Superconductivity in a system of fractional spectral dimension

Zygmunt Bak*
Institute of Physics, Pedagogical University of Czestochowa, 42-200 Czestochowa, al. Armii krajowej 13/15, Poland
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This work is concerned with an extension of the fractional-dimensional scheme to the study of superconductivity. We show that quasi-two-dimensional layered structure of the high-temperature superconductors can be described as a system for which both spatial and spectral dimensions have nonintegral values. We derive a general formula for critical temperature as a function of spectral dimension α, which for layered structures can vary within the $1<\alpha<4$ range. We show that for $\alpha>3$ there is an enhancement (when compared to the three-dimensional superconductor) of critical temperature up to the maximum determined by the pairing mechanism.

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I. INTRODUCTION

With recent progress in technology, the growth of systems consisting of alternate heteroepitaxial thin layers with controlled nanoscale thickness has become possible. This achievement offers the ability to produce strong spatial localization of electrons and holes between high-quality interfaces. As a result of the in-layer confinement of mobile charge carriers there arises size-quantization field, which can alter physical behavior of the initially free electron gas.[1] In this context it is important to find some universal relations between confined geometry of the system and physical phenomena. The idea of universality stimulates studies of physical behavior in real structures by means of model systems. Within this approach only a few parameters are sufficient to determine the relevant statistical properties of a wide class of systems. Among the most relevant parameters that characterize both single particle and collective behavior of any physical system is the dimensionality. In the following we will focus our attention on description of superconductors which exhibit multilayered structure. Special attention will be paid to the influence of confined geometry on the critical temperature of superconductivity (SC) in a stratified system. In the description of SC, most of the theoretical effort is focused on a search for microscopical mechanism responsible for electron pair formation. Since, till now, there is no unified picture of SC, detailed study of size effects is impossible. Fortunately, as we will show below, there is an alternative approach to description of the size effects, which is based on the dynamical properties of the electron-hole gas with no specific assumptions concerning the pairing.

It is a well-known fact that dimension of space D plays a crucial role in phase transitions. The critical value D, above which a continuous symmetry can be spontaneously broken at finite temperature, equals 2. To any physical systems various definitions of dimension can be proposed, thus it is important to determine which notion of dimensionality is relevant in the description of critical phenomena. In the case of nontranslation-invariant structures, it has been proven that spectral dimension is the best generalization of the Euclidean dimension of the system when dealing with dynamical or thermodynamical properties. The notion of spectral dimension opens the way for research of structures, which cannot be classified as a system having integer dimension. Typical examples of such systems are the heterepitaxial multilayers. In any laminar structure the interlayer tunneling responsible for the charge transfer along the growth direction z is the result of thermal fluctuations and has three-dimensional (3D) character[2] if

$$k_B T > t_z^2(T)/t_{xy}\,. \tag{1}$$

In Eq. (1) t_z and t_{xy} are the interlayer and in-layer hopping rates, respectively, while k_B is the Bolzmann constant. When the temperature is lowered to

$$k_B T \approx t_z^2(T)/t_{xy}\,, \tag{2}$$

the interlayer transfer is gradually limited and we have temperature driven dimensional 3D→2D crossover. Thus, laminar systems offer good testing ground to study the general relations between dimension of a system and physical phenomena.

In conclusion, the approximation of the Fermi gas in a quantum well, i.e., in a layered system, by a purely 2D or 3D system is seldom a reasonable choice. The purpose of the present paper is to formulate a simplified model of the SC in the intermediate region, when the dynamical dimensionality of the mobile charge carriers interpolates between 2D and 3D cases. In our approach we will treat the dimension of electron gas system as a continuous parameter. So let us first recall the concept of fractional dimensionality in the solid-state physics.

II. FRACTIONAL DIMENSIONALITY

The concept ''dimension of the system'' may have several meanings. It may describe the number of coordinates to be dealt with, e.g., in a problem of several quasiparticles. It can mean the dimension of the position (Euclidean) space embedding the particles. In this work we shall be interested in another definition of dimensionality, which is related to the motion of quasiparticles within a solid (dynamical space, spectral dimensionality). Within quantum formalism the states of mobile quasiparticles of a finite solid system (i.e., with periodic boundary conditions) are labeled by the $\vec{k}$-wave vectors, which form the reciprocal lattice. There is a

widespread conjecture that dimensions of the position space (lattice) and of dynamical space (reciprocal lattice) should be both equal and integer. However, there is experimental evidence that in many laminar systems at least one of the above-mentioned relations does not hold.

In many low-dimensional systems, e.g., superlattices or overlayers, the vibrational as well as the electron density of states, extracted from the experimental data correlates with those predicted for the systems of fractional dimension (FD).[3–5] Laminar systems, such as Ag/Cu(001) overlayer or $GaAs/Al_xGa_{1-x}As$ quantum wells and superlattices as the layer thickness decreases (see Refs. 4 and 6 and references therein), show continuous dimensional crossover from 3D to almost 2D behavior.[5] Generally, the dimension of these systems changes with the monolayer coverage, wire thickness, or temperature.

