Generation of triplications in transversely isotropic media

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(Received 26 March 2003; published 12 August 2003)

Triplications and cusp edges can occur in homogeneous transverse isotropy (TI) provided the strength of anisotropy exceeds a critical value. The critical strength of anisotropy is 9.50% for axial triplication, 9.71% for basal triplication, 8.86% for oblique triplication, and 9.72% for double triplication. No TI with strength less than the critical can display triplications. On the other hand, high values of the strength of anisotropy do not guarantee the existence of triplications. Hence, observations of triplications on the wave surface cannot serve as a unique criterion defining strong TI.

DOI: 10.1103/PhysRevB.68.054107 PACS number(s): 46.40.Cd, 62.30.+d

I. INTRODUCTION

Triplications and cusp edges on the wave front can significantly complicate modeling of wave fields. They produce energy focusing[1–9] and phase shifting of signals.[10,11] In homogeneous transverse isotropy with normal polarization, triplications can occur for the SV wave only and can be classified into four different types (see Fig. 1): (i) oblique, (ii) axial, (iii) basal, and (iv) double (axial and basal) triplications. The existence of triplications is conditioned by the existence of concave or saddle-shaped areas on the slowness surface (see Fig. 2). These areas are separated from the convex areas by parabolic lines, formed by points of zero Gaussian curvature.[12–15] Generally, the stronger the anisotropy, the larger the part of the slowness surface which may be concave or saddle shaped, and the more developed the triplications. If anisotropy is decreased, the concave or saddle-shaped areas are reduced and the triplications are restricted to a narrower interval of angles. If two cusps defining the triplication coalesce into one (the width of the triplication reduces to only one direction), we speak of ''incipient'' triplication. In this case, the medium represents a borderline between the media with and without a triplication.[16]

Triplication conditions in transverse isotropy (TI) yield inequalities, with which one can uniquely decide whether the TI under study triplicates or not.[17–24] The inequalities are, however, rather complicated and difficult to understand. Hence, it is possible to classify any specific TI, but it is not easy to establish simple generalizations. We expect that no triplication can occur in a sufficiently weak TI,[15] but we have no understanding of which combinations of elastic parameters generate triplication and how strong anisotropy must be to generate triplications. Furthermore, it is not clear whether the occurrence of a triplication can be used as a criterion for distinguishing weak from strong TI.

II. TRIPLICATION CONDITIONS

We consider TI media, which satisfy the ''stability'' conditions[16,25]

$$a_{33}>0, \quad a_{44}>0, \quad a_{66}>0, \quad a_{11}-a_{66}>0,$$

$$a_{33}(a_{11}-a_{66})>a_{13}^2, \tag{1}$$

and the conditions that prevent the P and SV slowness or phase-velocity surfaces to intersect one another:

$$a_{11}-a_{44}>0, \quad a_{33}-a_{44}>0, \tag{2}$$

$$a_{13}+a_{44}>0, \tag{3}$$

where a_{kl} are the density normalized elastic parameters in the Voigt notation. Equation (3) is also the condition for the so-called ''normal polarization'' of P and SV waves.[26] For the analysis under less restrictive conditions, see Payton[20] or Alshits and Chadwick.[22]

The following equation conditions the axial triplications [see Musgrave,[17] Eq. (8.3.4)]

$$(a_{13}+a_{44})^2-a_{11}(a_{33}-a_{44})\geqslant 0; \tag{4}$$

the basal triplications [see Musgrave,[17] Eq. (8.3.5)],

$$(a_{13}+a_{44})^2-a_{33}(a_{11}-a_{44})\geqslant 0; \tag{5}$$

and the oblique triplications [see Dellinger,[27] Eq. (2.19); Thomsen and Dellinger,[24] (Eq. 9)],

$$3a_{44}^2-(a_{13}+a_{44})^2-a_{44}(a_{33}+a_{11})+3a_{11}a_{33}$$
$$-2\sqrt{(a_{33}-a_{44})(a_{11}-a_{44})}\frac{a_{11}a_{33}-a_{44}^2}{a_{13}+a_{44}}\leqslant 0, \tag{6}$$

where the equality sign stands for the incipient triplication. The slowness angle θ_i of the incipient triplication is $\theta_i=0°$ for the axial triplication, $\theta_i=90°$ for the basal triplication, and

$$\sin^2\theta_i=\frac{a_{33}-a_{44}}{a_{11}+a_{33}-2a_{44}}, \quad \cos^2\theta_i=\frac{a_{11}-a_{44}}{a_{11}+a_{33}-2a_{44}} \tag{7}$$

for the oblique triplication. Note that Eq. (6) is the exact opposite of the approximate conditions proposed by various authors.[18,19,21–23]

III. ANISOTROPY PARAMETERS

Triplication conditions (4)–(6) can also be expressed in terms of parameters ε, γ, σ, κ, and a_{44}, which represent an alternative parametrization of TI:[28,29]

