Ising and dimer models in two and three dimensions

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Motivated by recent interest in 2+1 dimensional quantum dimer models, we revisit Fisher's mapping of two-dimensional Ising models to hardcore dimer models. First, we note that the symmetry breaking transition of the ferromagnetic Ising model maps onto a non-symmetry breaking transition in dimer language—instead it becomes a deconfinement transition for test monomers. Next, we introduce a modification of Fisher's mapping in which a second dimer model, also equivalent to the Ising model, is defined on a generically different lattice derived from the dual. In contrast to Fisher's original mapping, this enables us to reformulate frustrated Ising models as dimer models with positive weights and we illustrate this by providing a new solution of the fully frustrated Ising model on the square lattice. Finally, by means of the modified mapping we show that a large class of three-dimensional Ising models are precisely equivalent, in the time continuum limit, to particular quantum dimer models. As Ising models in three dimensions are dual to Ising gauge theories, this further yields an exact map between the latter and the quantum dimer models. The paramagnetic phase in Ising language maps onto a deconfined, topologically ordered phase in the dimer models. Using this set of ideas, we also construct an exactly soluble quantum eight vertex model.

I. INTRODUCTION

Dimer models have long been of interest to statistical mechanicians.[1–5] In addition to their interest in various physical contexts, they have the striking feature that they are exactly soluble on any planar graph.[2] Following this insight, Fisher[6] constructed a general mapping—reviewed below—from two-dimensional Ising onto planar dimer models, thereby relating the solvability of the one to that of the other. In particular he related the partition function of the ferromagnetic Ising model on the square lattice to that of the dimer model on the (now) Fisher lattice, which is sketched in Fig. 1. Earlier work by Kasteleyn[2] and Stephenson[7] had related some Ising models to dimer models on nonplanar decorated lattices.

More recently, *quantum* dimer models (QDMs) have been formulated and studied.[8] These live in Hilbert spaces spanned by dimer configurations of a given lattice and their Hamiltonians contain kinetic and potential energies that are naturally defined in this basis. These models were introduced to capture the dynamics of valence bond dominated phases of quantum antiferromagnets, with the particular intent of finding Anderson's hypothesized resonating valence bond (RVB) liquid[9]—a hope realized recently.[10]

In this context, attention has been focused on the nature of the quantum dimer phases, in particular their topological properties and low-energy excitations, and a recurring and useful theme has been their interpretation in gauge theoretic terms, particularly the identification of the RVB phase as a deconfined phase.[11–15] In Ref. 14 this identification was made as an exact reduction of an "odd" Ising gauge theory to a quantum dimer model on the same lattice, in the extreme strong coupling limit. As Ising gauge theories in $d=2+1$ are dual to Ising models, this allows dimer models to be exactly

related to frustrated quantum (transverse field) Ising models in $d=2+1$ in the limit of weak transverse fields and thence to a set of ideas and techniques for obtaining their phase diagrams.[16] Very recently, Misguich *et al.* have constructed an exactly soluble dimer model on the kagome lattice. This model maps onto a transverse field Ising model with *zero* exchange, thereby allowing the entire spectrum and correlations to be determined.[15]

In this note, we further explore the above connections between Ising and dimer models in two and three dimensions. We begin by reviewing Fisher's construction in two dimensions which utilizes the loop model generated by the Ising high-temperature expansion, and maps its configurations onto those of a dimer model on a decorated lattice. We note that the symmetry breaking transition in the two-dimensional ferromagnetic Ising model maps onto a nonsymmetry breaking, deconfinement transition of test monomers in the dimer model. As an aside we point out that the dimer formulation provides an immediate insight into how the Ising
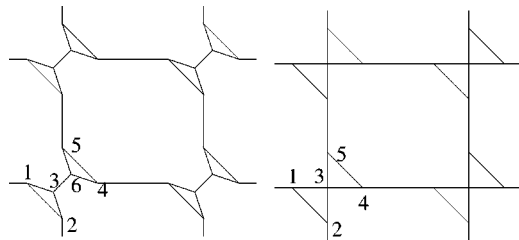

FIG. 1. (Left) The Fisher lattice, a decorated square lattice. The six sites in the basis correspond to one square lattice site. A dimer model on this lattice can be used to calculate the partition function of an Ising model on the (direct) square lattice. (Right) The modified Fisher lattice, used to solve the fully frustrated Ising model on the square lattice as a positive weight dimer model.
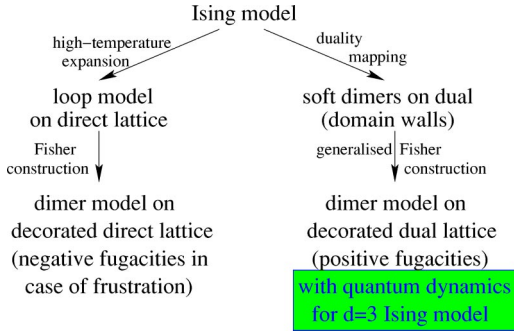
Ising model

high−temperature
expansion

duality
mapping

loop model
on direct lattice

soft dimers on dual
(domain walls)

Fisher
construction

generalised Fisher
construction

dimer model on
decorated direct lattice
(negative fugacities in
case of frustration)

dimer model on
decorated dual lattice
(positive fugacities)

with quantum dynamics
for d=3 Ising model

FIG. 2. The mappings of the Ising model onto dimer models.

model yields a lattice theory without doubled fermions. While Fisher's construction suffices to solve any planar Ising model by dimer methods, it has two features that obstruct its extension to making contact between quantum Ising and quantum dimer models. The first is that it does not relate a given spin configurations to a specific dimer configuration. The second is that for frustrated Ising models it leads to negative weights in the equivalent dimer model—which is a problem as quantum dimer models are equivalent to classical dimer models with positive weights at high temperatures. To get around these problems, we introduce a modification of Fisher's construction which proceeds via an intermediate map to a generalized (nonhardcore) dimer model on the dual lattice which is then decorated to produce the hardcore constraint. As an illustration of this new construction, we solve the fully frustrated Ising model on the square lattice as a purely positive weight dimer model on a modified Fisher lattice.

