volume bound by what I called Luttinger surface (LS) $|p|=P_L$, the Luttinger momentum P_L defines the point where $G(0,p)$ changes sign passing through either infinity or zero.

Finally, one may apply formula (1) to a MI. Now there is a gap in excitation spectrum $\epsilon_{ex}(p)$ and imaginary part of propagator $G''(\epsilon,p)$ has zero value in a finite range around chemical potential $\epsilon=0$. Due to analytic properties of propagators, G (see Sec. II below) cannot have singularities on real axis inside the gap and zeros of $G(0,p)$ are required to account for bare particle's density according to form (1). It defines the Luttinger surface $|p|=P_L$ by $G(0,P_L)=0$. A model propagator of this kind is

$$G(\epsilon,p)=\frac{\epsilon+v(|p|-P_L)}{\epsilon^2-\Delta^2-v^2(|p|-P_L)^2}, \tag{3}$$

easily recognizable as BCS electron propagator with v and Δ being the Fermi velocity and superconducting gap and P_L coinciding with the Fermi momentum P_F. Unfortunately there is no meaningful analog of the Luttinger theorem in superfluid states due to breaking of the particle number conservation law (phase or gauge-invariance breaking).

Obviously there are two types of insulating states. First, a *compressible* insulator where P_L and the density are smooth functions of the chemical potential μ. This state is reminiscent of a superconductor or a superfluid state in liquid He3, albeit without Bose condensation. Second, a physical mechanism may fix the density at which a gap is opened. Only one such *incompressible* insulator is known in a genuine liquid: the quantum liquid of fractional quantum Hall effect where the famous Laughlin *commensurability* between density and magnetic flux is the reason. Of course, commensurability can also be provided by the underlying crystalline lattice. A known example is a one-dimensional metal at half filling where Umklapp scattering opens the gap and fixes the density.

Simple as it is, this approach to MI's is not irreproachable.[5,6] The trouble is that the zeros of G on real axis signify poles in self-mass operator $\Sigma(\epsilon,p)$ and the Luttinger derivation of Eq. (1) (see Sec. II) involves integration of Σ over frequencies ϵ. Obviously the result depends on how these singular integrals are regularized. Altshuler *et al.*[5] employed a regularization which amounts to radical redefinition of the ground state introducing a sort of chiral anomaly in the theory and as a result derived a formula for density different from Eq. (1).

In what follows I present another regularization. My way is to start at a finite temperature T and to perform the Luttinger calculations in Euclidean sector of energy–momentum space (imaginary energies) where there are *no poles* in self-mass. At the end, by turning T to zero I arrive at the old Luttinger result (1). Of course, this does not mean that the state of Altshuler *et al.* is impossible, but that it exists only at zero temperature and may be reached only via zero-temperature phase transition.

In Sec. II the Luttinger formula (1) is derived for fermionic liquids and crystals. In Sec. III some consequences for MI's and NFL's are discussed.

II. THE LUTTINGER THEOREM

The Luttinger derivation of formula (1) is substantially based on analytic properties of propagators in the energy ϵ complex plane. We will use three kinds of propagators at finite temperature; the retarded propagator $G_r(\epsilon,p)$ analytic in the upper half plane of complex ϵ, the advanced propagator analytic $G_a(\epsilon,p)$ analytic in the bottom half plane, and the special finite temperature Matsubara propagator $M(\xi_n,p)$ defined at discrete "imaginary energies" $i\xi_n = i\pi T(2n+1)$; where n are integers. [The corresponding definitions and inter-relations are easily available, see, e.g., Ref. 2.]

