

Nonperturbative renormalization group approach to the Ising model: A derivative expansion at order ∂^4

Léonie Canet,¹ Bertrand Delamotte,¹ Dominique Mouhanna,¹ and Julien Vidal²

¹Laboratoire de Physique Théorique et Hautes Energies, CNRS UMR 7589, Université Pierre et Marie Curie Paris 6,
Université Denis Diderot Paris 7, 2 place Jussieu, 75252 Paris Cedex 05, France

²Groupe de Physique des Solides, CNRS UMR 7588, Université Pierre et Marie Curie Paris 6, Université Denis Diderot Paris 7,
2 place Jussieu, 75251 Paris Cedex 05, France

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On the example of the three-dimensional Ising model, we show that nonperturbative renormalization group equations allow one to obtain very accurate critical exponents. Implementing the order ∂^4 of the derivative expansion leads to $\nu=0.632$ and to an anomalous dimension $\eta=0.033$ which is significantly improved compared with lower orders calculations.

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Many problems in high-energy as well as in statistical physics call for nonperturbative methods. On the one hand, several physical systems are described by field theories in their strong coupling regime so that the usual perturbative techniques become troublesome. They fail either because only the first orders of perturbation are computed and do not suffice, or because, even when high orders are known, standard resummation techniques do not provide converged results. On the other hand, some phenomena such as confinement in QCD or phase transitions induced by topological defects are genuinely nonperturbative.

Apart from some methods restricted to specific dimensions or situations, very few nonperturbative techniques are available. During the last years, the Wilson approach¹ to the renormalization group (RG) has been turned into an efficient tool.²⁻⁴ This nonperturbative RG can be implemented in very general situations and, in particular, in any dimension, so that it has allowed one to study several issues difficult to tackle within a perturbative framework among which the three-dimensional Gross-Neveu model,⁵ frustrated magnets,⁶ the randomly dilute Ising model,⁷ and the Abelian Higgs model.⁸

This method relies on a nonperturbative renormalization of the effective action Γ , i.e., the Gibbs free energy. It consists in building an effective action Γ_k at the *running* scale k by integrating out only fluctuations greater than k . At the scale $k=\Lambda$, Λ^{-1} denoting the spacing of the underlying lattice, Γ_k coincides with the Hamiltonian H since no fluctuation has yet been taken into account while, at $k=0$, it coincides with the standard effective action Γ since *all* fluctuations have been integrated out. Thus, Γ_k continuously interpolates between the microscopic Hamiltonian H and the free energy Γ . The running effective action Γ_k follows an exact equation which controls its evolution with the running scale k^2 :

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_t R_k(q) \{ \Gamma_k^{(2)}[\phi(q)] + R_k(q) \}^{-1}, \quad (1)$$

where $t=\ln(k/\Lambda)$ and $\Gamma_k^{(2)}[\phi]$ is the second functional derivative of Γ_k with respect to the field $\phi(q)$. In Eq. (1),

$R_k(q)$ is an infrared cutoff function which suppresses the propagation of the low-energy modes without affecting the high-energy ones.

Although exact, Eq. (1) is a functional partial integrodifferential equation which cannot be solved exactly. To handle it, one has to *truncate* Γ_k . A natural and widely used truncation is the derivative expansion, which consists in expanding Γ_k in powers of $\partial\phi$, keeping only the lowest order terms. Physically, this truncation rests on the assumption that the long-distance physics of a given model is well described by the lowest derivative terms, the higher ones corresponding to less relevant operators. Up to now, only truncations up to order ∂^2 have been considered since, in many cases, they turn out to be sufficient to get a satisfying qualitative and even sometimes quantitative description of both universal and nonuniversal behaviors.⁹

Nevertheless, several important issues concerning the reliability of the method remain open. The first one concerns the convergence of the derivative expansion. This point is particularly delicate since, within this kind of truncation, there is no expansion parameter in terms of which the series obtained can be analyzed and controlled. It has been moreover suggested that the expansion could be only asymptotic.¹⁰ Actually, this question of convergence has only been addressed within the perturbative context. In Refs. 11,12, it has indeed been shown that the two-loop perturbative result for the $O(N)$ model can be recovered from a summation of the derivative expansion. However, in its full generality, this problem still appears as a major challenge. The second issue, the truly interesting one from a practical point of view, concerns the accuracy of the results provided by low orders truncations. This has been thoroughly studied only at order ∂^0 within the $O(N)$ model and at order ∂^2 within the Ising model through the optimization of the cutoff function R_k (Refs. 13,14) (see also related studies with the Polchinski equation^{15,16} and within the proper time RG formalism¹⁷). Let us emphasize that, even for these models, the anomalous dimension η remains poorly determined. This likely originates in the crudeness of the order ∂^2 truncation that fails to capture the essential momentum dependence of the two-point correlation function. In this respect, an impor-