Finally, we turn to a class of three-dimensional classical Ising models—these are general (frustrated or unfrustrated) two-dimensional Ising magnets stacked ferro- or antiferro-magnetically in the third direction, which can therefore be mapped onto transverse field Ising magnets in $2+1$ dimensions by taking a continuum limit in the third direction. Armed with the modified Fisher construction we are able to map these onto quantum dimer models where the quantum dynamics induced by the transverse field translates to a particular "resonance" dynamics of the kind studied by Misguich *et al.* This is particularly interesting from the viewpoint of the dimer models since it allows known lore on Ising models to be transcribed into statements about the former. Quite generally, the paramagnetic phases of the Ising models map to deconfined, topologically ordered phases in the dimer models. Specifically, for the case of the ferromagnetic Ising model we find a dimer transition characterized solely by confinement and the loss of topological order. We also observe that the dimer models are exactly equivalent to Ising gauge theories, as they must be since the latter are dual to Ising models in $2+1$ dimensions. Consequently their spectrum in the deconfined phase is understood in terms of Ising vortices or visons. The relationships between these mappings are displayed in Fig. 2.

We close with some remarks concerning possible extensions of this work, including the construction of an exactly soluble quantum eight-vertex model along the lines of Ref. 15. Such a model may also be of interest in the context of

orbital current models, such as the d-density wave models considered in Ref. 17.

II. THE ISING TRANSITION IN $d=2$ AS A DECONFINEMENT TRANSITION

In this and further sections, we consider classical Ising models defined by the Hamiltonian

$$H = -\sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z, \qquad (2.1)$$

where σ^z is a Pauli matrix, and J_{ij} is the strength of bond $\langle ij \rangle$. The sites of the lattice, labeled by $i=1, \ldots, N$, together with the bonds $\langle ij \rangle$ with nonzero J_{ij} form a graph, $\mathcal{G}$, the interaction graph. In the following, we will use the following definitions: $\beta = 1/k_B T$; $K_{ij} = \beta J_{ij}$; $v_{ij} = 1/w_{ij} = \tanh K_{ij}$. Although we have formally written a quantum Hamiltonian, the problem is, of course, still classical.

We first briefly review Fisher's mapping.[6] Its starting point is the high temperature expansion of the partition function, $Z = \text{Tr} \exp(-\beta H)$, of the Ising model,

$$Z = 2^N \left[\prod_{\langle ij \rangle} \exp(-K_{ij}) \cosh(K_{ij}) \right] \Upsilon(v_{ij}; \mathcal{G}). \qquad (2.2)$$

Here, $\Upsilon(v_{ij}; \mathcal{G})$ is the crucial quantity: it is the sum over all loop coverings, labeled by $\Gamma(\mathcal{G})$, of the graph $\mathcal{G}$. These loops can intersect one another, provided an even number of links emanate from each site. The weighting of a particular covering is given by the product of v_{ij} of all the links of its loops, so that

$$\Upsilon(v_{ij}; \mathcal{G}) = \sum_{\Gamma(\mathcal{G})} \prod_{\langle kl \rangle \in \mathcal{G}} v_{kl}. \qquad (2.3)$$

Fisher's mapping turns the resulting loop model on the graph $\mathcal{G}$ into a dimer model on a decorated graph, $\mathcal{D}$, by a decoration procedure outlined in Ref. 6, see in particular its Fig. 6 for the general decoration rule and Fig. 7 for the explicit example of the square lattice magnet. The resulting lattice is depicted in Fig. 1. There are now two types of bonds, the original ("external") ones and the new ("interior") ones.

The crucial property of Fisher's mapping is that, if $\mathcal{G}$ is a planar graph, so is $\mathcal{D}$. The partition function of the dimer model on $\mathcal{D}$, $Z_{\mathcal{D}}$, which is related to that of the original Ising model in a simple way, can in that case be evaluated by Pfaffian methods; by Kasteleyn's theorem,[2] one can assign directions to each bond of $\mathcal{D}$ such that the product of orientations when traversing any elementary plaquette in a clockwise direction is odd. These orientations can be used to define a matrix, A, which has entries $A_{ij} = +1(-1)$ if bond $\langle ij \rangle$ is oriented for i to j (from j to i). With fugacities of the external and internal bonds given by w_{ij} and 1, respectively, one has $Z_{\mathcal{D}} = \text{Pf} A$.