Actually all three have a common Hilbert representation:

$$G_{r,a}(\epsilon)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho(x)}{x-\epsilon}dx; \quad \begin{matrix}\mathrm{Im}\epsilon>0\\ \mathrm{Im}\epsilon<0\end{matrix}. \tag{4}$$

The real function $\rho(x)$ is obviously connected with imaginary parts of $G_{r,a}$ at real energies ϵ:

$$\rho(\epsilon)=G_r''(\epsilon)=-G_a''(\epsilon). \tag{5}$$

For fermions, G_r'' and ρ are *negative* for all energies[2] and as a consequence $G_r(\epsilon)$ does not have zeros in upper half plane and $G_a(\epsilon)$ is zeroless in bottom half plane (see Appendix A). Finally, the Matsubara propagator is given by

$$M(\xi_n)=\begin{cases}G_r(i\xi_n), & \xi_n>0\\ G_a(i\xi_n), & \xi_n<0.\end{cases} \tag{6}$$

The density N/V as a function of chemical potential and temperature is (see Ref. 2)

$$\frac{N}{V}(\mu,T)=\frac{2T}{(2\pi)^3}\sum_n e^{i\xi_n\tau}\int d^3p\, M(\xi_n,p,T), \tag{7}$$

where $\tau\to+0$. At zero temperature the summation over n goes into integration:

$$\frac{N}{V}(\mu)=\frac{2}{(2\pi)^4}\int_{-\infty}^{\infty}d\xi\int d^3p\, e^{i\xi\tau}\mathbf{M}(\boldsymbol{\xi},\mathbf{p},\mathbf{0}). \tag{8}$$

The sum in Eq. (7) converges to Eq. (8) smoothly enough if there is no phase transition at $T=0$ (see Appendix B).

Now we may transpose step by step the calculations done by Luttinger[2,3] in the framework of the Feynman field theory at real time to the Matsubara field theory at imaginary time. Density (8) is expressed as

$$\frac{N}{V}=-\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty}d\xi\, e^{i\xi\tau}\int d^3p\, M(\xi,p) \times\left(\frac{\partial}{\partial\xi}M^{-1}(\xi,p)+\frac{\partial\sigma}{\partial\xi}(\xi,p)\right), \tag{9}$$

where $\sigma(\xi,p)$ is the self-mass in the Matsubara theory: $M^{-1}=i\xi-\epsilon(p)-\sigma$. The integral of the second term here actually is zero as it was in the real time version. Namely,

$$\int d_\xi d^3p\, M\frac{\partial\sigma}{\partial\xi}=\int d_\xi dp\frac{\partial}{\partial\xi}M\sigma-\int d_\xi dp\,\sigma\frac{\partial M}{\partial\xi}.$$

The above integration by parts is perfectly legal here because even in the case of a Mott insulator the Mitsubara self-energy σ does *not* have poles in $-\infty<\xi<\infty$ and the objections of Altshuler *et al.*[5] are not held. The first integral on the right is zero because at $\xi\to\infty$ propagator $M\to 1/i\xi\to 0$ and $\sigma\to$const. The second integral disappearance is proven by the same effective arguments of Luttinger.[2,3] M and σ being nonsingular the Luttinger construction is solid.

Thus, we arrive at the formula for density:

$$\frac{N}{V}=\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty}d\xi e^{i\xi\tau}\int d^3p\frac{\partial}{\partial\xi}\ln M. \tag{10}$$

Formula (10) may be rewritten in terms of an integral along a path in complex energy ϵ plane of two analytic propagators $G_r(\epsilon)$ and $G_a(\epsilon)$. Namely,

$$\frac{N}{V}=\frac{2i}{(2\pi)^4}\int d^3p\left\{\int_{C_+}e^{\epsilon\tau}d\epsilon\frac{\partial}{\partial\epsilon}\ln G_r(\epsilon) +\int_{C_-}e^{\epsilon\tau}d\epsilon\frac{\partial}{\partial\epsilon}\ln G_a(\epsilon)\right\}, \tag{11}$$

where C_+ and C_- are correspondingly the top and the bottom half of the imaginary axis (Fig. 1).