In the case of rough interfaces, a noninteger dimension of the stratified system can be interpreted in terms of fractal geometry (Haussdorff dimension[7]), but fractional dimensionality has been observed in a system not having fractal structure.[8] In principle, FD originates from restrained motion of mobile particles or quasiparticles in the stratified media. As it has been shown in Ref. 9, the observed FD of a given physical system is based on the physical strength rather than on the geometrical effects. This can be easily understood, numerous physical problems involve basic objects, which are usually described by shrinking or stretching the shape of some characteristic functions. This fact modifies the energy spectrum of the mobile quasiparticles, which, in turn, determines the spectral dimension.[4] Invoking a FD space in description of such a system offers a convenient alternative to computational techniques.[6] In this case single parameter—the spectral dimensionality—contains all of the information about the perturbation. We adopt the approach by He,[4] who has shown that the anisotropic interactions in 3D space become isotropic ones in lower FD space, where the dimension is the Hausdorff dimension and is determined by the degree of anisotropy. Evidently, when the potential that causes the in-plane confinement is infinite, the system is purely 2D. However, in the case of finite quantum wells the envelope functions of free electrons (holes) spread into the barrier region and partially restore the 3D character of the motion. Consequently, the system exhibits behavior, which is somewhere in between 2D and 3D. Different notions of dimensionality manifest themselves in many ways, e.g., one can imagine a 3D spin structure on a 2D lattice.[10]

In case of the laminar systems (thin films, superlattices, or overlayers) there arises a question whether the dynamical states of quasiparticles can be labeled by the wave vectors **k**, i.e., whether the **k**-space formalism is valid. Since in laminar systems the translational symmetry is broken, one would expect that the proper answer is no, on the other hand the existence of band structure in liquid metals and random alloys suggests that a positive answer is possible. This is given by Tsallis and Maynard,[11] who have shown that the **k**-space formalism can be applied to the description of dynamical states in the fractal system, when only statistical invariance under translation is observed. The metallic superlattices possess crystalline in-plane symmetry and the fractional dimensionality arises not from intrinsic disorder, but due to the restraints imposed onto motion of free particles. The most prominent example of this presents the metallic magnetic superlattices. RKKY-reminiscent indirect interaction is mediated by mobile electrons confined within metallic layer. The effective interlayer coupling between magnetic layers is still well described in terms of **k** vectors spanning the Fermi surface of the spacer layer. This gives experimental evidence of the applicability of **k**-space formalism in these systems and indicates that the arguments of Tsallis and Maynard[11] still hold. The direct observations show that many real systems with a mobile charge carriers, as extracted from excitation statistics, exhibit fractional spectral dimensionality.[3] This confirms both FD of the systems and applicability of **k**-space formalism in layered systems.

Let us mention here that FD space is not, in general, a vector space.[12] However, one can trace a number of mutually perpendicular lines, which can be regarded as *pseudocoordinate* axes. It is worth mentioning that the largest number of mutually perpendicular axes (pseudocoordinates) can be even larger than FD α. Since in our case we assume the **k** space to be of FD this implies to renounce the use of any vector property. In the following we will use the term **k** space instead of **k** wave vectors.

The method by He[4] postulates that the electron quantum states are homogenously distributed in the αD **k** space and a surface of constant energy is an αD spherical shell. Suppose further that the energy dispersion is parabolic, $E-E_o \approx k^2$, we obtain the expression for the density of states in αD **k** space[4] as

$$n(E)dE \approx (E-E_o)^{\alpha/2-1}dE, \tag{3}$$

where E_o is the band gap. This means although the ionic (mass) distribution position space of dimensionality β shows no peculiarities, the density of free particle eigenstates shows (sometimes fractional) power-law scaling (with effective spectral dimension $\alpha \neq \beta$).[4]

The effective spectral dimensionality of laminar system can be easily determined provided that energy spectrum of mobile particles within the layer is known. As an example, let us consider a semiconductor with planar doping, often used for forming V-shaped potential wells with quasi-two-dimensional (quasi-2D) electron gas. When the deposition of impurities can be represented by the Dirac δ function, it is called δ doping.[13] The enhanced mobility of the 2D electron gas in V-shaped, δ-doped semiconducting multilayers can be described by the following Hamiltonian:[14]

$$H = -\alpha_1 \nabla^4 - \alpha_2 \nabla^2 + V, \tag{4}$$

where

$$\alpha_1 = \frac{\hbar^4}{4E_g}\left(\frac{1}{m^*}-\frac{1}{m_o}\right)^2, \qquad \alpha_2 = \frac{\hbar^2}{2m^*}, \tag{5}$$

while m_o and m^* denote the bare and effective electron mass, respectively, while E_g is the bottom of the valence band (for detailed description of the model see Refs 14 and 15, and references therein). $V=V(z)$ is the confinement po-

tential that includes electron-electron interaction. At the same time $V(z)$ is the conduction band-edge profile. The eigenfunctions and eigenvalues of Eq. (4) are given in the form

$$\Psi_{nk}(\vec{r}) = \frac{1}{2\pi}\exp(i\vec{k}\cdot\vec{\rho})h_{nk}(z),$$
$$E_{nk} = E(n,k) + \alpha_2 k^2 - \alpha_1 k^4, \tag{6}$$

where $\vec{\rho}=(x,y)$, $\vec{k}=(k_x,k_y)$. Both h_{nk} and $E(n,k)$ satisfy the 1D Schrödinger equation $\hat{H}h_{nk}=E(n,k)h_{nk}$,[15] where $\hat{H}$ is given by Eq. (4). E_{nk} represents the nth two-dimensional subbands of the 2D electron gas within the quantum wells of a laminar system. When limited to the single band model, E_{nk} reduces to

$$\epsilon_k = \alpha_2 k^2 - \alpha_1 k^4 - \mu, \tag{7}$$

where μ is the Fermi energy. Having the mobile quasiparticle spectrum (7) known we can calculate the density of states as follows:[16]

$$n(k)dk \approx \pi(\epsilon - \epsilon_o)^{-1/2} d\epsilon. \tag{8}$$

Expression (8) can be fitted to the general formula (3) by setting $\alpha=1$. This means that we can model such a planar semiconductor with nonparabolic dispersion (7) by a 1D system with parabolic dispersion.