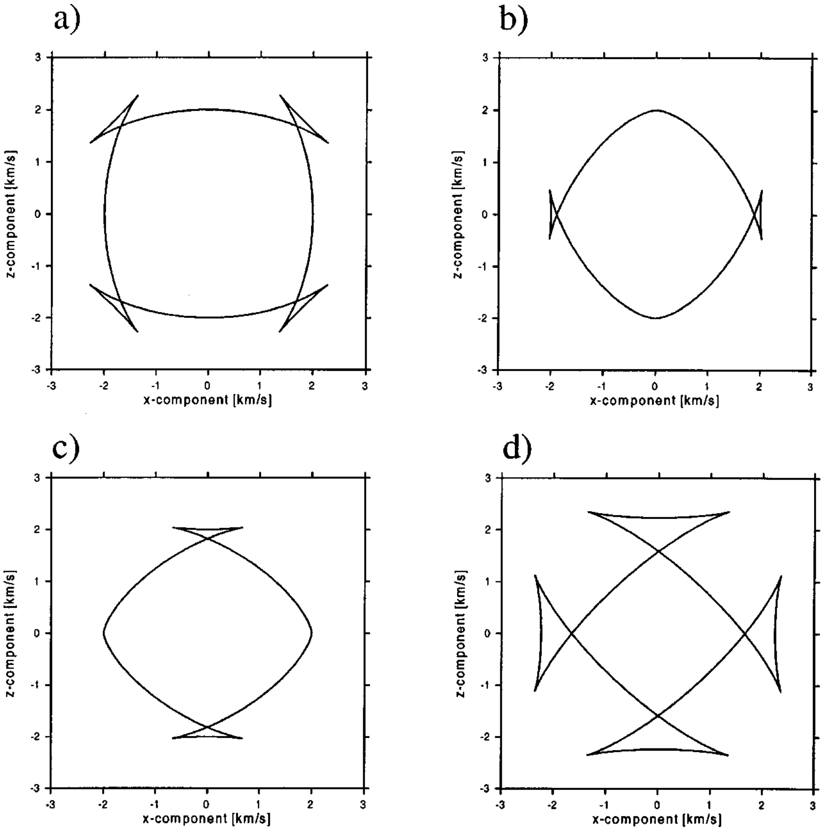

FIG. 1. Types of triplications in transverse isotropy. Vertical sections of wave surfaces are shown for (a) oblique, (b) basal, (c) axial, and (d) double triplications.

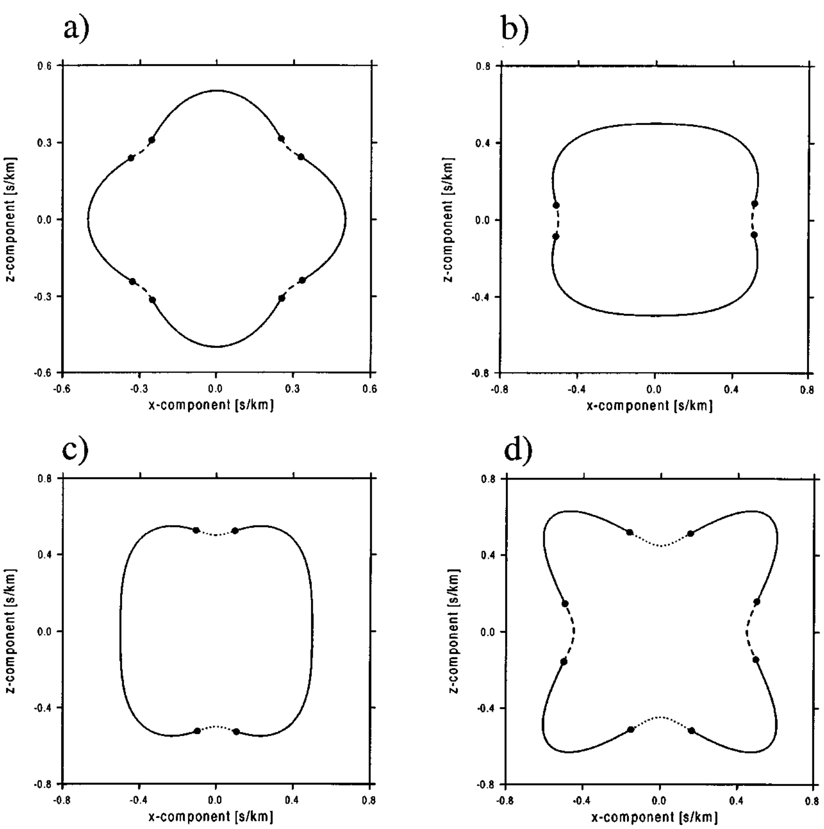

FIG. 2. Vertical sections of slowness surfaces generating different types of triplications. For details, see the caption of Fig. 1. Parabolic points (marked by dots) separate convex (solid line), concave (dotted line), and saddle-shaped (dashed line) areas.