The exponential factor $e^{\epsilon\tau}$ in Eq. (11) permits to turn the paths $C_\pm$ to the left. Moreover, $\ln G_r$ and $\ln G_a$ being analytic in the respective half planes ($G_{r,a}$ do not have zeros, see above) there are no obstacles on the way and the integrations in Eq. (11) are reduced to the integrations just above (C_{0+}) and just below (C_{01}) the negative half of real axis. We have

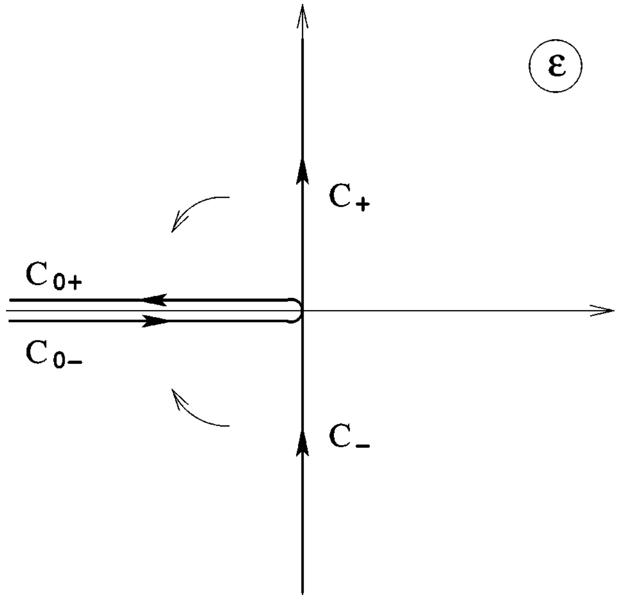

FIG. 1. Integration path in Eq. (11).

$$\begin{aligned}\frac{N}{V}&=\frac{2i}{(2\pi)^4}\int d^3p\int_{-\infty}^{0}d\epsilon\frac{d}{d\epsilon}\ln\frac{G_r}{G_r^*}\\&=-\frac{2}{\pi(2\pi)^3}\int d^3p\int_{-\infty}^{0}d\epsilon\frac{d}{d\epsilon}\phi_r\\&=\frac{2}{(2\pi)^3}\int d^3p\frac{1}{\pi}[\phi_r(-\infty)-\phi_r(0)],\end{aligned}\tag{12}$$

where ϕ_r is the phase of retarded propagator. $G_r(-\infty)$ is obviously real and negative ($\phi_r=\pi$) and we arrive at the Luttinger theorem (1).

Strictly speaking the absence of winding is implicit here but it is precluded by the fact that $I_m G_r$ does not change sign on the way from $-\infty$ to 0.

The proof just presented is good for any state of our system: FL, NFL, or MI [in other words, either of poles or zeros can be used to change the sign of G_r in Eq. (1)]. The only way to incapacitate the Luttinger theorem in form (1) is to assume that the limit $T\rightarrow 0$ is discontinuous. Actually, one has to require that the whole line $T=0$ is a line of phase transitions.

Extension of the above considerations to crystalline states of fermions is a mere technicality.[3] Propagators become matrices $\hat{G}_{r,a}(\epsilon,k)$ in the space of Bloch functions $\psi_{nk}(x)$, where k is corresponding quasimomentum and $n=0,1,\ldots$ numerates bands. We will use the single notation n for both band number n and spin projection σ:$n=(n\sigma)$, thus in the case of a liquid $\hat{G}_{r,a}$ are matrices in spin variables only.

Actually the Bloch set ψ_{nk} may be chosen at one's convenience (see below). The important thing here is that in the large energies limit propagators are unity matrices:

$$\hat{G}(\epsilon,k)\simeq\frac{1}{\epsilon}\hat{1},\quad \epsilon\rightarrow\pm\infty.\tag{13}$$

The physical density now is an average over the unit-cell volume v. In terms of the Matsubara propagator density,

$$\begin{aligned}\frac{N}{V}&=\sum_{\sigma}\int_v\frac{d^3x}{v}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi\tau}d\xi M_{\sigma\sigma}(x,x;\xi)\\&\equiv\frac{1}{(2\pi)^4}\int d^3k\int_{-\infty}^{\infty}e^{i\xi\tau}d\xi\,\mathrm{Tr}M(\xi,k);\end{aligned}\tag{14}$$

Tr here includes the summation over spin projection.

