Reply to "Comment on 'Renormalization-group picture of the Lifshitz critical behavior' "

Marcelo M. Leite*
Departamento de Física, Instituto Tecnológico de Aeronáutica, Centro Técnico Aeroespacial, 12228-900, São José dos Campos, SP, Brazil

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We reply to the preceding Comment by Diehl and Shpot [Phys. Rev. B **68**, 066401 (2003)] criticizing a new approach to the Lifshitz critical behavior just presented [M. M. Leite, Phys. Rev. B **67**, 104415 (2003)]. We show that this approach is free of inconsistencies in the ultraviolet regime. We recall that the orthogonal approximation employed to solve arbitrary loop diagrams worked out in the criticized paper even at the three-loop level is consistent with homogeneity for arbitrary loop momenta. We show that the criticism is incorrect.

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Diehl and Shpot (DS) (Ref. 1) recently formulated a criticism of a new renormalization-group (RG) picture of the Lifshitz critical behavior.[2] It is based on a scaling hypothesis with two independent relevant length (momentum) scales, which characterize the spatial axes without competition, as well as those in competing space directions.[3] In momentum space, the Feynman diagrams are calculated up to two-loop order using two different approximations. The theory is renormalized using dimensional regularization in two different renormalization schemes. We first use normalization conditions with two distinct symmetry points characterizing different momenta directions. Then, we check our results using a minimal subtraction scheme.

The dissipative approximation was used to calculate some critical exponents along directions perpendicular to the (quartic) competing directions.[4] The main point of the criticism by DS in Ref. 5 to this approximation was the impossibility to treat the isotropic case. However, it was pointed out in Ref. 6 that it is a good approximation for the anisotropic behaviors, since it preserves the homogeneity of the Feynman integrals in the external quadratic momentum components perpendicular to the competition axes.

The orthogonal approximation to perform loop integrals introduced in Ref. 2 is the most general one consistent with the physical principle of homogeneity. It can address both isotropic and anisotropic cases since the loop integrals are homogeneous functions of arbitrary external momentum scales perpendicular to or along the competing axes. Therefore, the main point of DS in Ref. 5 no longer applies for the orthogonal approximation presented in Ref. 2.

The criticism in Ref. 1 has a different nature: the authors claim that "(i) Leite's renormalization scheme does not yield an ultraviolet finite renormalized theory, and the structure of the RG he formulates is incorrect." Let us show now why this statement is wrong. Recall that each vertex part in Ref. 2 has a subscript $\tau=1,2$ [see, for example, Eq. (6)]. If $\mathbf{k}$ is a vector along the m competing directions and $\mathbf{p}$ is a vector along the noncompeting $(d-m)$ directions, the vector $\mathbf{q}=(\mathbf{k},\mathbf{p})$ is the most general d-dimensional momentum. When $\tau=1$, the vertex part $\Gamma_{R(1)}^{(2,0)}(\mathbf{q})$ has nonvanishing external momentum components only along directions perpendicular to the competing axes, i.e., $\mathbf{q}=(\mathbf{0},\mathbf{p})$. The associated renormalization factors $Z_{\phi(1)},Z_{\phi^2(1)}$ and renormalized coupling constant u_1 are defined through (2a)–(2e) in such a way that the renormalized vertex part with $\Gamma_{R(1)}^{(2,0)}(\mathbf{p})$ [given by Eq. (193a) for $\tau=1$] in Ref. 2 is ultraviolet finite. When $\tau=2$, the vertex part $\Gamma_{R(2)}^{(2,0)}(\mathbf{q})$ has nonvanishing external momenta only along directions parallel to the competing axes, i.e., $\mathbf{q}=(\mathbf{k},\mathbf{0})$. *The authors miss that in addition* one has the renormalization factors $Z_{\phi(2)},Z_{\phi^2(2)}$ and renormalized coupling constant u_2 which are defined through (3a)–(3e) with renormalized vertex $\Gamma_{R(2)}^{(2,0)}(\mathbf{k})$ [given by Eq. (193a) for $\tau=2$] which is ultraviolet finite. In minimal subtraction, for $\tau=1$, the functions $Z_{\phi(1)}$, $Z_{\phi^2(1)}$, and u_1 are defined in Eqs. (192a)–(192c). Equation (193a) defining the renormalized vertex $\Gamma_{R(1)}^{(2,0)}(\mathbf{p})$ and Eqs. (194a) and (195d) expressing the bare vertex $\Gamma_{(1)}^{(2,0)}(\mathbf{p})$ eliminate the ultraviolet pole proportional to p^2/ϵ_L, making the renormalized vertex $\Gamma_{R(1)}^{(2,0)}(\mathbf{p})$ ultraviolet finite as shown explicitly there. The pole proportional to k^4/ϵ_L of the bare vertex $\Gamma_{(2)}^{(2,0)}(\mathbf{k})$ is explicitly eliminated in VIB2 using a similar reasoning with $\tau=2$. These explicit cancellations in this minimal subtraction scheme first appeared in Ref. 2. Indeed, the minimal subtraction was carried out up to three-loop level for $\Gamma_{R(\tau)}^{(2,0)}(\mathbf{q})$.