$$\varepsilon = \frac{a_{11} - a_{33}}{2a_{33}}, \tag{8}$$

$$\gamma = \frac{a_{66} - a_{44}}{2a_{44}}, \tag{9}$$

$$\sigma = \frac{1}{2a_{44}}\left[a_{11} - a_{44} - \frac{(a_{13} + a_{44})^2}{a_{33} - a_{44}}\right], \tag{10}$$

$$\kappa = a_{33}/a_{44}. \tag{11}$$

Parameters ε, γ, and σ are called the anisotropy parameters and are frequently used for describing weak TI. For example, they control the angular variations of phase velocities in weak TI as follows:

$$(c^P)^2 = \kappa a_{44}\left(1 + 2\varepsilon \sin^2\theta - 2\frac{\sigma}{\kappa}\sin^2\theta \cos^2\theta\right), \tag{12}$$

$$(c^{SV})^2 = a_{44}(1 + 2\sigma \sin^2\theta \cos^2\theta), \tag{13}$$

$$(c^{SH})^2 = a_{44}(1 + 2\gamma \sin^2\theta), \tag{14}$$

where θ is the angle between the slowness vector and the axis of symmetry. The anisotropy parameters become zero in isotropy and can thus serve as a measure of strength of TI. Therefore, expressing the triplication conditions using these parameters provides a better understanding of how strong the anisotropy must be for the triplication to occur.

Equations (1)–(3) limit the values of κ, ε, γ, and σ. Equation (2) yields for κ and ε

$$2\varepsilon + 1 - \frac{1}{\kappa} > 0, \quad \kappa > 1, \tag{15}$$

Eqs. (1), (2), and (10) yield for σ

$$\frac{1}{2a_{44}}\left[a_{11} - a_{44} - \frac{(\sqrt{a_{11}a_{33}} + a_{44})^2}{a_{33} - a_{44}}\right] < \sigma < \frac{1}{2a_{44}}(a_{11} - a_{44}), \tag{16}$$

and using Eq. (1), we obtain for γ

$$-\frac{1}{2} < \gamma < \frac{1}{2}\left(\frac{a_{11}a_{33} - a_{13}^2}{a_{33}a_{44}} - 1\right). \tag{17}$$

IV. TRIPLICATION CONDITIONS EXPRESSED USING ANISOTROPY PARAMETERS

Following Thomsen and Dellinger,[24] the conditions for the axial and basal triplications can be expressed as follows:

$$\sigma \leqslant \sigma_c, \tag{18}$$

with σ_c expressed for the axial triplication ($\theta_i = 0°$) as

$$\sigma_c = -\frac{1}{2}, \tag{19}$$

and for the basal triplication ($\theta_i = 90°$) as

$$\sigma_c = -\frac{1}{2} - \varepsilon\frac{\kappa}{\kappa - 1}. \tag{20}$$

Parameter σ_c is the critical value of σ under which the incipient triplication occurs. The condition for oblique triplication (6) yields the following cubic equation:

$$\sigma_c^3 + A\sigma_c^2 + B\sigma_c + C = 0, \tag{21}$$

where

$$A = \frac{1}{\kappa - 1}\left[\frac{3}{2}\kappa^2(2\varepsilon + 1) + \kappa(\varepsilon + 1) + \frac{3}{2}\right],$$

$$B = 2\frac{\kappa}{(\kappa - 1)^2}(\varepsilon + 1)[\kappa^2(2\varepsilon + 1) + 1],$$

$$C = -2\frac{\kappa^2}{(\kappa - 1)^2}[\kappa(2\varepsilon + 1)^2 - 2\varepsilon - 1]. \tag{22}$$

The solution of Eq. (21) yields the interval of values for the σ parameter under which the medium triplicates [for an equivalent solution in a different notation, see Payton,[20] Eq. (2.4.12)]

$$\sigma \geqslant \sigma_c, \tag{23}$$

where

$$\sigma_c = u + v - \frac{1}{3}, \tag{24}$$

$$u = \left[-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right]^{1/3},$$

$$v = \left[-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right]^{1/3}, \tag{25}$$

$$p = B - \frac{1}{3}A^2, \quad q = -\frac{1}{3}AB + \frac{2}{27}A^3 + C. \tag{26}$$

Note that the triplication conditions do not depend on parameter a_{44}, which is only a scaling factor, and on parameter γ, which controls the propagation of the *SH* wave.

V. BEHAVIOR OF CRITICAL σ

We consider only the basal and oblique triplications, since the condition for the axial triplication is elementary. Figure 3 shows σ_c dependent on ε and κ ranging in the intervals $1 < \kappa < 10$ and $-0.2 < \varepsilon < 0.2$. The total interval for κ is subdivided into two intervals, $1 < \kappa < 2$ (lower plots) and $2 < \kappa < 10$ (upper plots), because the variation of σ for small κ is very strong. Figure 3 indicates that σ_c attains negative values for the basal triplication, but positive values for the oblique triplication. Hence, the basal and oblique triplications cannot occur simultaneously under the same TI. The figure also shows that triplications can occur for σ very close to zero. If