The above Luttinger procedure gives here

$$\frac{N}{V}=\frac{i}{(2\pi)^4}\int d\xi\,e^{i\xi\tau}\int d^3k\sum_{n,m}(M^{-1}_{nm})\frac{\partial}{\partial\xi}\hat{M}_{mn}$$

or using determinant algebra,

$$\frac{N}{V}=\frac{i}{(2\pi)^4}\int_{-\infty}^{\infty}d\xi e^{i\xi\tau}\int d^3k\frac{\partial}{\partial\xi}\ln\det\hat{M}.\tag{15}$$

It is shown in Appendix C that determinants $\det\hat{G}_{r,a}(\epsilon)$ do not have zeros in the corresponding half planes at complex energies, which again permits to turn paths $C_\pm$ in Fig. 1 into paths $C_{0\pm}$ without obstacles. Finally, we arrive at the formula analogous to Eq. (12):

$$\frac{N}{V}=\frac{i}{(2\pi)^4}\int_{-\infty}^{0}d\epsilon\int d^3k\frac{\partial}{\partial\epsilon}\ln\frac{\det\hat{G}_r}{\det\hat{G}_a}.\tag{16}$$

Now it is the time to follow Luttinger[3] and to make a choice of the Bloch set ψ_{nk}. The matrices $\hat{G}_{r,a}$ at zero energy are Hermitian and have real eigenvalues $G^L_{nn}(0,k)$ corresponding to the diagonalizing set ψ^L_{nk}, L stands for Luttinger here. The procedure brings us to the trivial generalization of formula (1),

$$\frac{N}{V}=\frac{1}{(2\pi)^3}\sum_n\int_{G^L_{nn}(0,k)>0}d^3k.\tag{17}$$

Together with the definition of bands ψ^L_{nk}, formula (17) represents the general Luttinger theorem.

III. NON-FERMI LIQUIDS AND MOTT INSULATORS

The case of Fermi liquids was extensively discussed in the original Luttinger paper.[3] Propagators of NFL's are given by a general scaling formula (see, e.g., Ref. 4)

$$G(\epsilon,p)=\frac{1}{\epsilon^{1-\alpha}}f\left(\frac{|p|-P_L}{\epsilon^{1/\kappa}}\right);$$

$$G(0,p)=\frac{\mathrm{const}}{(|p|-p_L)^{\kappa(1-\alpha)}},$$

$$G(\epsilon,p_L)=\frac{\mathrm{const}}{\epsilon^{1-\alpha}},\tag{18}$$

where α,κ are positive scaling exponents. For simplicity we confine ourselves to $\kappa=1$.

At $0<\alpha<1$ the propagator at the chemical-potential level $\epsilon=0$ in Eq. (1) changes sign when it crosses the LS $|p|=P_L$. On the surface itself G is infinite but singularities are somewhat weaker than in a genuine Fermi liquid $\alpha=0$.

At larger α, $1<\alpha<2$, the propagator is nullified on the LS but its behavior close to the LS remains substantially singular.

Finally, at $\alpha>2$ the propagator close to the LS is virtually analytic:

$$\mathrm{Re}\, G_r(\epsilon,p)\simeq A\epsilon+B(|p|-p_L)+\cdots,$$

$$\mathrm{Im}\, G_r(\epsilon,p)\simeq C\,\epsilon^2+\cdots. \tag{19}$$

In the case of a liquid Mott insulator (if it really exists) imaginary part of G is zero inside a gap: $-\Delta_-<\epsilon<\Delta_+$ and close to the LS,

$$G_a(\epsilon,p)=G_r(\epsilon,p)\simeq A\epsilon+B(|p|-p_L). \tag{20}$$

As it was discussed above, the Luttinger theorem is valid for both NFL's and MI's and Eqs. (18)–(20) give for density