tant remark is that in the critical theory, and at $k=0$, this function is nonanalytic, $\Gamma_{k=0}^{(2)}(q) \sim q^{2-\eta}$, so it appears non-trivial to retrieve η from a derivative expansion. However, for $k \neq 0$, the infrared fluctuations ($q \ll k$) are suppressed and $\Gamma_k^{(2)}(q)$ should become regular with the standard q^2 behavior. This means that the nonanalyticity builds up smoothly as k vanishes. Berges *et al.*⁹ have proposed that this function behaves approximately as $q^2(q^2 + ck^2)^{-\eta/2}$, where c is a constant. Roughly speaking, for $\Gamma_k^{(2)}$, the derivative expansion consists in expanding this function around $q=0$ and in computing η from its behavior in k^2 , instead of q^2 . It is not trivial that the resulting series for η converge since it amounts to correct the normal q^2 behavior with higher powers of q^2 . The aim of this paper is to investigate this question by including order ∂^4 terms in the derivative expansion of Γ_k for the three-dimensional Ising model.

The effective average action Γ_k of the Ising model truncated at order ∂^4 is written as

$$\Gamma_k[\phi] = \int d^d x \left(U_k(\rho) + \frac{1}{2} \{ Z_k(\rho) (\nabla \phi)^2 + W_k^a(\rho) (\Delta \phi)^2 + W_k^b(\rho) (\nabla \phi)^2 (\phi \Delta \phi) + W_k^c(\rho) [(\nabla \phi)^2]^2 \} \right), \quad (2)$$

where $\rho = \phi^2/2$ is the \mathbb{Z}_2 invariant. Compared with its expansion at order ∂^2 , Γ_k involves three new terms denoted $W_k^s(\rho)$ $s=a,b,c$, linearly independent with respect to the integration by parts. The evolution equation for the potential U_k is derived by evaluating Eq. (1) for a uniform field configuration. By contrast, the definition and, thus, the evolution of the functions Z_k and W_k^s are linked to a specific momentum dependence of the functional derivatives of Γ_k , in the limit of vanishing external momenta

$$Z_k(\rho) = \lim_{p_i \rightarrow 0} \partial_{p_1^2} \frac{\delta^2 \Gamma_k}{\delta \phi(p_1) \delta \phi(p_2)}, \quad (3)$$

$$W_k^a(\rho) = \lim_{p_i \rightarrow 0} \partial_{p_1^4} \frac{\delta^2 \Gamma_k}{\delta \phi(p_1) \delta \phi(p_2)}, \quad (4)$$

$$W_k^b(\rho) = - \frac{1}{2\sqrt{2}\rho} \lim_{p_i \rightarrow 0} \partial_{p_1^2 p_2^2} \frac{\delta^3 \Gamma_k}{\delta \phi(p_1) \delta \phi(p_2) \delta \phi(p_3)}, \quad (5)$$

$$W_k^c(\rho) = - \frac{1}{4} \lim_{p_i \rightarrow 0} \partial_{p_1^2 p_2^2 p_3^2} \frac{\delta^4 \Gamma_k}{\delta \phi(p_1) \delta \phi(p_2) \delta \phi(p_3) \delta \phi(p_4)}. \quad (6)$$

As usual, to find a fixed point, we use the associated dimensionless renormalized quantities $\bar{\rho}, u_k, z_k$, and w_k^s . The flow equations of these functions, derived from Eq. (1), are far too long to be displayed.