In principle, it is enough if the density of states fulfills relation (3) in a small energy window close to the Fermi energy. Extensive analytical discussion of how the effective spectral dimensionality is associated with the number of the free electron modes can be found in Ref. 17.

Although for the further considerations only the value of spectral dimension is important, we must point out that the spatial dimension can also be a fraction. Concerning the problem of mobile particle confined within a layer, the question is what is the spatial dimension α, which measures the anisotropy of the system. A possible answer in the case of superlattice is given in Ref. 18, where the FD is defined as $\alpha=2+\gamma=2+\mu_o/\mu_z$ where μ_o and μ_z are the on-axis reduced effective masses in the 3D crystal and in the superlattice, respectively. Another possible choice is to express α in terms of the effective quantum well width L_w^* in the case of excitons having the extension ξ is given by the expression $\alpha=3-e^{-L_w^*/\xi}$.[6] In magnetic systems the FD can be determined as viewed form the measurements of the Bloch exponent in the low-temperature magnetization $M(T)=M(0)(1-BT^{\alpha/2})$ or specific heat.[19]

III. SUPERCONDUCTIVITY

As we have shown above the dynamical states of mobile charge carriers in some laminar systems can be described properly with the help of a **k** space having fractional dimension. This concerns also the YBaCuO compounds for which the FD has been postulated.[20] Most of the theoretical approaches to the superconductivity rely of the **k**-space pairing; thus, it is reasonable to consider the problem of SC in a system of (spectral) FD. For the use of further considerations, it is not necessary to specify any peculiar mechanism of pairing. Experiments confirm that spectral FD case arise in various laminar systems involving polarons[21] (and thus bipolaronic SC[22]), excitons,[6] phonons,[21] or magnons.[17,19] This variety of quasiparticles and interactions covers almost all mechanisms postulated for description of SC (provided that real-space pairing theories are excluded).

In conclusion we assume that the Hamiltonian that is responsible for the Cooper pair formation is given by

$$H = \sum_{k,\sigma} (\epsilon_k - \mu) c^+_{k,\sigma} c_{\vec{k},\sigma} + \sum_{k,k_1} V_{kk_1} c^+_{k,\uparrow} c_{-k,\downarrow} c^+_{k_1,\uparrow} c_{-k,\downarrow}, \tag{9}$$

where $c^+_{k,\sigma}$ is the fermion creation operator labeled **k** and spin σ. The only difference when compared to conventional approaches is that the **k** states fill the space of nonintegral dimensionality. As we have mentioned above, we assume the SC transition as the Bose-Einstein condensation of preexisting boson pairs. It is well known fact that Bose-Einstein condensation produces a nonzero absolute temperature T_c, below which a macroscopic condensation emerges, only if $D>2$. Although there were recently reports on possibility of SC state in a systems with $D<2$ (Ref. 24), we must note that these results refer to the geometrical meaning of dimension. However, if we determine the value of spectral dimension αD [according to formula (3)], we can easily find that the reported system[24] still fulfills the $\alpha D>2$ condition. The conventional theory of boson condensation derived for systems of integral dimensionality[23] can be easily extended onto systems, which exhibit fractional spectral dimension α. The total number of bosons $N_B(T)$ in the system consists of the $N_{B,0}(T)$ ones that occupy the ground state ϵ_o ($\epsilon_o=0$ in the thermodynamic limit), while the others are distributed over higher-energy levels. In view of this we have[23]

$$N_B = N_{B,0}(T) + \sum_{\mathbf{k}\neq 0} \frac{1}{e^{\beta(\epsilon_k - \mu_B)} - 1}, \tag{10}$$

where $\beta=1/K_BT$ and $\mu_B<0$ is the chemical potential. Similarly as in Eq. (9) we assume that summation goes over the **k** states filling the fractional αD space. The sum over **k** in Eq. (10) can be converted to an integral over positive $k=|\mathbf{k}|$, where **k** fills the αD space. In view of the results of Stillinger[12] the integration over αD space (when $2<\alpha<3$) can be performed with use of the formula[21]

$$\sum_{\mathbf{k}} \rightarrow \frac{V_\alpha}{(2\pi)^\alpha} \frac{2\pi^{(\alpha-1)/2}}{\Gamma(\alpha-1/2)} \int_0^\infty dk \int_0^\pi k^{\alpha-1} (\sin\theta)^{\alpha-2} d\theta, \tag{11}$$

where $\Gamma(x)$ is the Euler Γ function.