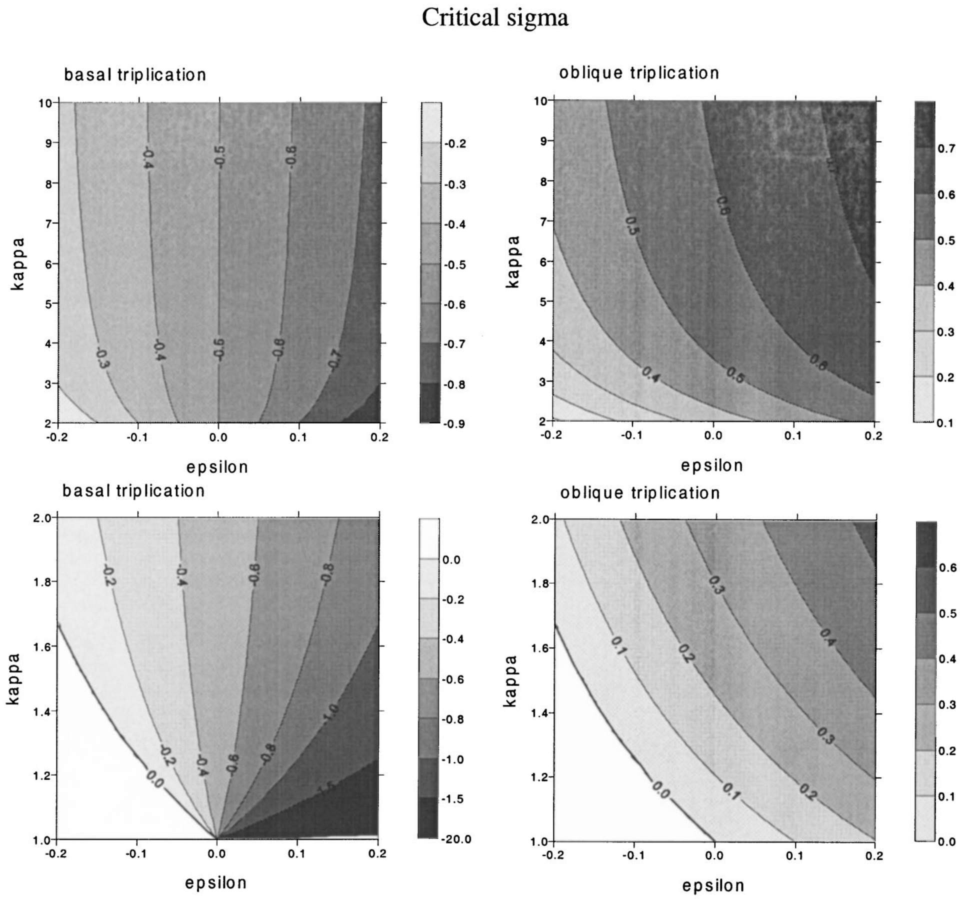

FIG. 3. Behavior of the critical value of σ. The blank areas in the left corner of the bottom plots are the areas in which Eq. (15) is not satisfied.