As in Ref. 14, we have implemented a further approximation which consists in expanding each running function u_k, z_k , and the w_k^s in powers of $\bar{\rho}$. The motivation which underlies this is twofold. First, in systems having a symmetry group smaller than $O(N)$, the number of functions analogous

to u_k, z_k , and w_k^s grows as well as the number of arguments, analogous to $\bar{\rho}$, on which they depend. In this case, dealing with the full field dependence at each order of the derivative expansion can be very demanding and the field expansion becomes almost unavoidable. Second, this expansion provides valuable indications about the orders in field necessary to correctly describe the critical behavior. Here, we expand the u_k, z_k , and w_k^s functions around the configuration $\bar{\rho} = \bar{\rho}_0$ that minimizes u_k since it leads to a better convergence than the expansion around $\bar{\rho}=0$ (Ref. 18)

$$\zeta_k = \sum_{j=0}^{p\zeta} \zeta_{j,k} (\bar{\rho} - \bar{\rho}_0)^j, \quad (7)$$

where ζ stands for u, z, w^a, w^b, w^c . The RG equation (1) then leads to a set of ordinary coupled differential equations for the coupling constants $\{\zeta_{j,k}\}$. The nonperturbative features of their flows with the running scale k are entirely encoded in a finite set of integrals, called threshold functions. There are six—three of them being specifically linked to the inclusion of the ∂^4 order terms—which are written

$$F_n^d = \int dy y^{d/2-1} \tilde{\partial}_t \left(f(y) \frac{1}{[p(y) + m^2]^n} \right), \quad (8)$$

where $\tilde{\partial}_t$ means that the derivative only acts on the cutoff function $R_k(q) = Z_{0,k} q^2 r(y)$ with $y = q^2/k^2$; $p(y) = y[1 + w_{0,k}^a y + r(y)]$, $m^2 = 2u_{2,k} \bar{\rho}_0$, and $f(y)$ can be either $y(\partial_y p)^i$ with $i=0, \dots, 4$ or $y \partial_y^2 p$. The occurrence of $\partial_y^2 p$ imposes the cutoff function R_k to be at least of class C^3 , which dismisses for instance the theta cutoff introduced in Ref. 19. Here, we choose the exponential cutoff defined by²

$$R_k(q) = \alpha \frac{Z_{0,k} q^2}{e^{q^2/k^2} - 1}, \quad (9)$$

which fulfills this condition and constitutes an efficient regulator. We remind that any truncation of Γ_k introduces a spurious dependence of the results on R_k . Here, we study this influence by varying the cutoff through the amplitude parameter α . For each truncation, the optimal α is determined through a principle of minimum sensitivity (PMS) which indeed corresponds to an optimization of the accuracy of the critical exponents.¹⁴

At each order of the derivative expansion, up to order ∂^4 , and for higher and higher order field truncations, we compute the fixed point and the associated critical exponents ν and η , as functions of α . Then, for each truncation, we determine the optimized exponents from the PMS, which are referred to, in the following, as PMS exponents. We first expand in fields the potential u_k and then z_k which, respectively, constitute the orders ∂^0 and ∂^2 of the derivative expansion. The corresponding PMS exponents are displayed as functions of $p_{u,z}$ —which denote the orders of the truncation in $\bar{\rho}$ of u_k and z_k —in the first two zones of Fig. 1. At this stage, it is worth emphasizing that strong oscillations occur at the first orders in the field expansion for both orders ∂^0

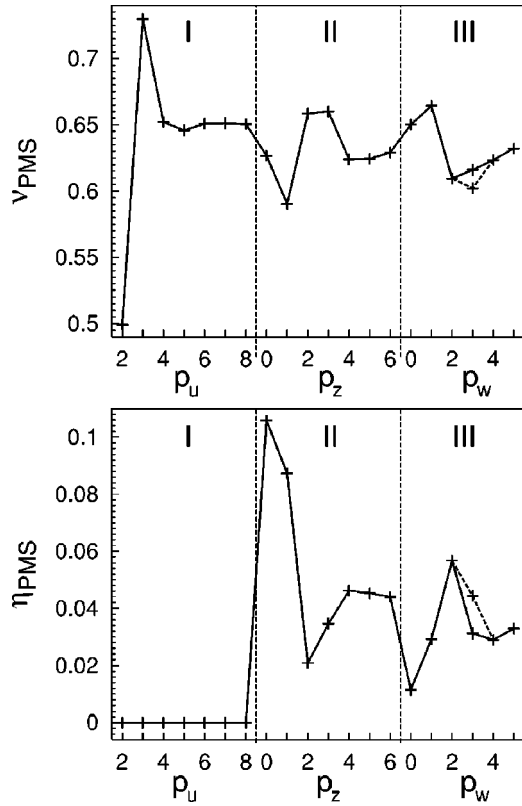


FIG. 1. ν_{PMS} and η_{PMS} as functions of the truncation. The three zones I, II, and III correspond to the expansions of u_k , z_k , and the w_k^s , respectively. In zone III, the two values at $p_{w^s}=3$ reflect the different choices of PMS solutions (see below).