Notice that if one tries to renormalize the theory using minimal subtraction using the vertex $\Gamma_R^{(N,L)}(\mathbf{k},\mathbf{p})$ with arbitrary momenta, without separating each subspace into independent RG transformations, the renormalized vertices are not finite in the ultraviolet regime. Had we not separated the renormalized vertices in that way we would have obtained a renormalized theory with bad ultraviolet behavior. This separation is possible, for the two coupling constants flow consistently to the same fixed point. Given these facts, the claim (i) is incorrect.

Next, the authors make the claim "(ii) Leite's insufficient choice of counterterms is biased towards giving the incorrect value $\theta=\frac{1}{2}$ for the anisotropy exponent $\theta=\nu_{L4}/\nu_{L2}$." As shown in the above paragraph, the choice of counterterms is not insufficient. Moreover, the critical exponents ν_{L4} and ν_{L2} are determined independently in the perturbative framework up to the two-loop level. The value $\theta=\frac{1}{2}$ is just a simple consequence of this analysis. In the following discussion of this claim, the authors insist that the choice of counterterms is insufficient. Hence, the claim (ii) is unwarranted.

The following claim is "(iii) Leite obtained incorrect hyperscaling relations because he missed the fact that θ is an independent exponent, not identical to $\frac{1}{2}$ for all $\epsilon_L>0$." The hyperscaling relation is derived from the specific heat vertex part above the critical Lifshitz temperature and relates the specific heat exponent to the space dimension and correlation length exponents, as stated in Eqs. (45) for the anisotropic cases.[2,8] We recall that the exponent β_L is obtained when performing the RG analysis *below* the critical Lifshitz temperature, as shown in Eqs. (54b) and (54d). Of course, since the specific heat critical exponent is the same above and below T_L, the magnetization exponent β_L can be related to α_L and γ_L through the Rushbrook law. In Ref. 2 it was explicitly demonstrated that the anisotropic scaling relations are identical to that in the seminal paper of Ref. 9. Equations (54b) and (54d) do satisfy Eq. (1) in Ref. 1 for $\theta=\frac{1}{2}$, which is the correct value of θ, at least at the two-loop level. From our scaling analysis θ is not an independent exponent. Therefore, claim (iii) is out of order.

The claim (iv) is about the role of σ. As shown in the text,[2] σ is not required, since we develop two independent sets of normalization conditions in each subspace. Furthermore, if σ is set to unity and the external momenta along the competing axes have the same canonical dimension as the components perpendicular to the competing axes, the quartic kinetic term in the Lagrangian accounting for the effect of the competition is inconsistent for it has the wrong canonical dimension (in mass units). This invalidates (iv).

The claim (v) says that the results obtained in Ref. 2 for the isotropic case are false. Let us first analyze the scaling laws. The scaling laws in Ref. 2 are identical to those obtained in the earlier work[10] for isotropic cases with arbitrary even momentum powers p^{2L} in the propagators when $L=2$. It is important to mention that the DS treatment was unable to derive these scaling laws.

Consider the one-loop Feynman integral $I_2(K')$, Eq. (150) from Ref. 2. It can be calculated to order ϵ_L^0 without any approximation as follows. Using Feynman parameters, Eq. (150) reads

$$I_2(K')=\Gamma(4)\int_0^1 dx\,x(1-x)\int\frac{d^mk}{[xk^2+(1-x)(k+K')^2]^4}. \tag{1}$$

Using the formula

$$\int\frac{d^mk}{(k^2+2kk'+m^2)^\alpha}=\frac{1}{2}\frac{S_m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\alpha-\frac{m}{2}\right)}{\Gamma(\alpha)}\times(m^2-k'^2)^{m/2-\alpha}, \tag{2}$$

we obtain

$$I_2(K')=\frac{1}{2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(4-\frac{m}{2}\right)S_m\int_0^1 dx\,x(1-x)\times[x(1-x)K'^2]^{m/2-4}. \tag{3}$$