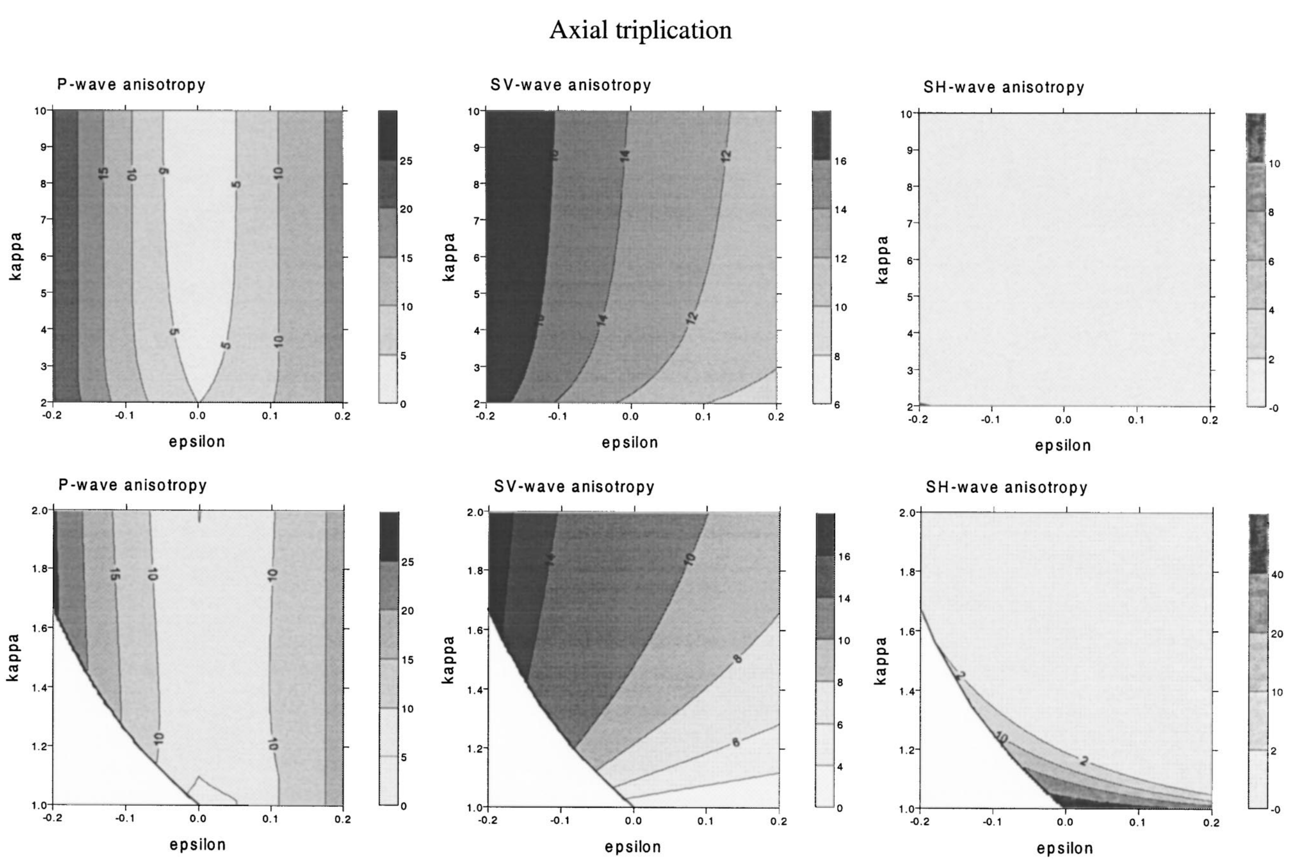

FIG. 4. Minimum values of the *P*-, *SV*-, and *SH*-wave anisotropy as a function of parameters κ and ε for the incipient axial triplication.

Basal triplication

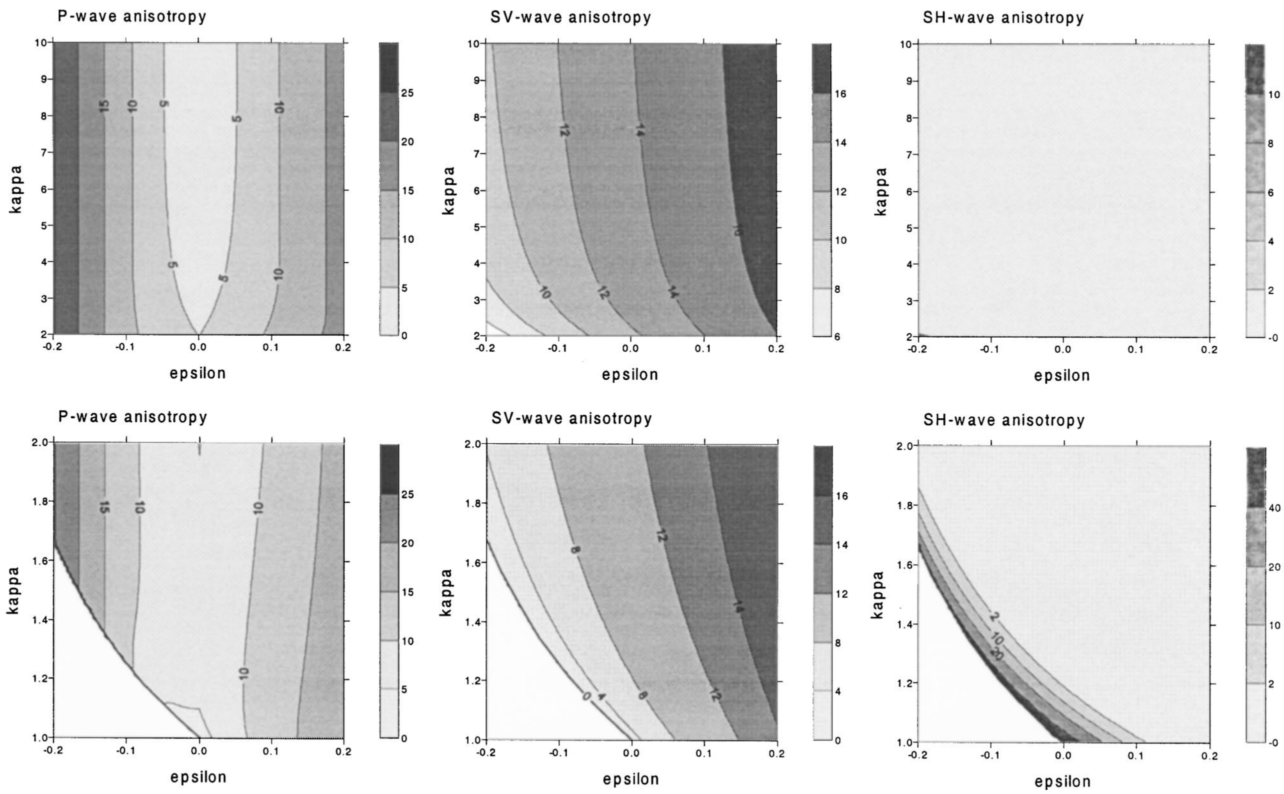

FIG. 5. Minimum values of the *P*-, *SV*-, and *SH*-wave anisotropy as a function of parameters κ and ε for the incipient basal triplication.