$$\frac{N}{V}=\frac{p_L^3}{3\pi^2}; \tag{21}$$

for FL's the Luttinger momentum p_L coincides with the Fermi momentum p_F. Now the question is how the Luttinger momentum p_L can be measured, if at all. Its value in FL's—the Fermi momentum p_F—is conveniently detectable by angle-resolved photoemission, which gives so-called spectral density $S_{(p,\epsilon)}$ [being just imaginary part of propagator $G''(p,\epsilon)$]. At a fixed $p<p_F$ and close enough to p_F the spectral density S as a function of energy ϵ exhibits sharp maximum at $\epsilon=v_F(p-p_F)$ [see Fig. 2(a)]. The maximum's width is defined by the excitations lifetime and its area by the residue Z of the FL propagator [see form (2)].

The tails in Fig. 2(a) at $\epsilon>0$ and at $\epsilon\ll v_F(p-p_F)$ testify to residual particle interactions.

In NFL states, there are no maxima that sharp as in Fig. 2 and a substantial part of spectral density at $p<p_L$ is transferred above the chemical-potential level $\epsilon=0$.

Finally, in NFL's with zeros in propagators $[G_r(0,p_L)=0$, see form (17)] and in MI's roughly 50% of spectral density at $p\approx p_L$ is transferred across the chemical-potential level $\epsilon=0$ across the gap in MI's, see Figs. 3(a,b). This can be seen directly from the spectral representation (3), written say, for MI's:

$$G_r(p,0)=\frac{1}{\pi}\int_{-\infty}^{-\Delta_-}\frac{G_r''(p,x)}{\mathbf{x}}dx+\frac{1}{\pi}\int_{\Delta_+}^{\infty}\frac{G_r''(p,x)}{x}dx. \tag{22}$$

A zero at the left-hand side at $p=p_L$ implies the complete compensation of the negative $(x<0)$ and the positive integrals $(x>0)$ at the right-hand side $[G_r''(\epsilon)\leqslant 0$ for all $\epsilon]$.

Granted, the *real* measurement of p_L requires the complete knowledge of spectral density for all values of p and ϵ. Nevertheless, the massive spectral density shift on the Luttinger surface seems to be a valid qualitative indicator.

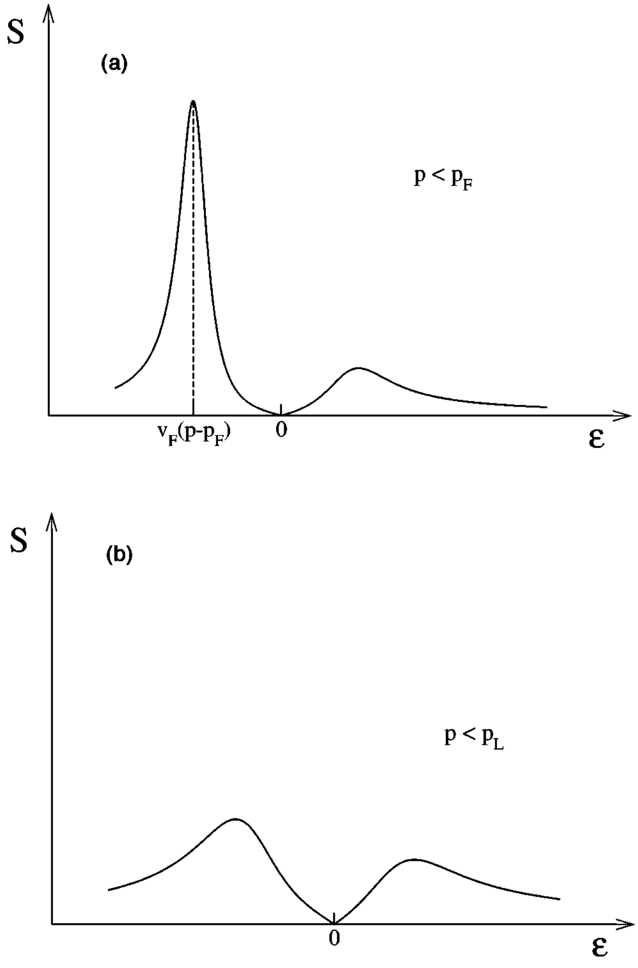

FIG. 2. Energy dependence of spectral density $S(\epsilon,p)$ at fixed momentum p for (a) FL state, (b) NFL state with small exponent $\alpha<1$.