However, if we recall the main idea of FD approach,[4,5] i.e., replacement of anisotropic 3D system by an isotropic (but lower) FD space, the integration over angle θ can be performed and relation (11) reduces to

$$\sum_{\mathbf{k}} \rightarrow \frac{V_\alpha}{2^{\alpha-1}\pi^{(\alpha/2)}\Gamma(\alpha/2)}\int_0^\infty k^{\alpha-1}dk. \tag{12}$$

Inserting relation (11) into Eq. (10) we obtain

$$N_B = N_{B,0}(0) = N_{B,0}(T) + \frac{V_\alpha}{2^{\alpha/2}\pi^{(\alpha/2)}\Gamma(\alpha/2)}\left(\frac{m^*}{\beta\hbar^2}\right)^{\alpha/2}\int_0^\infty \frac{x^{\alpha/2-1}}{e^x-1}dx, \tag{13}$$

where we have accounted for the fact that at $T=0$ all boson pairs form the condensate, i.e., $N_{B,0}(T=0)=N_{B,0}(0)=N_B$. In the calculations the parabolic energy spectrum of the quasiparticles is assumed. The integral in Eq. (13) can be expressed with help of Riemann ζ function[23] in the following form:

$$\int_0^\infty \frac{x^{\alpha/2-1}}{e^x-1}dx = \Gamma(\alpha/2)\zeta(\alpha/2). \tag{14}$$

Inserting this result into Eq. (13) we can calculate the condensate fraction $N_{B,0}(T)/N_{B,0}(0)$ as

$$\frac{N_{B,0}(T)}{N_{B,0}(0)} = 1 - \frac{V_\alpha}{N_B}\left(\frac{m^*}{2\pi\beta\hbar^2}\right)^{\alpha/2}\zeta(\alpha/2). \tag{15}$$

The condensate fraction falls off when the temperature is increased and eventually at T_c the condensate vanishes, i.e., $N_{B,0}(T)/N_{B,0}(0)=0$. From this condition and Eq. (15) we can derive the formula for the critical temperature T_c as a function of the effective spectral dimension α:

$$T_{c,\alpha} = \frac{m^*}{2\pi k_B\hbar^2}\left(\frac{V_\alpha}{N_B}\,\zeta(\alpha/2)\right)^{2/\alpha}. \tag{16}$$

The fact that the phase transitions are governed by the value of spectral rather than spatial dimension has been established long time ago when studying fractal systems.[25,26] However, in the case of bulk or laminar systems phase transitions still are improperly classified according to the value of their spatial (geometrical) dimension.

Let us discuss some consequences of Eq. (16). In conventional theories the ratio V_α/N_B is treated as the inverse boson pair concentration n_B^{-1}. Such interpretation is justified provided that spectral dimension α and dimension of real space β (position space) are equal. However, in systems of FD such interpretation is not valid. Suppose that in the system under consideration we have some characteristic length L, then the volume $V_\alpha \approx L^\alpha \approx (k_F)^{-\alpha}$. Simultaneously the volume of the system, i.e., volume filled with quasiparticles (boson pairs) can be expressed as $V_\beta \approx L^\beta$. In view of this, concentration n_B being the real-space quantity reads as $n_B = N_B/V_\beta$. Distinction between these different notions of dimensionality is often missed, but as it will be shown below, it is crucial in proper description of dimensional effects in SC. Inserting relation $V_\alpha \approx L^\alpha$ into Eq. (15) we have

$$T_{c,\alpha} = \frac{m^*L^2}{2\pi k_B\hbar^2}\left(\frac{\zeta(\alpha/2)}{N_B}\right)^{2/\alpha}. \tag{17}$$

Similarly, as in Ref. 23, from Eqs. (11)–(17) we have

$$\frac{N_{B,0}(T)}{N_{B,0}(0)} = 1 - \left(\frac{T}{T_c^\alpha}\right)^{(\alpha/2)}. \tag{18}$$

As we have mentioned in Sec. II, the FD arises due to the anisotropy forces and when external conditions (temperature, thickness, or fields) change the FD system undergoes (sometimes continuous) dimensional crossover. Let us consider a FD system in two states, which exhibit FD α and α', respectively. Moreover, let us assume that number of preexisting boson pairs is constant during this dimensional crossover. In view of Eq. (17) the hypothetical critical temperatures in both states fulfill the relation.

$$\frac{T_{c,\alpha'}}{T_{c,\alpha}} = \frac{m^*_{\alpha'}}{m^*_\alpha}\,\frac{\zeta(\alpha'/2)^{\alpha'/2}}{\zeta(\alpha/2)^{\alpha/2}}N_B^{(2/\alpha'-2/\alpha)}. \tag{19}$$

Let us study the variation of the critical temperature $T_{c,\alpha'}$ associated with the continuous dimensional crossover. We assume that in Eq. (18) $\alpha=3$, i.e., we take the 3D case as the reference system. First, let us note that ratio $m^*_{\alpha'}/m^*_\alpha$ is a factor of the order of unity. Value of $\zeta(\alpha'/2)/\zeta(\alpha/2)$ can be estimated as follows. Using the definition of Riemann ζ function

$$\zeta(\sigma) = \sum_n \frac{1}{n^\sigma}, \tag{20}$$

the value of Riemann ζ function for a given argument can be estimated as follows:

$$\int_1^\infty \frac{1}{x^\sigma}dx < \zeta(\sigma) < 1 + \int_1^\infty \frac{1}{x^\sigma}dx. \tag{21}$$

From Eq. (21), it follows that

$$\frac{1}{\sigma-1} < \zeta(\sigma) < 1 + \frac{1}{\sigma-1}. \tag{22}$$

In view of Eq. (22) the ratio [in Eq. (19)] of Riemann ζ functions for different α and α' can be estimated as follows:

$$\frac{2(\alpha-2)}{\alpha(\alpha'-2)} < \frac{\zeta(\alpha'/2)}{\zeta(\alpha/2)} < 1 + \frac{\alpha'(\alpha-2)}{\alpha(\alpha'-2)}. \tag{23}$$