The integral above is different from its analog in the standard ϕ^4 theory for the appearance of the extra factor $x(1-x)$. By taking $m=8-\epsilon_L$ and expanding the Γ functions, we find up to ϵ_L^0 with no approximation the result

$$I_2(K')=S_m 4\times 3\times 2\frac{\left(1-\frac{13\epsilon_L}{24}\right)}{\epsilon_L}\times\left[\frac{1}{6}-\frac{\epsilon_L}{2}\int_0^1 dx\,x(1-x)\ln[x(1-x)K'^2]+O(\epsilon_L^2)\right]. \tag{4}$$

Notice that the remaining logarithmic integral in the last equation is momentum dependent. In minimal subtraction this integral does not need to be calculated. Nevertheless, it has to be considered in order to show that the renormalization factors are momentum independent. This condition is achieved provided the cancellations of all the logarithmic integrals take place for arbitrary vertex parts. Thus, any attempt to solve the integrals without doing approximations has to take into account these basic facts.

It is clear from the last equation that the remaining logarithmic integrals in the isotropic case are not the same as those in the standard ϕ^4 theory. In Ref. 11 the validity of Eq. (A1) for arbitrary external momenta implies that the logarithmic integrals do not cancel out in the calculation of the renormalization factors, making them momentum dependent in contradiction to Eqs. (12)–(14) in Ref. 11. This shows that the results in Ref. 11 are inconsistent. On the other hand, the use of the orthogonal approximation in Ref. 2 provides all cancellations of logarithmic integrals for arbitrary vertex parts, making the renormalization factors momentum independent as explicitly shown there.

Let us compare our findings for the isotropic case using normalization conditions. Taking $K'^2=1$, the integral above can be easily calculated, giving the result

$$I_2(K'^2=1)=4S_m\frac{\left(1-\frac{\epsilon_L}{24}\right)}{\epsilon_L}. \tag{5}$$

In Ref. 2 it was incorrectly asserted that the choice of a convenient factor to be absorbed in the coupling constant would affect the universal quantities. Then, if we choose the factor $F_{m,\epsilon_L}=4S_m(1-7\epsilon_L/24)$, the exact result above and approximate form, Eq. (157), from Ref. 2 of $I_2(K'^2=1)$ are the same and only differ by an ultraviolet finite reparametrization of the theory. Thus, the orthogonal approximation for I_2 is the same as the exact solution up to a finite ultraviolet reparametrization which does not change universal amounts. This invalidates the sentences "...due to his incorrect calculation... he gets even the simple one-loop integral $I_2(K)$ defined in (150) wrong." The discussion above implies that (v) is false. The advantage of the orthogonal approximation is that it permits one to treat the anisotropic and isotropic loop integrals within the same mathematical footing.

In (vi), the authors actually "...*fail to see*... ϵ-expansion results qualify as acceptable approximations." In fact Ref. 2 achieved the two goals: (a) homogeneity is the physical principle which justifies the orthogonal approximation and (b) the independent flow of the two coupling constants along different momenta subspaces to the same fixed point in the anisotropic cases is consistent and yields a well-defined theory. The critical exponents and other universal amounts for the anisotropic cases[12,13] reduce correctly to the cases $m=0$ using this approximation.

DS see no need to make approximations, but there is a point in their formulation that deserves at least an update to correct a wrong result. Consider the two loop integral $I_4(\mathbf{P},\mathbf{K}')$ contributing to the coupling constant at two loops. Since $\mathbf{P}$ is a $(d-m)$-dimensional momentum vector perpendicular to the competing axes and $\mathbf{K}'$ is a momentum vector parallel to the m-dimensional competing axes, Eqs. (148) and (137) for the solution of this integral using the orthogonal approximation in Ref. 2 depends on *both* external momenta. In Ref. 7, Eq. (B14), the integral $I_4(\mathbf{P},\mathbf{K}')$ only depends on P when ''performing'' the calculations either in momentum space (as they ''did'' in Appendix B) or in coordinate space (see Appendix C). This is obviously incomplete and wrong, since the most general situation should include both momentum scales. In Ref. 11 they tried to defend their result with a falacious argument in Appendix B. The simple orthogonal approximation presented in Ref. 2 for this integral, Eq. (148), simply rules out Eq. (B14) in Ref. 7 as a valid equation, making it unacceptable. As was already pointed out in Ref. 6 the incorrect behavior of this integral, for instance, prevents the transition from the anisotropic to the isotropic case.

To summarize, the renormalized field theory in Ref. 2 is free of ultraviolet pathologies for both the isotropic and anisotropic cases. DS's misconceptions of the method proposed in that reference lead them to make an incorrect criticism.

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*Electronic address: leite@fis.ita.br

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