The case of electrons in crystals does not entail complications: one discusses the states listed above: FL, NFL, and MI for each of the Luttinger bands in form (17). The case of atomic quantum Fermi crystals, notably He3, deserves a comment. In a genuine quantum crystal there is no reason to require that each crystal site (unit cell) is fully occupied, the occupation numbers may well be less than 1 and the corresponding density less than the volume of unit cell of reciprocal lattice being necessarily equal to the volume confined by the Luttinger surface in Eq. (17).

The Luttinger surface, of course, may turn out just the Fermi surface of a Fermi-liquid state which was visualized decades ago as a gas of quantum-mechanically delocalized vacancies. The signature of such a state is the existence of mass flow always absent in a classical harmonic crystal.

In a MI state a mass flow is forbidden. One may say that the above-mentioned vacancies are quantum mechanically (not classically) localized. Intermediate NFL states will exhibit a variety of peculiar nonstationary mass flow phenomena.

To conclude, the Luttinger surface as defined by Eq. (1) or

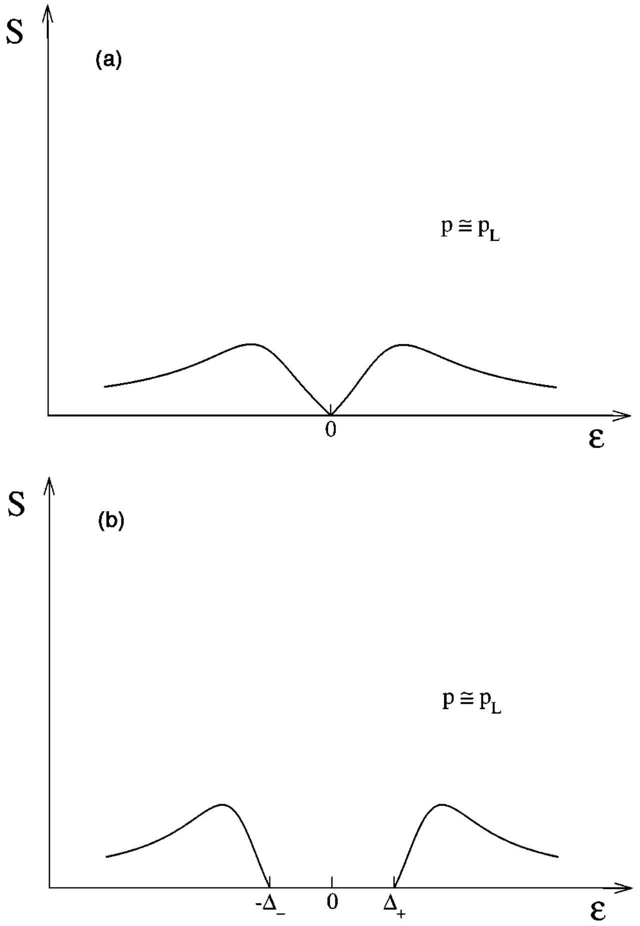

FIG. 3. Energy dependence of spectral density $S(\epsilon,p)$ at fixed momentum p close to p_L for (a) NFL state with large $\alpha>1$, (b) MI state.

(17) is a meaningful concept. Not being as sharp and measurable feature as the Fermi surface it is easily recognizable by the massive spectral density transfer in its vicinity and, I believe, has nonzero heuristic potential.