and ∂^2 . It follows that the PMS exponents become almost steady only from $p_{u,z}=4$. As discussed in Ref. 14, the truncation $p_u=8$ and $p_z=6$ allows one to obtain a very accurate approximation of the order ∂^2 results. Indeed, the corresponding exponents $\nu_{\text{PMS}}=0.6291$ and $\eta_{\text{PMS}}=0.0440$ differ by less than 1% compared with their ‘‘asymptotic’’ values obtained for large $p_{u,z}$ (see Table I). Note also that, already at this order ∂^2 , ν_{PMS} agrees well with the best known values whereas, as mentioned above, this is not the case for η_{PMS} .

Let us come to the role of the order ∂^4 terms. We choose to simultaneously expand in fields the three functions w_k^s , $s=a,b,c$, up to $p_{w^s}=5$, while fixing $p_u=8$ and $p_z=6$. Ac-

TABLE I. Critical exponents of the three-dimensional Ising model.

method	ν	η
LPA ^a	0.6506	0
∂^2 ^a	0.6281	0.0443
∂^4 ^b	0.632	0.033
7-loop ^c	0.6304(13)	0.0335(25)
MC ^d	0.6297(5)	0.0362(8)

^a Effective average action method (field expansion) (Ref. 14).

^b Present work.

^c 7-loop calculations (Ref. 20).

^d Monte Carlo simulations (Ref. 21).

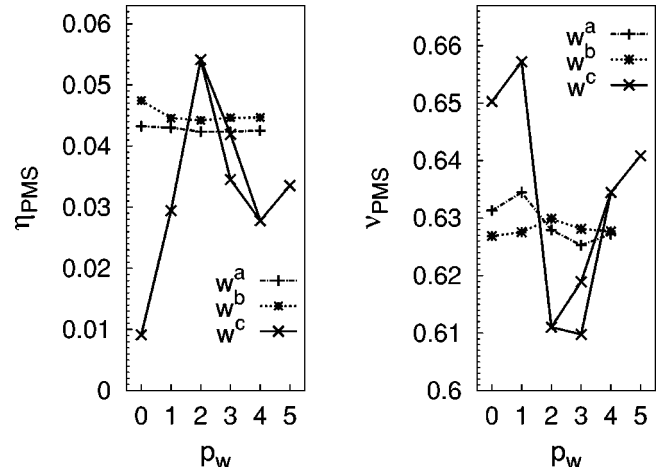


FIG. 2. ν_{PMS} and η_{PMS} as functions of the order of the field truncation for each function w_k^s separately. For w^c , the two values at $p_{w^c}=3$ reflect the different choices of PMS solutions (see below).

tually, the highest truncation corresponds to $p_{w^c}=5$ and $p_{w^{a,b}}=4$ for the following reason. Figure 2 displays, for each w_k^s considered independently, the evolutions of the PMS exponents with the order of the field truncation. It shows that the exponents associated with w_k^a or w_k^b have almost converged, up to a few tenths of percent, as soon as $p_{w^{a,b}}=3$. On the contrary, η_{PMS} and ν_{PMS} related to w_k^c still oscillate at this order. We have checked that, within the simultaneous expansion of the three w_k^s , w_k^c indeed dictates the variations of the critical exponents $w_{j,k}^a$ and $w_{j,k}^b$ exerting a minor influence for $j \geq 3$. This, together with the fact that we encounter here the limits of our computational capacities, justifies our choice $(p_{w^a}, p_{w^b}, p_{w^c}) = (4, 4, 5)$ for the last truncation.

We can now concentrate on the behavior of the exponents at the order ∂^4 . At the low order field truncations, corresponding to $p_{w^s}=0, 1$, and 2, each exponent exhibits a single PMS solution, ν_{PMS} , and η_{PMS} , which are thus unambiguously defined. As displayed in Fig. 3, several PMS solutions appear for the next two truncations, corresponding to $p_{w^s}=3, 4$. This renders the optimization procedure in these cases (see discussion below) unclear.