A nonanalyticity in Z bequeathes one to $Z_{\mathcal{D}}$. For instance, the Fisher lattice dimer model obtained from the square lattice ferromagnetic Ising model must exhibit a phase transition as the ratio of external to internal fugacities is varied. Nevertheless the Ising character of the transition seems to

have disappeared *en route*—there no longer is an Ising symmetry to break. In the following paragraphs, we will see that the dimer model does not break any symmetries whatsoever. Instead the Ising transition has turned into a confinement–deconfinement transition in which the free energy to separate two test monomers is the appropriate diagnostic. The universality classes of the two transitions coincide, as they must, by virtue of both of them mapping onto the theory of a single gapless Majorana Fermion.

As $\mathcal{D}$ is periodic, one can find the partition function by Fourier transformation; $Z = \mathrm{Pf}\, A = \sqrt{\det A} = \sqrt{\Pi_q \det \tilde{A}(q)}$, where $\tilde{A}(q)$ is the Fourier transform of A at wave vector q. The specific free energy, $\mathcal{F}$, is given by

$$-N\beta\mathcal{F} = \ln Z_{\mathcal{D}} = (1/2)\sum_q \ln(\det \tilde{A}(q)). \qquad (2.4)$$

This expression does indeed reproduce Onsager's formula for the square lattice Ising model for any coupling strength K.[6,18] In particular, nonanalyticities occur when $\det \tilde{A}(q) = 0$. With[6]

$$\det \tilde{A}(q) = (1+w^2)^2 - 2w(w^2-1)(\cos q_x + \cos q_y),$$

this happens only for $q_x = q_y = 0$ and $w = w_c = 1 + \sqrt{2}$ so that $K_c = \ln(1+\sqrt{2})/2$, as it should.

What is the nature of the phases on either side of w_c? For $w \to \infty$, only one dimer configuration survives, namely, one in which all external bonds and the bond linking the two internal sites are occupied. This configuration breaks no lattice symmetries, and is in that sense not a crystal. In the opposite limit, $w = 1$, all dimer configurations have equal weight and their ensemble also respects all lattice symmetries. Evidently the transition does not involve symmetry breaking.

Instead, the two phases differ in their response to the insertion of a pair of test monomers (sites not part of a dimer). It is not hard to see that the high temperature phase is confining. If one places a monomer on one end of an external leg, the site on its other end has to pair up with another site in its cluster (group of sites obtained from one original site by decoration), which in turn leaves the partner of the latter site unpaired. Two monomers placed a distance L apart therefore exact a cost in free energy proportional to the minimal number of unoccupied external bonds, which is proportional to L. This point is thus in a confined phase. Intuitively, the low temperature phase involves a true dimer fluid which should therefore allow monomers to be separated with finite free energy cost.

These statements can be made precise by tracking the spin–spin correlation, $\langle \sigma_i \sigma_j \rangle$, from the spin formulation into the dimer formulation. The spin correlator can, in the context of the high-temperature expansion, be expressed as a loop model on the square lattice containing, along with the closed loops, one open string running between the sites i and j.[19] This loop-string model can in turn be cast in terms of a dimer model with two monomers (Fig. 3). The vertices at the ends of the string can be encoded in the monomer–dimer model by summing over four partition functions, Z_{md}, with the
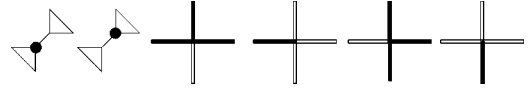

FIG. 3. The open string configurations (four rightmost plots) are generated with a monomer (filled circle) placed on the upper interior point of the cluster. The other monomer location (leftmost plot) generates the remaining four string configurations, obtained by a reflection of the four displayed ones along a diagonal.

monomers in the clusters i,j being placed independently onto one of two interior points of the cluster. The partition function of the loop-string model is thus given by the sum over the four monomer–dimer partition functions, $Z_{ls}(i,j) = \Sigma Z_{md}(i,j)$.

Next, one uses $\langle \sigma_i \sigma_j \rangle = Z_{ls}(i,j)/Z_{\mathcal{D}}$. It follows that in the high temperature (paramagnetic) phase, where $\langle \sigma_i \sigma_j \rangle$ vanishes exponentially, all the monomer pairs are confined. Conversely, in the low temperature phase, $\langle \sigma_i \sigma_j \rangle$ decays to a constant, which implies that at least one (and probably all) monomer pairs are deconfined.

Two comments on related issues are worth making. First, dimer models generally exhibit topologically disconnected sectors under local moves—a classical precursor of the notion of topological order for quantum problems[12] which we shall invoke below for quantum dimer models. On the Fisher lattice there are four such sectors on a torus corresponding to the various combinations of an even or odd number of dimers intersecting the two nontrivial loops. The Fisher map represents the Ising partition function as a sum over all four sectors of the dimer model. Evidently, all Ising spin configurations in a finite system are accessible from one another by local spin flips, and the topological distinction in the dimer model is an artifact of the bookkeeping from the perspective of the Ising model.

Second, dimer problems are solved by lattice Majorana fermions. As the unit cell of the Fisher lattice has six sites, microscopically the dimer formulation leads to as many Majoranas. However, at the transition fugacity w_c, two eigenvalues of $\tilde{A}(0)$ vanish. These two combine to form one single Majorana Fermion, to yield the known critical theory as in the Ising model. We note that given the lore on fermion doubling,[20] it is somewhat surprising that we obtain only a single Majorana Fermion from our lattice problem. While this has been commented on from a different perspective before,[21] we note that in the dimer formulation this conclusion arises from the manifestly asymmetric nature of our lattice, no longer invariant under reflections along both x and y axes. This leads to lattice derivatives different from the ones normally encountered when discretizing the continuum.