From Eq. (23), it follows that for $\alpha=3$ (our reference system) and $\alpha'>2.5$ ratio (23) of Riemann ζ functions is a number of the order of unity. The factor that shows strongest influence on ratio (19) of critical temperatures in different states of the system under consideration (i.e., in the states that exhibit different values of effective spectral dimension) is the last term, namely, $N_B^{(2/\alpha'-2/\alpha)}$. In the case $\alpha=3$, $\alpha'=2.8$, and $N_B=10^{20}$ this factor can be estimated as $N_B^{(2/\alpha'-2/\alpha)}=10^{-1}$, while for $\alpha=3$, $\alpha'=2.5$, it takes value

$N_B^{(2/\alpha'-2/\alpha)}=10^{-3}$. This means that when the effective dimension is decreased the critical temperature decreases in a very rapid manner. Contrary to the previous remark, if $\alpha=3$ and $\alpha'>3$, one would expect an elevated critical temperature. This point is important per se, independently of quantitative predictions since it allows us to draw general conclusions concerning the role of dimension in formation of SC phase. In connection with previous remarks, there arises a question why in some laminar systems or overlayers one can observe enhancement of SC above critical temperature characteristic for bulk systems? The possible explanation is that the effective spectral dimension of the copper oxide system is higher than 3. At first sight conclusion that the dimension of **k** space for the boson gas with in layer confinement can be higher than three appears to be counterintuitive, one would rather expect $2<\alpha<3$. To show that our conclusion can be correct, let us discuss the relation between the dimensionalities of position and **k** spaces.

IV. SUPERCONDUCTIVITY IN A SYSTEMS WITH $D>3$

In the conventional, isotropic systems both dimensionality of **k** space and the dimensionality of the position space are equal 3. One would expect that due to the constraints imposed onto free electron (hole) motion within the quantum well, the effective dimensionality should be lower than 3. However, the variational (numerical) calculations of the quasiparticle mobility in the $GaAs/Ga_{1-x}Al_xAs$ heterostructure suggest smaller confinement than in the isotropic 3D situation.[27] This means that there may arise situation with a smaller than isotropic medium confinement described by an effective dimension $D>3$.[27] This is not just a computational error. To show this let us consider a parabolic quantum well fabricated of semimagnetic semiconductors,[28] a system being an object of interest of nanotechnology. Due to the nonrectangular quantum-well barriers, the mobile electrons (holes) exhibit nonconventional quantization in the growth direction. Assuming that z denotes the direction of the planar parabolic quantum well, the Hamiltonian that describes the electronic structure within the envelope function formalism and effective mass approximation reads[28]

$$H_n=-\frac{\hbar^2}{2m_n}\nabla^2+\frac{D_n}{2}z^2+\epsilon_n. \tag{24}$$

Here, n represents the band index, ϵ_n and m_n the band-gap energy and the effective mass at the center of the parabolic, respectively. D_n is the curvature of the parabola potential profile, which is assumed to be infinitely high. Thus, the electronic states within the parabolic quantum-well are given in the form

$$\Psi_{nk}(\vec{r})=\frac{1}{2\pi}\exp(i\vec{k}\cdot\vec{\rho})h_{nk}(z), \tag{25}$$

where $h_{nk}(z)$ are the standard harmonic oscillator states and, consequently, the electron spectrum is given by[28]

$$E_{mk}^n=E_n+\frac{\hbar^2k^2}{2m_n}+\hbar\omega_n\left(m+\frac{1}{2}\right), \tag{26}$$

with $\vec{\rho}=(x,y)$ and $\vec{k}=(k_x,k_y)$. E_{nk}^m represents the nth two-dimensional subbands of the quasi-2D electron gas within the planar parabolic quantum-well (PQW). In the following we will focus our attention on the single-band model for which E_{mk}^n reduces to

$$E_{m,k}=\frac{\hbar^2k^2}{2m}+\hbar\omega\left(m+\frac{1}{2}\right)-\mu, \tag{27}$$

where μ is the Fermi energy. Using the mobile charge carrier energy spectrum given by Eq. (27) we can find the density of dynamic eigenstates[29]

$$n(E)\approx|E-E_F|, \tag{28}$$

fitting it to general formula (3) with $\alpha=4$.[29] This means that the electron (hole) gas within the parabolic quantum-well exhibits effective spectral dimension $\alpha=4$. One may suspect that such result indicates limitations of the effective dimensionality approach. On the other hand, there is experimental evidence that the dimension of reciprocal lattice in the case of quasi-2D system can be higher than 3.

Let us remind the case of modulated superstructures. In those systems x-ray diffraction pattern spots form $(n+d)D$ reciprocal lattice,[30] which is described by the theory of supercrystals.[30,31] In this picture the modulated commensurate or incommensurate phases are described by $(n+d)D$ Euclidean superspace containing nD subspace, called position space. Such a picture can be achieved after the crystal has been embedded in higher-dimensional Euclidean space and the additional dimensions are connected with the internal degrees of freedom. A typical example of a displacive modulation arises for those heteroepitaxial systems for which the elemental constituents exhibit significant ionic radii mismatch. In such a quasi-2D systems a $(2+1)D$ (Ref. 32) or $(2+2)D$ superstructure [and thus $(2+1)D$ and $(2+2)D$ **k**-space] is to be expected.