Concerning the largest truncation, η_{PMS} is unambiguously determined from the unique PMS solution. For ν , several PMS solutions exist. However, provided the field expansion has almost converged at this order, a unique PMS solution can also be selected for ν . The argument underlying this choice originates from the fact that when no truncation in derivatives is performed, the results are independent of the cutoff. Therefore, the best cutoff is the one achieving the weakest sensitivity of the results with respect to the order of the derivative expansion, i.e., leading to the fastest convergence (see Ref. 14 for a detailed discussion). In practice, this consists in minimizing the difference between the values of ν determined at order ∂^m and at order ∂^{m+1} . In our case, this selects the PMS solution located at $\alpha \approx 0.6$ (see Fig. 3).

Let us now discuss the convergence of the field expansion. To this end, we first examine the two truncations $p_{w^s}=3, 4$ for which multiple PMS solutions exist for both exponents. There is no argument to clearly settle between the

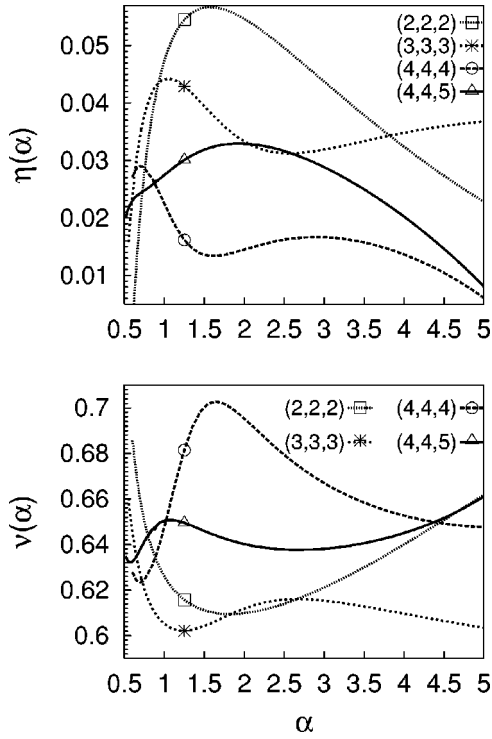


FIG. 3. Variations of the critical exponents ν and η with the cutoff parameter α , Eq. (9), for the highest orders truncations labeled by (p_w^a, p_w^b, p_w^c) .

PMS solutions. We present two sensible way to favor reasonable PMS solutions. First, one can choose to minimize, for these orders, the oscillations induced by the field expansion. This, in turn, corresponds to improving the rapidity of convergence of the field expansion. This choice corresponds to the full line in the third zone of Fig. 1. Alternatively, one can decide to follow a given PMS solution (characterized by its

concavity and its location α), order by order in the field expansion, starting from the low orders $p_w^s=0,1,2$, where the PMS solutions are unique. This corresponds to the dashed line in Fig. 1. Note that both criteria lead to the same PMS solution for $p_w^s=4$. Finally, the important features of the exponents evolution remain essentially unchanged whatever choice is adopted: low orders generate strong oscillations that tend to vanish after a few orders. Indeed, the results for $p_w^s=4$: $\nu_{\text{PMS}}=0.6234$ and $\eta_{\text{PMS}}=0.0289$, are very close to those for $p_w^s=5$: $\nu_{\text{PMS}}=0.6321$ and $\eta_{\text{PMS}}=0.0330$. Although the exponents are not rigorously steady, this suggests that the asymptotic regime is just entered. This is consistent with the fact that, at orders ∂^0 and ∂^2 , the oscillations die down for the same order of truncation $p_{u,z} \simeq 4$. This legitimates our former assumption of field convergence. We therefore approximate the order ∂^4 results by the $p_w^s=5$ estimates (see Table I). To summarize, we have computed the critical exponents of the three-dimensional Ising model up to the ∂^4 order in the derivative expansion. The successive contributions significantly decrease with the order, which supports good convergence properties of this expansion and, in particular, a correct behavior of its implementation around $q=0$. We emphasize that the exponent ν is almost unaltered at order ∂^4 compared with its value at the order ∂^2 , whereas η undergoes a substantial correction which drives it within a few percents of the best known values. This confirms the statement that the inclusion of the ∂^4 order terms allow one to improve the anomalous dimension. Note that although fully converged results would require to handle the full field dependence of u_k , z_k , and w_k^s ,²² this study shows that the truncation in fields constitutes a reliable way to compute critical exponents. Finally, the present work brings out convincing evidence of the ability of the effective average action method to provide very accurate estimates of physical quantities.

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