Oblique triplication

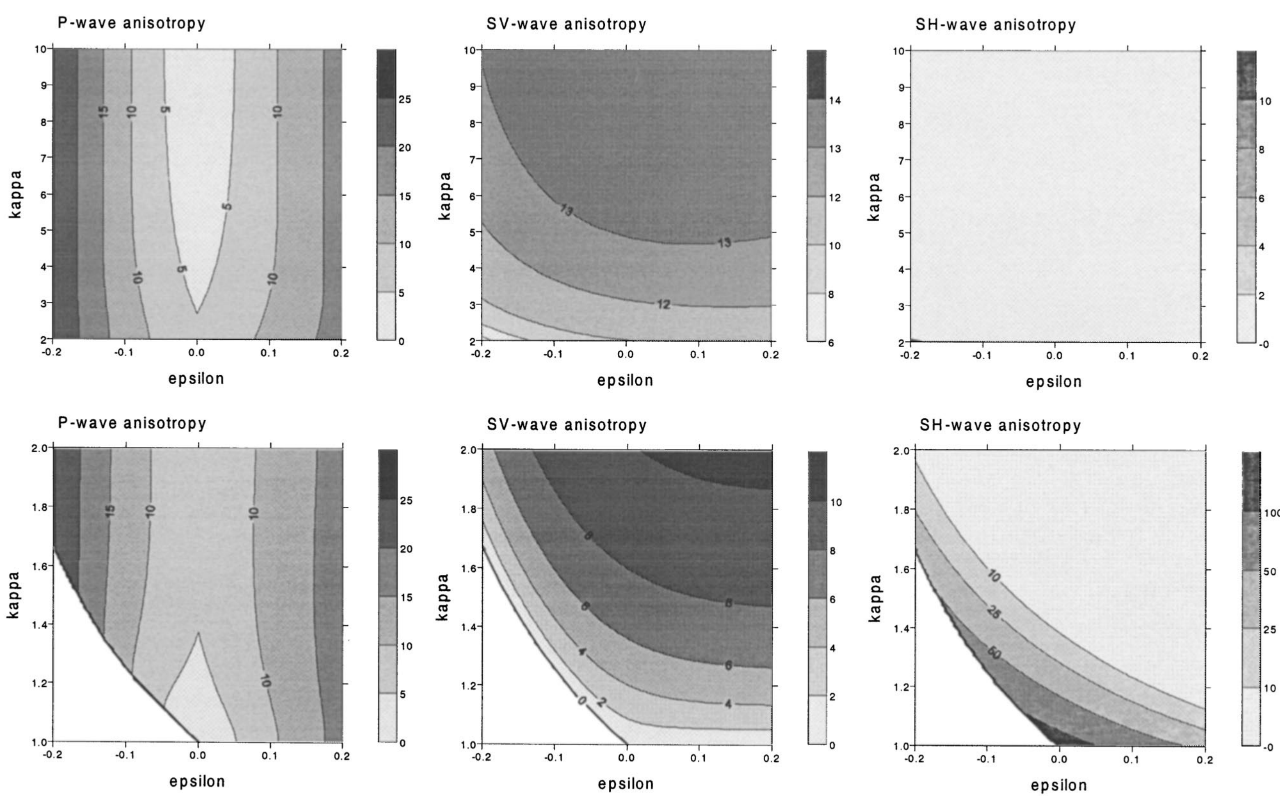

FIG. 6. Minimum values of the *P*-, *SV*-, and *SH*-wave anisotropy as a function of parameters κ and ε for the incipient oblique triplication.