APPENDIX A: Absence of zeros of $G_{r,a}$ in complex plane of ϵ

The easiest way to prove the absence of zeros is the direct use of representation (3). If $\epsilon=\epsilon'+i\epsilon''$ is a complex zero, $G_{r,a}(\epsilon)=0$, then

$$\int_{-\infty}^{\infty}\frac{(x-\epsilon')\rho(x)dx}{(x-\epsilon')^2+(\epsilon'')^2}=0,$$

$$\int_{-\infty}^{\infty}\frac{\rho(x)dx}{(x-\epsilon')^2+(\epsilon'')^2}=0.$$

These two lines are obviously incompatible if ρ is negative for all x.

APPENDIX B: The temperature dependence in formula (6)

The sum over Matsubara frequencies n in Eq. (6) may be written as a complex integral:

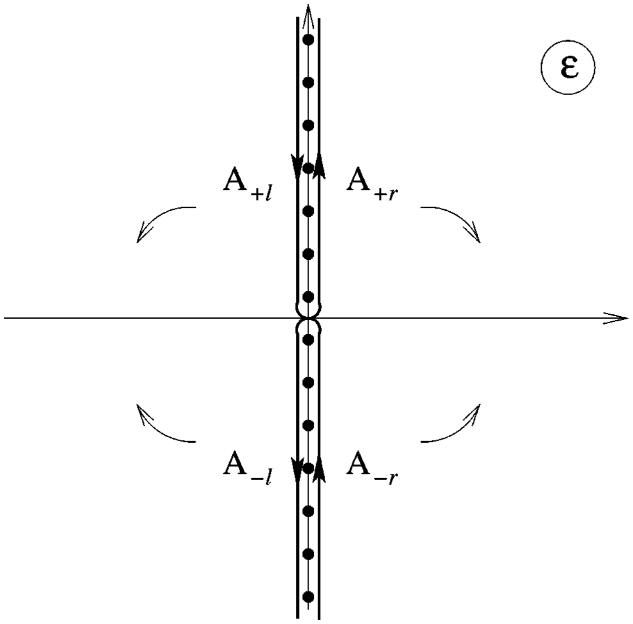

FIG. 4. Integration path in Eq. (B1).

$$T\sum_n e^{i\xi_n\tau}M(\xi_n,T)=\frac{1}{4\pi i}\int_{A_{+r}+A_{+L}}e^{\epsilon\tau}G_r(\epsilon,T)\tanh\frac{\epsilon}{2}d\epsilon+\frac{1}{4\pi i}\int_{A_{-r}+A_{-l}}e^{\epsilon\tau}G_a(\epsilon,T)\tanh\frac{\epsilon}{2T}d\epsilon. \tag{B1}$$

The paths $A_{\pm r,l}$ are shown in Fig. 4 where dots indicate the points $i\xi_n=i2\pi T(2n+1)$. Relations (5) between $M(\xi)$ and $G_{r,a}(\xi)$ were used here.

Subtracting the integral in Eq. (7),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\, e^{i\xi\tau}M(\xi,0),$$

from the right-hand side of Eq. (B1) and using relations (5) again we represent the temperature-dependent part of Eq. (B1) as

$$\frac{1}{4\pi i}\int_{A_{+L}}\left(G_r(\epsilon,T)\tanh\frac{\epsilon}{2T}+G_r(\epsilon,0)\right)d\epsilon$$
$$+\frac{1}{4\pi i}\int_{A_{-L}}\left(G_a(\epsilon,T)\tanh\frac{\epsilon}{2T}+G_a(\epsilon,0)\right)d\epsilon$$
$$+\frac{1}{4\pi i}\int_{A_{+r}}\left(G_r(\epsilon,T)\tanh\frac{\epsilon}{2T}-G_r(\epsilon,0)\right)d\epsilon$$
$$+\frac{1}{4\pi i}\int_{A_{-r}}\left(G_a(\epsilon,T)\tanh\frac{\epsilon}{2T}-G_a(\epsilon,0)\right)d\epsilon.$$