III. FRUSTRATED ISING MODELS AS DIMER MODELS WITH POSITIVE FUGACITIES

A. The modified Fisher construction

Fisher's construction is quite general. The high temperature expansion generates a loop model on any interaction graph $\mathcal{G}$, and the rules for generating the clusters out of the

vertices depend only on the number of legs of each given vertex. All resulting models are in principle exactly soluble.

As mentioned in the Introduction, an unfortunate feature of the Fisher mapping, in the context of quantum dimer models discussed below, is that it leads to negative dimer fugacities in the case of frustrated models: there, not all interactions can be chosen to be ferromagnetic, and hence the $v_{ij} = 1/w_{ij} = \tanh K_{ij}$ are negative for some of the bonds.

Here, we present a way of addressing this problem via a modified Fisher construction on the dual of a two-dimensional planar lattice. It proceeds by mapping each Ising spin configuration onto a link configuration, $\{\tau\}$, on the dual lattice.

A given link of the dual lattice is occupied ($\tau=1$) if and only if the bond of the direct lattice it crosses is frustrated. Such a link configuration has the property that the site of the dual lattice at the center of a frustrated plaquette has an odd number of occupied links emanating from it; for an unfrustrated plaquette, this quantity is even. Clearly a spin configuration and its Ising reversed counterpart map onto the same link configuration on the dual lattice.

Such soft dimer configurations on the dual lattice can be converted into hardcore dimer coverings by suitably decorating each site. Depending on the even or oddness of the number of occupied links at the site, two different decoration operations are required, leading to two different types of clusters. The decoration transformation for odd sites is shown in Fig. 4. For even sites, one requires a slightly different prescription as in Fisher's original construction, as that included interchanging empty and occupied links. The corresponding construction is also given in Fig. 4.

The dimer fugacities are determined by the Boltzmann factors which come with the presence of a frustrated bond. We find it convenient to add a constant term to the Hamiltonian so that an unfrustrated bond has energy 0 and a frustrated bond has energy $2J$, so that the dimer fugacity of link $\langle ij \rangle$ is given by $u \equiv \exp(-2K_{ij})$, which is always positive and between 0 and 1. As in Fisher's original construction, Ising models on planar lattices lead to planar dimer models, which can hence be solved using Kasteleyn's theorem.

Three comments on the differences between the Fisher construction and the modified Fisher construction are in order. First, as is appropriate for a dual construction, high and low temperatures trade places when we compare the original Fisher construction to the modified construction, e.g., the equal fugacity dimer model corresponds to zero and infinite temperature, respectively. Second, in the modified construction the spin model maps onto a single topological sector of the dimer model. The remaining sectors are generated by considering different boundary conditions for the spin model. Third, the original mapping does not relate individual spin configurations to dimer configurations but the modified mapping does, upto a twofold ambiguity coming from global Ising reversal. This will be important in making a connection between Ising models and quantum dimer models below. But first we apply the modified construction to the solution of a classical frustrated Ising model.

B. The fully-frustrated Ising model as a dimer model

As an illustration of the above technique, we now use it to solve Villain's odd model, also known as the fully frustrated Ising model on the square lattice.[22] This model is defined for spins on the square lattice with nearest neighbor interactions such that each plaquette has an odd number of antiferromagnetic interactions. This can for example be achieved by choosing all horizontal bonds to be ferromagnetic and the vertical bonds in ferromagnetic rows alternating with antiferromagnetic ones. Transcribed to the dual lattice, which is again square, this corresponds to an odd number of occupied links emanating out of each plaquette. The corresponding decorated lattice (Fig. 1) has five sites per cluster.

The Kasteleyn matrix on this lattice can be partially diagonalized by means of a Fourier transform. While the lattice itself has five sites per unit cell (see Fig. 5), choosing appropriate signs for the matrix requires that we consider a doubled unit cell in one direction which we take to be the horizontal direction. This yields the 10×10 Fourier transformed matrix, $\tilde{A}(q)$,

$$\tilde{A}(q) = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -ue^{-iq_x} & 0 \\ 1 & 0 & -1 & 0 & ue^{-iq_y} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -u & 0 & 0 & 0 & 0 \\ 0 & -ue^{iq_y} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -ue^{-iq_y} \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ ue^{iq_x} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & ue^{iq_y} & -1 & 1 & 0 \end{pmatrix} \tag{3.1}$$
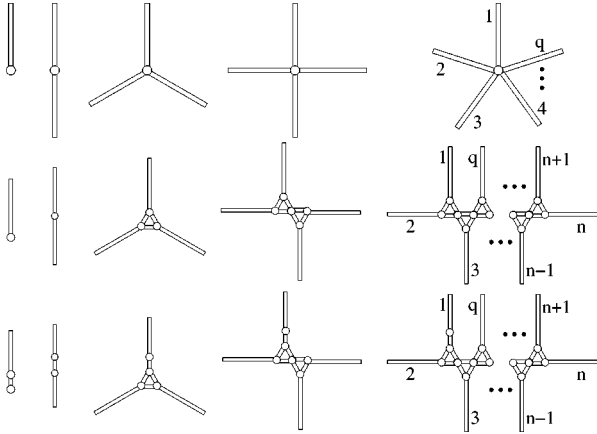
FIG. 4. Decoration of vertices of coordination q (top) into clusters to generate the lattice on which a frustrated (middle) or unfrustrated (bottom) Ising model is represented by a hardcore dimer model with positive fugacities.