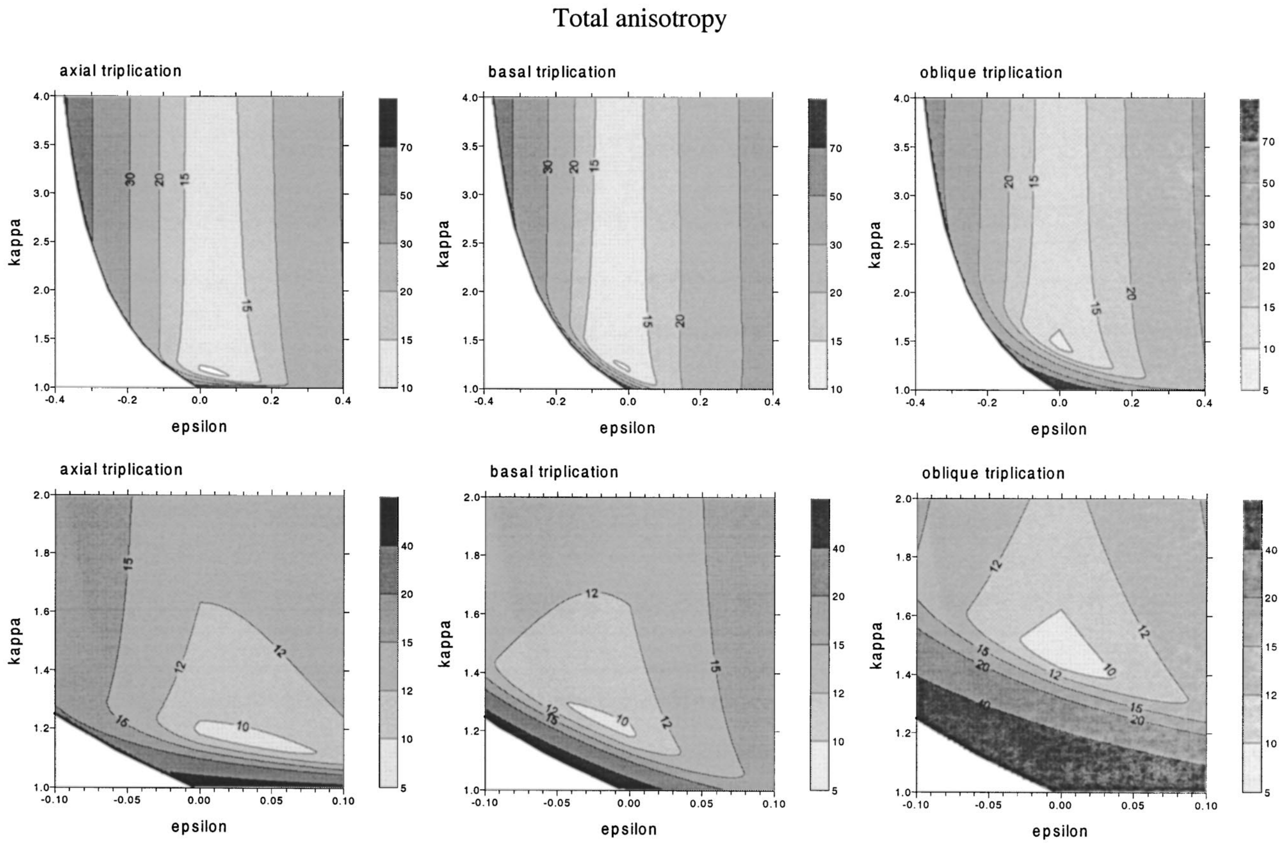

FIG. 7. Minimum values of the total anisotropy as a function of parameters κ and ε for the incipient axial (left), basal (middle), and oblique (right) triplications. The lower plots show the detailed behavior of anisotropy near its minimum.

$\kappa \to 1$ and $\varepsilon = 0$, then $\sigma_c \to 0$. This means that no threshold value for σ_c exists for either triplication.

VI. CRITICAL STRENGTH OF ANISOTROPY

We now ask how strong transverse isotropy must be to generate triplications. Figures 4–6 show the critical values of the *P*-, *SV*-, and *SH*-wave anisotropy for the occurrence of the axial, basal, and oblique triplications as a function of parameters κ and ε. The strength of anisotropy is defined as

$$a^W = 2\,\frac{c_{\max}^W - c_{\min}^W}{c_{\max}^W + c_{\min}^W} \cdot 100\%, \tag{27}$$

where W denotes the type of wave (*P*, *SV*, or *SH*), and $c_{\max}$ and $c_{\min}$ denote the maximum and minimum phase velocities of the *P*, *SV*, or *SH* waves. The total strength of anisotropy sums the strength of anisotropy of all waves in the following way:

$$a = \sqrt{(a^P)^2 + (a^{SV})^2 + (a^{SH})^2}. \tag{28}$$

Interestingly, the pattern of the critical strength of the *P*-wave anisotropy is very similar for all types of triplications. The strength is sensitive to ε, but almost insensitive to κ. For ε close to zero, the *P*-wave anisotropy is very small irrespective of κ. For $\kappa \to 1$ and $\varepsilon = 0$, the *P*-wave anisotropy tends to zero. On the contrary, the pattern of the *SV*-wave anisotropy depends on the type of the triplication. But the critical strength of the *SV* anisotropy can also be zero. It occurs for $\kappa \to 1$ and $\varepsilon = 0$ independently of the type of triplication. For basal and oblique triplications, zero *SV* anisotropy is observed also for κ and ε close to the borderline, delimiting the area of their permissible values. Hence, we observe that *P* and *SV* anisotropies can simultaneously attain values very close to zero for $\kappa \to 1$ and $\varepsilon = 0$. This means that the triplications can occur even under infinitesimally weak *P* and *SV* anisotropies and no *P* and *SV* anisotropy threshold exists for the occurrence of triplications in TI. However, where *P* and *SV* anisotropy simultaneously attain values close to zero, the *SH* anisotropy should be nonzero (see Figs. 4–6, right-hand plots). This follows from the stability conditions (17), which constrain the values of the *SH* anisotropy. Remember that the triplication conditions yield no other constraints on the *SH* anisotropy.