Now we are in a position to turn left paths $A_{\pm L}$ towards negative real axis and right paths $A_{\pm r}$ towards positive real axis. The final results for the temperature dependence of density are

$$\frac{N}{V}(T,\mu)=2\int\frac{d\xi}{2\pi}\frac{d^3p}{(2\pi)^3}e^{i\xi\tau}M(\xi,p,0)$$
$$+2\int_{-\infty}^{0}\frac{d\epsilon}{2\pi}\int\frac{d^3p}{(2\pi)^3}\biggl(G_r''(\epsilon,p,T)\tanh\frac{\epsilon}{2T}$$
$$+G_r''(\epsilon,p,0)\biggr)+2\int_0^{\infty}\frac{d\epsilon}{2\pi}\int\frac{d^3p}{(2\pi)^3}$$
$$\times\biggl(G_r''(\epsilon,p,T)\tanh\frac{\epsilon}{2T}-G_r''(\epsilon,p,0)\biggr). \qquad \text{(B2)}$$

The temperature-dependent integrals in Eq. (B2) apparently converge to zero *if* there is no phase transition at $T=0$.

APPENDIX C: Analyticity of $\ln\det\hat{G}_r(\epsilon)$

We have to prove that $\det\hat{G}_r$ does not have zeros in the upper half plane of ϵ knowing that the matrix $\hat{G}_r(\epsilon)$ and correspondingly the propagator in coordinate-frequency representation $G_r(x,x',\epsilon)$ are operators analytical in the upper half plane.

Let us pick up a function of $x-\Psi(x)$ and define a convoluted operator $C(\epsilon)$ depending only on ϵ:

$$C(\epsilon|\Psi)=\int dx\,dx'\,\Psi^*(x)G_r(x,x',\epsilon)\Psi(x'). \qquad \text{(C1)}$$

The function $C(\epsilon)$ is analytic in our domain. The definition of G_r in terms of field operators ensures that $C(\epsilon)$ is represented by the same Hilbert type formula (3) with negative density (see, e.g., Ref. 2). Thus, $C(\epsilon,\Psi)$ does not have zeros at complex energies ϵ.

Assume that $\det\hat{G}_r$ might be zero at some complex ϵ_0 and consider the eigenvalue problem:

$$\int dx'\,G_r(x,x',\epsilon_0)\Psi_n(x',\epsilon_0)=\lambda_n(\epsilon_0)\Psi_n(x,\epsilon_0). \qquad \text{(C2)}$$

Eigenvalues λ_n are complex because $G_r(\epsilon_0)$ is not a Hermitian operator. From Eq. (C2) it follows that

$$\lambda_n(\epsilon_0)\int dx|\Psi_n(x,\epsilon_0)|^2$$
$$=\int dx dx'\,\Psi_n^*(x,\epsilon_0)G_r(x,x',\epsilon_0)\Psi_n(x',\epsilon_0)$$
$$\equiv C(\epsilon_0|\Psi_n(\epsilon_0)). \qquad \text{(C3)}$$

Now the convoluted function

$$C(\epsilon|\Psi_n(\epsilon_0))$$

is zeroless for any complex ϵ including ϵ_0:

$$C(\epsilon_0|\Psi_n(\epsilon_0))\neq 0$$

and (C3) gives

$$\lambda_n(\epsilon_0)\neq 0.$$

Finally,

$$\det\hat{G}_r(\epsilon_0)\equiv\prod_n\lambda_n(\epsilon_0),$$

which completes the proof.

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[1] L.D. Landau, JETP **3**, 920 (1956).

[2] A.A. Abrikosov, L.P. Gor'kov, and I.E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).

[3] J.M. Luttinger, Phys. Rev. **119**, 1153 (1960).

[4] I.E. Dzyaloshinskii, J. Phys. I **6**, 119 (1996).

[5] B.L. Altshuler, A.V. Chubukov, A. Dashevskii, A.M. Finkelstein, and D.K. Morr, Europhys. Lett. **41**, 401 (1998).

[6] F.H.L. Essler and A.M. Tsvelik, Phys. Rev. B **65**, 115117 (2002).