Its determinant, $\det \widetilde{A}(q)$, is given by

$$\det \widetilde{A}(q) = 2u^2(2(1+u^2)^2 + (-1+u^2)^2 \cos(q_x) - (-1+u^2)^2 \cos(2q_y)), \quad (3.2)$$

which yields the dimensionless free energy per site,

$$-\beta \mathcal{F} = \frac{\ln u}{2} + \frac{1}{4} \int \int_0^{2\pi} \frac{dq_x dq_y}{(2\pi)^2} \ln \det\left(\frac{\widetilde{A}(q)}{u^2}\right). \quad (3.3)$$

The first term is the ground state energy, as there is one frustrated bond for each pair of sites. At zero temperature, the second term gives the ground state entropy, which integrates to the well-known result G/π, where G is Catalan's constant.[2,24] After a little algebra, our expression Eq. (3.3) can be shown to agree with Villain's result for the partition function given in Appendix 2 of Ref. 22 at all temperatures.[26]

A direct examination of Eq. (3.3) shows that the model is critical only at $T=0$, where $u=0$. As a final exercise we will now compute the divergence of the correlation length as $T \rightarrow 0$ by means of our solution.

To find its behavior in the critical regime, we consider the case of small u. Near $u=0$, we can expand $\det(\widetilde{A}(q))/(2u^2)$ to find that it varies as

$$2 + \cos(q_x) - \cos(2q_y) + 2u^2(2 - \cos(q_x) + \cos(2q_y)). \quad (3.4)$$

In particular, at $u=0$, it vanishes at $p_1^{(0)} = (\pi, 0)$ and $p_2^{(0)} = (\pi, \pi)$.

To find the correlation length, it is necessary to compute the Green function by inverting A. This is done by inverting $\widetilde{A}(q)$ to obtain and then carrying out the inverse Fourier transform on $\widetilde{G}(q) = \widetilde{A}^{-1}(q)$. If one is only interested in the correlation length and not the details of the correlations, it suffices to do the Fourier integral $\int\int d^2q \exp(iqr)\widetilde{G}(q)$ asymptotically, using the property that due to the inversion process, the structure of $\widetilde{G}(q) = \widetilde{g}(q)/\det \widetilde{A}(q)$, where $\widetilde{g}(q)$ denotes a cofactor. We thus have to do integrals of the type

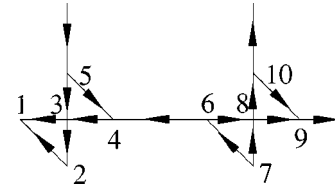FIG. 5. Arrow convention for the Kasteleyn matrix of the frustrated Fisher lattice.

$$G \sim \int \int d^2q \exp(iq \cdot \hat{r}R)/f(q), \quad (3.5)$$

where the zeros of $\det \widetilde{A}(q)$ determine the locations of the poles of the integrand—we have checked that the cofactors do not all vanish at the locations of the poles. In this equation, we have emphasized the two-dimensional nature of $q = (q_x, q_y)$ by writing it as $\mathbf{q}$ for the time being. $\hat{r} = (\cos\theta, \sin\theta)$ is a unit vector in the direction of which the correlations are to be computed, and $R \rightarrow \infty$ is the quantity in which we will evaluate the integral asymptotically, so that $(x,y) = R\hat{r}$.

Let us first carry out the integral over q_x:

$$I_x \equiv \int dq_x \exp(Riq_x \cos\theta)/f(q). \quad (3.6)$$

From Eqs. (3.2) and (3.4), we find that the location of the poles can be written in terms of $q' = (2q_x', q_y')$ so that $p_i = p_i^{(0)} + q'$, where we have inserted a factor of 2 in front of q_x' to make the square symmetry more apparent,

$$q'^2 = q_x'^2 + q_y'^2 = -4u^2 \equiv -\alpha^2 u^2, \quad (3.7)$$

so that the poles lie at $q_x' = \pm\sqrt{\alpha^2 u^2 + q_y'^2}$. Depending on the sign of $\cos\theta$, we therefore choose an integration contour that runs through the following points in the complex q_x plane: $(0,0) \rightarrow (2\pi, 0) \rightarrow (2\pi, \pm\infty) \rightarrow (0, \pm\infty) \rightarrow (0,0)$. Due to the periodicity of the integrand, the contribution from the vertical contours cancel. The contribution from the horizontal contour at $\pm I\infty$ vanishes for $R \rightarrow \infty$ as $\exp(-R|\cos\theta|)$, so that I_x only picks up a contribution from the enclosed poles, where we denote the residue as $1/f'[p_i(q_y')]$.

To find G, we then need to determine

$$G \sim \int_0^{2\pi} dq_y I_x$$

$$= \int dq_y' \left\{ \frac{(-1)^x}{f'[p_1(q_y')]} + \frac{(-1)^{x+y}}{f'[p_2(q_y')]} \right\}$$

$$\times \exp[-R(|\cos\theta|\sqrt{\alpha^2 u^2 + q_y'^2} - iq_y' \sin\theta)]. \quad (3.8)$$

This integral can be treated asymptotically using Laplace's method. The dominant contribution comes from points where the argument of the exponential is stationary,

$$\frac{d}{dq'_y}\{|\cos\theta|\sqrt{\alpha^2 u^2 + q'^2_y} - iq'_y\sin\theta\} = 0. \qquad (3.9)$$

One finds $q'_y = i\alpha u\sin\theta$, so that at large distances, G decays exponentially as $\exp(-R/\xi)$, where the correlation length $\xi^{-1} = 2u$. As $T \to 0$, the correlation length of the Ising model therefore displays a divergence proportional to $\exp(2J/k_B T)$, in agreement with the result given in Ref. 23, which was obtained using the transfer matrix technique.