Generally a system with at least two interpenetrating, modulated crystalline subsystems that have incommensurate repeat distances along some common crystallographic distances has the reciprocal lattice, in which wave vectors are given by[33]

$$\vec{k}=\sum_{i=1|}^{n}h_i\vec{a}_i^*, \tag{29}$$

with h_i being integers, while $\vec{a}_i^*$ span the reciprocal space. In this quasiperiodic structures, being an intersection of a few incommensurate periodic structures the value of n in Eq. (29) is larger than the dimension of the position space β. In such systems the density of charge in physical space is given by

$$\rho(\vec{r})=\rho_o\left[1+\sum_{i=1}^{n}\eta_{\vec{q}_i}e^{i\vec{q}_i\cdot\vec{r}}\right], \tag{30}$$

with $n>\beta$. We must point out here that any q modulation of the charge density opens a gap at the Fermi surface. However, there still remain portions of the Fermi surface, which are weakly influenced by the change-density wave modulations, so our general conclusions (at least those qualitative)

concerning the role of spectral dimension in the formation of a SC phase remain unchanged.

From Eq. (19) we have drawn conclusion that for $\alpha>3$ an elevated critical temperature should occur. This is not just mechanical interpretation of mathematical formula; to prove that let us discuss possible mechanism behind this effect. In the case of the 3D k space the number of electron states at the Fermi surface can be estimated as $N_3(\epsilon_F)\approx N^{2/3}$. Generally the number of states that occupy a hypersphere of constant $N_\alpha(\epsilon_F)$ energy can be estimated as $N_\alpha(\epsilon_F)\approx N^{(\alpha-1/\alpha)}$, where N is the total number of quasiparticles in the system and α is the dimensionality of the k space. From this formula the ratio $N_4(\epsilon_F)/N_3(\epsilon_F)$ that describes relative occupation of the Fermi level in 4D and 3D k spaces can be estimated as $N_4(\epsilon_F)/N_3(\epsilon_F)\approx N^{1/12}$. In the case of $N=10^{23}$ the ratio is a number of the order of 10^2. This means that the number of electron states at the Fermi level is significantly higher in the 4D k space compared to the conventional 3D case. In many systems the spectral dimension α is a function of external fields (e.g., temperature) and with varying external conditions a gradual dimensional crossover is observed. Suppose that α is a fraction $\alpha=3+\epsilon$ (with $0<\epsilon\leqslant 1$) then the ratio of relative occupations of the Fermi level can be given as $N_\alpha(\epsilon_F)/N_3(\epsilon_F)=N^{3/3(3+\epsilon)}$. As we can see considerably enhanced densities of states near the Fermi level are predicted, which is expected to yield enhanced SC over that there are of standard 3D, provided that no large changes in the electron-ion matrix and phonon frequency contribution.[34]

From Eq. (19) it follows that ratio $T_{c,\alpha}/T_{c,3D}$ can achieve values of the order of 10^2 for $\alpha>3$, which is far from being realistic. We must pointout, however, that there is another limit set on critical temperature. The $T_{c,\alpha}$ enhancement predicted by Eq. (19) is valid provided that assumptions under which this equation was derived hold. The most restrictive in that we have assumed condensation of boson pairs, which exist above critical temperature. However, stability of electron (hole) pairs is limited by the mechanism of pairing. Suppose that the energy of electron-electron binding is given by E_{bond}. When temperature is increased to the value $k_BT_1\approx E_{bond}$, the boson pairs dissociate and Eq. (19) is no longer valid. This means that when the spectral dimension increases (dimensional crossover) the system can achieve only the maximal critical temperature allowed by the particular mechanism of charge carrier pairing, i.e., $k_BT_{c,\alpha}\leqslant E_{bond}$. This situation resembles to some extent the BCS picture where there is no preexisting bosons and they appear just at T_c. The fact that boson pairs are breakable suggests that we should consider the situation in which unpaired but pairable fermions coexist with the boson pairs.[23,24] However, such generalization is of less importance and cannot change the general conclusion presented above.

V. REMARKS ON HTC SC

The concept of our considerations is based on ideas drawn from models derived for structured semiconductors. Till now we have avoided any remarks concerning high-T_c (HTC) oxide SC although they share many common features with the previous systems. The copper oxides are (i) principally semiconducting, (ii) exhibit layered structure, and (iii) exhibit dimensional crossovers. Let us now, basing on the obtained till now results, pay some attention to the HTC materials. Since the advent of high-temperature superconductivity (HTC SC) a variety of mechanisms that could be responsible for this phenomenon has been postulated. Many elaborate models of reasonable accuracy present finite domains of validity and often remain out of scope of the experimentalists and none of them explains satisfactorily all aspects of this phenomenon. Thus, despite an enormous theoretical effort over the years and quite a variety of treatments, a complete theory of HTC SC still does not exist. Nevertheless, accumulated experimental data provide support for a widespread conjecture that superconductivity, in general, is a Bose-Einstein condensation of the charged Cooper pairs observed also in conventional superconductors.[24] Also the recently discovered MgB_2 seems to be a simple BCS superconductor without any of the other mechanisms for the HTC SC that are hypothesized for the cuprate superconductors. However, the specific mechanism behind this pairing remains unknown. This suggests that the theoretical description of the HTC systems should focus on more general properties rather than on microscopical mechanisms responsible for the pairing. The characteristic feature of the copper oxides (in principle of any HTC systems) is their layered structure. In metallic phase this results in anisotropy of conductivity and itinerant charge carrier concentration. The conductivity within CuO_2 plane is much higher compared to that of measured along the z axis, perpendicular to this plane. In the $YBa_2Cu_3O_{x_n}$ copper oxide the magnetoresistivity measurements with reduced x_n indicate gradual dimensional crossover from an anisotropic 3D to quasi-2D system.[35] Since all HTC systems exhibit anisotropic, quasi-2D mobility of the charge carriers, it is evident that this property is essential in the formation of SC state. In view of the previous remarks, there arises a question whether the copper oxides, which are commonly believed to be quasi-2D superconductors with parabolic dispersion, can be described by spectral dimension $\alpha D>3$? At first sight suggestion that the dimension of the **k** space for a boson gas confined within the CuO_2 layers can be higher than three seems to be errorneous, our intuition guided by the notion of spatial dimension suggests rather $2<\alpha<3$. To show that such situation may really occur, let us recall two experimental facts indicating that in some HTC materials the value of spectral dimension can exceed 3.