Figure 7 shows the total strength of anisotropy, which sums the strength of all waves. The figure shows that the minimum strength of anisotropy for the occurrence of a triplication is close to 10%. This applies to all types of triplications. Specifically, the minimum strength of anisotropy is 9.50% for the axial triplication ($\varepsilon = 0.04$ and $\kappa = 1.16$), 9.71% for the basal triplication ($\varepsilon = -0.01$ and $\kappa = 1.22$), 8.86% for the oblique triplication ($\varepsilon = 0$ and $\kappa = 1.46$), and

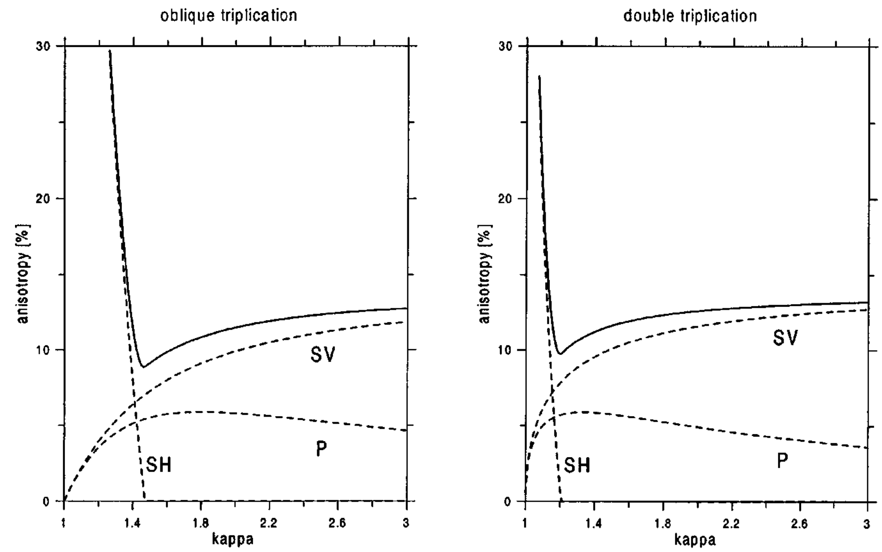

FIG. 8. Minimum values of the *P*-, *SV*-, and *SH*-wave anisotropy (dashed lines) together with the total anisotropy (solid line) as a function of parameter κ for the double (left-hand plots) and oblique (right-hand plots) triplications. Parameter ε equals zero.

9.72% for the double triplication ($\varepsilon=0$ and $\kappa=1.20$). These values represent the global minima of the critical strength of anisotropy. No triplication can occur in TI whose total strength of anisotropy is less than these minima. The minima are, however, rather shallow, because the strength of anisotropy increases very slightly with increasing κ. On the contrary, if κ decreases ($\kappa\rightarrow 1$), the strength of anisotropy rapidly increases.

A closed view of the strength of anisotropy as a function of κ is shown in Fig. 8. Parameter ε is fixed at zero. The critical *P*-wave anisotropy steeply increases from zero for $\kappa\rightarrow 1$, reaching its maximum at 6% for $\kappa=1.4$ (double triplication) or for $\kappa=1.8$ (oblique triplication). For higher values of κ, the *P* anisotropy decreases to less than 2% for $\kappa=10$. Also the critical *SV* anisotropy steeply increases from zero ($\kappa\rightarrow 1$) to 12% for $\kappa=3$. Thereafter the *SV* anisotropy increases very slowly, being less than 14% for $\kappa=10$. On the contrary, the *SH* anisotropy is high for $\kappa\rightarrow 1$, but rapidly decreases with increasing κ. The *SH* anisotropy is zero for $\kappa>1.2$ (double triplication) or for $\kappa>1.5$ (oblique triplication).

VII. DISCUSSION

The simplest triplication condition in TI is the condition for axial triplication. This condition requires the σ parameter to be less than -0.5. No other parameters control the occurrence of this triplication. The condition for the basal triplication is more involved. This triplication is controlled by anisotropy parameters σ and ε, and by parameter κ. The triplication condition requires σ to be negative. Therefore, if σ is less than -0.5, then double (axial and basal) triplication can occur. For $\varepsilon=0$, the condition for the axial and basal triplications becomes identical, hence they cannot be observed separately but only as the double triplication. For $\varepsilon\neq 0$ and $\sigma>-0.5$, only basal triplication can occur. The condition for this triplication yields

$$\sigma_c=-\frac{1}{2}\,\frac{a_{11}-a_{44}}{a_{33}-a_{44}}. \tag{29}$$

Taking into account Eq. (2), one can readily see that σ_c can never be positive. The maximum value that σ_c can attain is zero.

The most complicated condition is established for the oblique triplication. This triplication is controlled by anisotropy parameters σ and ε and by parameter κ. The triplication requires σ to be positive. A more detailed analysis would show that TI conditioned by Eqs. (1)–(3) could never produce the oblique triplication with negative values of σ. Hence, the basal and oblique triplications or axial and oblique triplications cannot occur simultaneously under the same TI.[20]

Analyzing the strength of anisotropy under which TI triplicates, we conclude that the critical strength of anisotropy is 9.50% for axial triplication, 9.71% for basal triplication, 8.86% for oblique triplication, and 9.72% for double triplication. No TI whose strength is less than the critical can display triplications. On the other hand, high values of the strength of anisotropy do not guarantee the existence of triplications. This can be illustrated on TI with elliptical angular dependence of phase velocities. Even an extremely high strength of such anisotropy produces no triplications. Hence, observations of triplications on the wave surface cannot serve as a unique criterion defining strong TI.

ACKNOWLEDGMENTS

The study was supported by the Consortium Project ''Seismic Waves in Complex 3D Structures,'' and by the Grant Agency of the Academy of Sciences of the Czech Republic, Grant No. A3012309.

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