IV. THE THREE-DIMENSIONAL ISING MODEL AS A TWO-DIMENSIONAL QUANTUM DIMER MODEL

In this section we use the modified Fisher construction to reformulate a class of three-dimensional Ising models as quantum dimer models in 2+1 dimensions. The interest of this mapping is that it yields nontrivial information on the quantum dimer models by transcribing known lore on the Ising models—no advances in solubility are entailed.[25]

The prototypical example of the Ising models of interest is the nearest neighbor ferromagnetic Ising model on the cubic lattice. As is well known, this is equivalent to a 2+1-dimensional transverse field Ising model in the "time continuum" limit. Briefly, this proceeds by the recognition that the phase structure of the model is unchanged if we take the cubic lattice to be a set of stacked square lattices and allow unequal couplings in the plane and in the stacking direction, $H(J_\tau, J_s) = -J_\tau \Sigma_{\{ij\}} \sigma^z_i \sigma^z_j - J_s \Sigma_{\langle ij \rangle} \sigma^z_i \sigma^z_j$, where the sums $\{ij\}$ and $\langle ij \rangle$ run over nearest neighbor sites in the stacking and planar directions. The anisotropic scaling limit $\exp(2K^\tau) \to \infty$ can be identified with the Trotter-Suzuki decomposition of the imaginary time path integral of the two-dimensional transverse field Ising model with the quantum Hamiltonian $\hat{H} = -J_s \Sigma_{\langle ij \rangle} \hat{\sigma}^z_i \hat{\sigma}^z_j - \Gamma \Sigma_i \hat{\sigma}^x_i$.

The Hilbert space of this quantum model is spanned by all classical Ising configurations of the two-dimensional classical model and the transverse field moves the system between these configurations. As the classical configurations can be related to dimer configurations on the Fisher lattice by the modified Fisher construction, it should be intuitively clear that the model can equally well be cast as a resonance dynamics in the space of dimer configurations.

The quantum dynamics induced in this way can be visualized by noting that the effect of the transverse field is to flip individual spins. In the modified Fisher construction, this corresponds to replacing empty *external* links surrounding the spin by occupied ones, and vice versa. In the dimer model, this also entails moving internal bonds to accommodate the change in external bonds. Here, it is important to note that the configuration of internal bonds of a cluster, given a set of external bonds, is unique and determined only by the external bonds belonging to that cluster. The quantum dimer Hamiltonian is therefore strictly local, although it includes a number of kinetic energy terms and loop flips of varying length with exactly the same strength—all nonzero off-diagonal matrix elements equal $-\Gamma$. An example of a dimer move present in the current Hamiltonian is given in Fig. 6. In addition the Ising nearest neighbor interaction—the
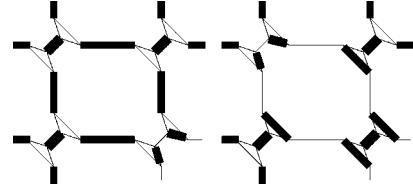

FIG. 6. One kinetic energy term of the quantum dimer Hamiltonian generated by the simple cubic Ising ferromagnet has a matrix element between the two configuration shown here.

number of frustrated bonds—translates into a potential energy for dimers on the external bonds.

The resulting quantum dimer Hamiltonian of this model can be schematically written as

$$H_D \equiv \hat{V} + \hat{T}$$

$$= -\sum_{\langle ij \rangle} J_{ij}\hat{\tau}^x_{ij} + \Gamma \sum_\square \prod_\square \hat{\tau}^\pm_{ij}. \qquad (4.1)$$

The J_{ij} of internal bonds are zero and the sum $\Sigma_\square$ runs over all closed loops made up of the external bonds surrounding a spin of the direct lattice and any of the bonds of the clusters on which these bonds terminate. $\Pi_\square \hat{\tau}^\pm_{ij}$ stands for alternating raising and lowering operators, $\cdots \tau^+_{ij}\tau^-_{jk}\tau^+_{kl}\tau^-_{lm}\cdots$ as one goes around the loop; this form preserves the hardcore condition.

In the above we have ignored one subtlety, namely that the map between Ising configurations and dimer configurations is two-to-one and the Ising dynamics *does* connect a state and its Ising reversed counterpart. The solution to this lies in considering combinations of a given state and its reversed counterpart,[16]

$$|\{\sigma_i\}\rangle_{\text{e/o}} = \frac{1}{\sqrt{2}}(|\{\sigma_i\}\rangle \pm |\{-\sigma_i\}\rangle) \qquad (4.2)$$

that are even and odd under global Ising reversal—a property respected by the Ising dynamics. In order to fix the "up" states we choose those so that a particular spin is always up in them. Now both sets of states can be mapped onto dimer states but the dimer dynamics is slightly (but importantly) different in the two sectors. In the even sector it is exactly what we described above. In the odd sector most matrix elements are the same but the ones that involve the chosen spin acquire an extra minus sign. Since Γ can be chosen negative without loss of generality, by the Perron-Frobenius theorem the ground state will always be in the even sector, although the first excited state need not be. So for the purposes of determining phase structure one can ignore this complication entirely. From the perspective of the dimer model, which is what we will take in the remaining, its dynamics will be represented solely by the even states of the Ising model.