After extensive discussion of the modulated systems, given in Sec. IV, let us point that the copper oxides exhibit lateral lattice modulations. The extended Hubbard model applied to the HTC systems predicts charge stripes (due to the phase separation). When the local Coulomb repulsion is taken into account, critical charge fluctuations drive the system towards phase separation. Such density instability evolves into an incommensurate charge density waves when nonlocal Coulomb forces are taken into account.[36,37] It is well-known fact that CDW transition cannot occur on a rigid lattice. In any case CDW formation is accompanied by (positively charged) ion displacement, which produces lattice constant modulation with the same as for CDW period. At the advent of the HTSC research the fractional (position)

dimensionality $d=2{,}03$ of cuprates has been suggested[20] as more realistic since it reflects inter-CuO-layer coupling (see Ref. 24). If we add 1D charge density modulation a $(2+\epsilon+1)$ electron system is generated and description of copper oxides within $\alpha=3+\epsilon$ **k** space formalism is justified. Let us give another experimental argument indicating that copper oxides can exhibit spectral dimension having value $\alpha D>3$.

Different types of SC can be classified according to the point-group symmetry of pair states. In classical SC there is no low-energy excitations, while some unconventional SC with a line of nodes (i.e., with zero gap along some directions, to this class belong, e.g., spin singlet pairs states with $d_{x^2-y^2}$ or d_{xy} symmetries[39]) are expected to have a zero-field density of states $N(E)\approx|E-E_F|$.[40] The density of states $N(E)\approx|E-E_F|$, similarly as the electron system within parabolic quantum well [see Eq. (28)], can be fitted to general formula by setting the spectral dimension $\alpha=4$. Calculations predict that the density of states $N(E)\approx|E-E_F|$ leads to the characteristic specific-heat term $c_e=\rho T^2 \approx \gamma_n T^2/T_c$, where γ_n is the coefficient of the linear-T term in the normal state. The most important fact is that the experimental data confirm presence of T^2 term in specific heat of the optimally doped YBCO system.[40] Thus, we can assume that there is experimental hallmark of 4D SC in this system. We must stress, however, that despite the above-mentioned experimental evidences, they cannot be assumed as a proof as discussed by us that mechanism plays active role in SC transition in HTC material. In description of the HTC SC, it is crucial to answer the question: what is the microscopical mechanism responsible for pair formation with binding energy exceeding 100 K? Since till now there is no consensus concerning the microscopical mechanism of HTC SC we can make only some model studies to reach better understanding of SC and to prepare testing ground for alternative models. Another argument why we should be aware when extending our results onto cuprates is that the layer thickness in structured semiconductors we are basing on is at least by an order higher than in the copper oxides. However, one can expect that some of the above conclusion should be valid in the case of multilayers or superlattices fabricated of HTC materials.

VI. SUMMARY AND DISCUSSION

There are many attempts of search for a general principle or a special (maybe hidden) symmetry that facilitates understanding of SC in systems with restricted geometry. In our approach, basing on the common belief that layered structure and lateral charge inhomogeneities are in the heart of SC phase we have adopted the concept of fractional spectral dimensionality to model the SC system. Eqs. (4)–(8) and (24)–(27) show that detailed knowledge of the low-energy excitations and the Fermi surface topology is essential to understand the unusual properties of the HTC SC and to shed light on the SC mechanism. On very general ground systems that are both interacting and nonhomogenous should exhibit deviations from single-particle Fermi surface. If for the modified Fermi surface predicts that dynamical states density in a narrow energy window close to E_F fulfills relation (3) with noninteger α, then description within fractional dimension picture is fully justified. Using systematic procedures we have shown that depending on the low-energy excitation spectra the effective spectral dimension varies within the $1<\alpha<4$ range. Although the value of spectral dimension in some systems can be calculated directly from the electron gas spectra we can assume α as a phenomenological parameter with values extracted from experimental data.[3] Our approach enables us to derive the formula for critical temperature being a function of spectral dimension. From our consideration it follows that systems with $\alpha>3$ produce increased population of the Fermi level compared to the conventional 3D systems. We have shown in a system for which the effective spectral dimension α exceeds three one can expect enhancement of critical temperature up to its maximum set by the pair formation energy. This conclusion is confirmed by recent experimental data, which report enhancement of T_c due to the quantum-confinement.[38] Basing on the theory of modulated systems (supercrystals), we show that layered system with 1D charge density modulation (i.e., situation as in the HTC superconductors) can be described within $\alpha=3+\epsilon$ **k** space. This means that conclusions concerning enhancement of critical temperature for $\alpha>3$ apply, at least in part, to the copper oxides. However, the main area of application we expect to be the multilayered SC systems, especially those with short coherence length. We believe that our results are of relevance for many other low-dimensional quasi-1D organic superconductors, $(1+\epsilon)D$ Bechgaard salts or $(2+\epsilon)D$ ET salts.[23] Furthermore, since the effective spectral dimension is sensitive to the interface stress or external forces, there arises possibility to fabricate systems with required effective spectral dimension, which gives optimum of critical temperature.

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*Email address: z.bak@wsp.czest.pl

[1] P. Bruno, Phys. Rev. B **52**, 411 (1995).
[2] G.G. Sergeeva and L. Vakula, cond-mat/0301037 (unpublished).
[3] D. Mo, Z.J. Jiang, and N. Ke, Solid State Commun. **114**, 277 (2000).
[4] X.F. He, Solid State Commun. **75**, 111 (1990).
[5] X.F. He, Phys. Rev. B **43**, 2063 (1991).
[6] P. Lefebvre, P. Christol, and H. Mathieu, Phys. Rev. B **48**, 17308 (1993).
[7] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[8] A. Kiryluk, J. Ferré, and D. Renard, Europhys. Lett. **24**, 403 (1993).
[9] R. Elber and M. Karplus, Phys. Rev. Lett. **56**, 394 (1986).
[10] Ph. Kurz, G. Bihlmayer, K. Hirai, and S. Blügel, Phys. Rev. Lett. **86**, 1106 (2001).

[11] C. Tsallis and R. Maynard, Phys. Rev. A **129**, 118 (1988).
[12] F.H. Stillinger, J. Math. Phys. **18**, 1224 (1977).
[13] *Delta-Doping of Semiconductors*, edited by E. F. Schubert (Cambridge University Press, London, 1996).
[14] J. Krupski and M. Pietka, Solid State Commun. **107**, 141 (1998).
[15] L.R. Gonzalez, J. Krupski, M. Pietka, and T. Szwacka, Phys. Rev. B **60**, 7768 (1999).
[16] Z. Bak, Czech. J. Phys. **52**, A25 (2002).
[17] Z. Bak, J. Jaroszewicz, and W. Gruhn, J. Magn. Magn. Mater. **213**, 340 (2000).
[18] P. Lefebvre, P. Christol, and H. Mathieu, Phys. Rev. B **46**, 13603 (1992).
[19] V. Ilković and S. Tuleja, Czech. J. Phys. **52**, A57 (2002).
[20] X.-G. Wen and R. Kan, Phys. Rev. B **37**, 595 (1988).
[21] A. Matos-Abiague, Phys. Rev. B **65**, 165321 (2002).
[22] A.S. Alexandrov, Physica C **363**, 231 (2001).
[23] M. Casas, A. Rigo, M. de Lano, O. Rojo, and M.A. Solis, Phys. Rev. A **245**, 55 (1998).
[24] M. Casas, M. de Lano, A. Puente, A. Rigo, and M.A. Solis, Solid State Commun. **123**, 101 (2002).
[25] P. Buonsante, R. Burioni, D. Cassi, and A. Vezzani, Phys. Rev. B **66**, 094207 (2002).
[26] Yu Shi and Changde Gong, Phys. Rev. E **49**, 99 (1994).
[27] E. Reyes-Gomez, A. Matos-Abiague, C.A. Perdomo-Leiva, M. de Dios-Leyva, and L.E. Oliveira, Phys. Rev. B **61**, 13104 (2000).
[28] W. Geisselbrecht, A. Masten, O. Gräbner, M. Forkel, G.H. Döhler, C. Campman, and A.C. Gossard, Superlattices Microstruct. **23**, 93 (1998).
[29] Z. Bak, Solid State Commun. **118**, 43 (2001).
[30] J. Kociński, *Commensurate and Incommensurate Phase Transitions* (Elsevier, Amsterdam, 1990).
[31] P. M. de Wolff, Acta Crystallogr., Sect. A **A30**, 777 (1974).
[32] Z. Bak and W. Gruhn, J. Alloys Comp. **219**, 296 (1995); R. Jaroszewicz, Z. Bak, Acta Phys. Pol. A **91**, 325 (1997).
[33] T. Janssen, O. Radulescu, and A.N. Rubtsov, Eur. Phys. J. B **29**, 85 (2002).
[34] In Gee Kim, J.I. Lee, B.I. Min, and A.J. Freeman, Phys. Rev. B **64**, 020508 (2001).
[35] E. Osquiqull, Z.X.Z. Gao, M. Maenhardt, and B. Wu, J. Phys.: Condens. Matter **5**, A385 (1993).
[36] R. Citro and M. Marinario, Eur. Phys. J. B **22**, 343 (2002).
[37] S. Monastra, F. Manghi, and C. Ambrosch-Draxl, Phys. Rev. B **64**, 020507 (2001).
[38] L. Menon, D. Yu, P.F. Williams, S. Bandyopadhyay, Y. Liu, and B. Jayaram, Solid State Commun. **117**, 615 (2001).
[39] C.C. Tsuei and J.R. Kirtley, Rev. Mod. Phys. **72**, 969 (2000).
[40] K.A. Moler, D.L. Sisson, J.S. Urbach, M.R. Kapitulnik, D.J. Baar, R. Liang, and W.N. Hardy, Phys. Rev. B **55**, 3954 (1997).