At zero temperature, there is one parameter in this problem, namely the ratio of transverse field to bond strength, Γ/J_s and two phases that meet at a critical point. In Ising language, the two phases are, of course, the ferromagnet and paramagnet. As the reader may anticipate from our previous
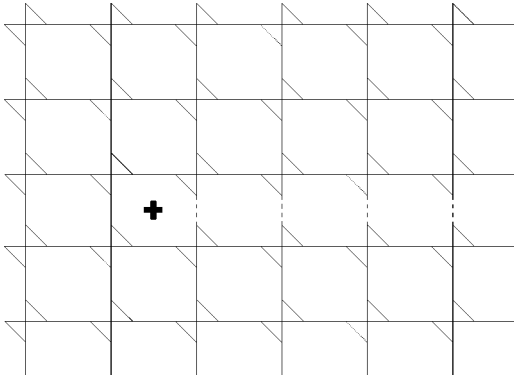
FIG. 7. A spin antialigned with the transverse field at the center of the plaquette denoted by the cross translates into a vortex excitation in the quantum dimer model.

classical considerations, the dimer transition is between a deconfining phase at large Γ/J_s and a confining phase at small Γ/J_s neither of which break any lattice symmetries.

The deconfined phase is of particular interest. Its point $\Gamma/J_s = \infty$ is the "Rokhsar-Kivelson" point of the model where the ground state wave function is the equal amplitude sum over all dimer configurations, which is the canonical short ranged RVB state. As discussed in Ref. 15, the elementary Ising excitation is a spin antialigned with the transverse field with energy 2Γ, and the entire spectrum is composed of them. In the dimer model the flipped spin translates into a vortex in which dimerconfigurations pick up a minus sign if the number of dimers on a string extending from the chosen plaquette to infinity is odd (labeled by dotted bonds in Fig. 7). The dimer model is exactly equivalent to the standard Ising gauge theory, and the vortex is therefore an Ising vortex (vison). For a system on a torus, these vortices need to be created in pairs, so that the minimal excitation energy is in fact 4Γ.[15]

The Ising states with a single flipped spin have disappeared in the course of the duality transformation. In fact, together with the other states with an odd number of flipped spins, they make up the sector, odd under global Ising reversal, which was discarded *en route* to the Ising model. The ground state at the Rokhsar-Kivelson point in this sector is the first excited state of the Ising model and has a degeneracy proportional to the system size. The equal amplitude superposition of the configurations in that sector corresponds to the reference spin being antialigned with the field.

As observed before there are four dimer sectors on the torus (and 4^g on lattices of genus g) which cannot be connected by local dimer moves. As these sectors correspond to different boundary conditions in the Ising problem, and the latter have only an $O(e^{-L/\xi})$ effect on the ground state energy in the paramagnetic phase (ξ is the correlation length), it follows that they lead to four exponentially degenerate ground states. Hence the deconfined phase displays topological order in the sense of Wen.[12]

Finally we observe that the dimer model on the Fisher lattice is exactly equivalent to the standard Ising gauge theory on the square lattice—dubbed the "even" Ising gauge theory in Ref. 14 due to the nature of its constraint. That

constraint states, in the Hamiltonian formulation, that the number of units of Ising electric flux entering/leaving a site (the distinction is irrelevant for Ising variables) must be even. If we identify a dimer on the external bonds of the Fisher lattice with such a flux, we recover this constraint. This conclusion is only to be expected since the even Ising gauge theory is dual to the transverse field Ising model.

Interestingly, the dimer constraint on the Fisher lattice, that the number of dimers coming out of a given site be one, is a U(1) constraint. As discussed in Ref. 14, this gives the dimer model the character of a U(1) gauge theory at the lattice scale. As in the case of the triangular lattice QDM, the topological sectors indicate a low energy structure with Ising character—which would be the general expectation for a deconfined (RVB) phase. What is special here is that, unlike the triangular QDM, we are able to put all states of the Fisher lattice QDM under consideration into correspondence with states in the Ising gauge theory.

Our basic considerations in this section can be generalized to a wide class of Ising models that permit a time continuum limit to be taken. These include models that can be viewed as identical planes stacked ferromagnetically or antiferromagnetically (or in any alternating combination). Depending on the frustration in the planes, we will obtain a QDM on a specific lattice obtained by the decoration procedure. For example, the FFIM stacked ferromagnetically will give rise to a QDM on the lattice shown in Fig. 1, which will be exactly equal to the "odd" Ising gauge theory in which the number of Ising fluxes leaving a site is odd. This QDM will exhibit a transition from a deconfined phase to a confining phase that does break lattice symmetries, as first discussed in the language of Ising models in Ref. 27.

V. COMMENTS ON THE SOLUBLE KAGOME QUANTUM DIMER MODEL

The dimer model we obtain at $\Gamma/J_s \rightarrow \infty$ has much in common with the kagome dimer model discussed by Misguich *et al.*, their quantum dimer model also has a range of kinetic terms around a given hexagonal plaquette, and it displays an analogous excitation spectrum consisting of Ising vortices. Nevertheless, the solubility and beautiful simplicity of their model does not derive from a Fisher construction. Instead, their problem has another very useful ingredient, namely the mapping of hardcore dimers to arrows on the kagome lattice,[28] and indeed any other lattice consisting of corner-sharing triangles.

Not until they have executed this mapping does our discussion of the Rokhsar-Kivelson point parallel their model, as one can consider the arrows as *bond* variables of a triangular lattice Ising model (or equivalently, link variables τ^x of a honeycomb gauge theory) which, crucially, has zero exchange strength. The quantum dynamics they study consists of the Wilson loop operation $\Pi_\bigcirc \tau^z$.

An exactly soluble quantum eight vertex model

To illustrate this correspondence, and because it is of interest to ask if the arrow mapping used by Misguich
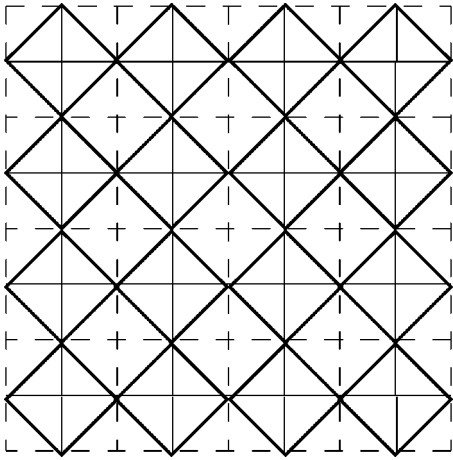
FIG. 8. A lattice of corner sharing squares (thick lines), which is the medial lattice of the lattice denoted by thin lines, which is in turn dual to the lattice denoted by dashed lines.
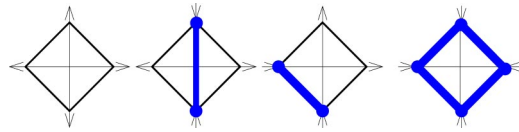

FIG. 9. Possible interpretation of the eight vertex configurations (denoted by arrows on thin lines) in terms of singlet bonds (denoted by dimers) between spins 1/2 on the corners of the square. The rightmost figure denotes an overall singlet of the four spins, of which there are two linearly independent ones.

et al. can be generalized, we note that their approach can be used to define an exactly soluble quantum eight vertex model. This model can also be given an interpretation in terms of singlet bonds.

The basic geometric object of this model are squares (instead of triangles), which are again arranged to share corners. At the corners of the squares reside Ising degrees of freedom (arrows), which point either in or out. In addition, we impose the constraint that an even number of arrows point out.

The simplest case occurs when the squares (denoted by fat lines in Fig. 8) are arranged on a square lattice (thin lines) so that they share corners. The different arrow configurations can be mapped onto configurations of the eight vertex model on the dual square lattice by identifying the arrows (which by construction live on the links of the dual lattice) with the arrows of the vertices of the eight-vertex model.

One can further interpret the arrows as link variables of an Ising model. The sites defining this model live on a further dual lattice (dashed lines). For the case of a bipartite thin lattice, one can label a bond frustrated if the arrow points from sublattice A to sublattice B and unfrustrated otherwise. (In the case of a nonbipartite thin lattice, one can choose a reference arrow and a reference spin configuration and identify the two.) The constraint of the eight-vertex model then becomes a constraint on the product of exchanges, $\Pi_{\square} J_{ij} = 1$, around a plaquette of the dashed lattice. However, this only imposes a constraint on allowed states—the strength of the exchanges vanishes. From this method, one can easily visualize the known result that the number of eight-vertex configurations on the square lattice equals 2^N, as there are no constraints on the Ising model on the dashed lattice. This result is straightforwardly generalized to any lattice defined by the midpoints of corner-sharing squares, as explained in Ref. 15 for lattices defined by midpoints of corner-sharing triangles.

This eight-vertex model can again be endowed with a quantum dynamics generated by a Wilson loop action on the links of the elementary plaquette of the thin square lattice.

This becomes the zero-exchange transverse field Ising model on the dashed square lattice.[29] The results, such as on the gap, 4Γ, or on the ultra-short correlation, follow just as they did before.[15]

A possible interpretation of the different eight-vertex configurations in terms of singlet bonds is as follows (see Fig. 9). Let spins $S = 1/2$ reside on the vertices of the fat square lattice. If no arrows point into a given square, there are no singlet bonds between any pair of spins of the square. If two point in, there is a singlet bond between the two. If four point in, the spins on the square form some collective singlet state. The choice of equal fugacities for all vertices amounts to disregarding the entropic contribution of the two linearly independent collective singlets.

As a related (and known) combinatorial result, we notice in passing that the number of nonoverlapping (but possibly intersecting) loop configuration on a lattice can be trivially evaluated in the same spirit. One can think of the loops as domain walls of an Ising model on the dual lattice. Up to a factor 2 from the global Ising symmetry, the number of domain wall configurations is then given by the size of the configuration space of the Ising model, which is of size 2^N for a dual lattice with N sites.

VI. SUMMARY

Two-dimensional Ising models are intimately related to dimer models. By means of the modified Fisher construction introduced in this paper, this connection can be exhibited configuration by configuration. The Ising transition in two dimensions maps onto a deconfinement transition in the dimer language as does the Ising transition in three dimensions, which now appears in a quantum dimer model. This last equivalence provides an instructive example of a topologically ordered RVB phase and a transition from it to a confining phase, and one for which all the details are known. These ideas can be extended straightforwardly to generate an exactly soluble quantum eight vertex model.

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