Possibility of *c*-axis voltage steps for a cuprate superconductor in a resonant cavity

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Very anisotropic cuprate superconductors, such as $BiSr_2Ca_2CuO_{8+x}$, when driven by currents parallel to the *c* axis, behave like stacks of underdamped Josephson junctions. Here, we analyze the possibility that such a stack can be caused to phase lock, to exhibit self-induced resonant voltage steps (SIRS's), and hence to radiate coherently when placed in a suitable resonant electromagnetic cavity. We analyze this possibility using equations of motion developed to describe such SIRS's in stacks of artificial Josephson junctions. We conclude that such steps might be observable with a suitably chosen cavity and resonant frequency.

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I. INTRODUCTION

Barbara *et al.*[1] have recently shown that underdamped Josephson-junction arrays can be made to phase lock, and to radiate efficiently, if placed in a suitable resonant cavity. This phase locking is believed to occur because of interactions between the Josephson junctions and the cavity resonant mode. This interaction causes each junction to lock to the cavity frequency, creating an indirect interaction between any two junctions. Because any two junctions are interacting, the locking tendency grows with increasing numbers of junctions. If there are more than a critical number of junctions, there is global phase locking, the array current-voltage (IV) characteristic shows so-called self-induced resonant steps (SIRS's), analogous to the usual single-junction Shapiro steps, and there is efficient radiation into the cavity. Several authors have developed models for this process.[2–4] Some of these models appear to reproduce most of the features of the experiments, although certain features related to the observed threshold number of junctions may be at variance with experiment.[2]

This paper considers whether the type of locking observed by Barbara *et al.*[1] can be caused to occur in a *naturally occurring* Josephson-junction array, namely, a single-crystal sample of cuprate superconductor. Some of these materials, notably $BiSr_2Ca_2CuO_{8+x}$ near optimal doping, behave experimentally like stacks of underdamped Josephson junctions.[5] Further evidence of this Josephson behavior comes from effects of Josephson plasmon resonances (JPR's),[6] such as terahertz radiation from high-T_c materials resulting from resonant excitation of JPR's.[7] By analogy with Ref. 1, it should therefore be possible to place such materials in a suitable resonant cavity, and generate phase locking, self-induced resonant steps, and efficient radiation into the cavity. We will use the model of Ref. 3 in order to investigate the parameters which may apply to the most anisotropic cuprate superconductors.

The remainder of this paper is arranged as follows. In Sec. II, we briefly review the model, and estimate parameters for the more anisotropic cuprates. Section III describes some model calculations carried out using these parameters. In Sec. IV, we give a brief concluding discussion, discussing possible model limitations and connections to other approaches.[8]

II. MODEL EQUATIONS AND ESTIMATE OF MODEL PARAMETERS

A. Basic equations

As was first shown by Kleiner *et al.*,[5] the c-axis IV characteristics of $BiSr_2Ca_2CuO_{8+x}$ resemble those of a stack of underdamped Josephson junctions, exhibiting such key characteristic features of underdamped junctions as Shapiro steps and hysteresis in the IV characteristics.

Suppose that a sample of such a cuprate superconductor is placed in a suitable resonant cavity. We assume that a current I is injected parallel to the c axis into one face of the sample, and extracted from the other face. The sample is considered as a stack of N underdamped Josephson junctions. According to the model of Ref. 3, the combined system of junctions and cavity satisfy the following equations of motion:

$$\ddot{\gamma}_j+\frac{1}{Q_{Jj}}\dot{\gamma}_j+\sin\gamma_j=\frac{I}{I_c(1+\Delta_j)}-2\ddot{\tilde{a}}_R, \tag{1}$$

$$\ddot{\tilde{a}}_R+(\Omega')^2\tilde{a}_R=-\tilde{g}\frac{\Omega'}{\Omega}\sum_{j=1}^{N}(1+\Delta_j)\ddot{\gamma}_j. \tag{2}$$

Here γ_j is the gauge-invariant phase difference across the jth junction, Q_{Jj} is its quality factor, and $I_c(1+\Delta_j)\equiv I_{cj}$ is its critical current. The dots represent derivatives with respect to a dimensionless time $\overline{\omega_p}t$. $\overline{\omega_p}$ is the average of the Josephson plasmon frequencies $\omega_{pj}=\sqrt{2E_{Cj}E_{Jj}}/\hbar=\sqrt{2eI_{cj}/(\hbar C_j)}$, where $E_{Cj}=(2e)^2/(2C_j)$ is the capacitive energy of the jth junction, C_j is its capacitance, and $E_{Jj}=\hbar I_{cj}/(2e)$ its Josephson coupling energy. In the resistively and capacitively shunted junction (RCSJ) model,[9] $Q_{Jj}=\omega_{pj}R_jC_j$, where R_j is the jth shunt resistance. $\tilde{a}_R=\sqrt{g}a_R$, where $a_R=(a+a^\dagger)/2$, a and $a^\dagger$ being the standard Bose creation and annihilation operators for the cavity mode. The equations of motion (1) and (2) are applicable in the "classical" limit when there are many photons in the cavity. In this regime, the operators a, $a^\dagger$, and a_R can be treated as c numbers.[3]

Finally,

$$g=(\hbar c^2/\Omega)[(2\pi)^3/\Phi_0^2]\left[\int_j \mathbf{E}(\mathbf{x})\cdot\mathbf{d}\ell\right]^2, \tag{3}$$

where $\Phi_0 = hc/(2e)$ is the flux quantum, the line integral is taken across the junction, and $\mathbf{E}(\mathbf{x})$ is proportional to the cavity mode electric field, normalized so that $\int_V |\mathbf{E}(\mathbf{x})|^2 d^3x = 1$, where V is the cavity volume, and Ω is the cavity mode frequency. The renormalized coupling constant $\tilde{g} = gE_J/(\hbar\overline{\omega}_p)$, and in terms of $\tilde{g}$, $(\Omega')^2 = \Omega^2/[1+2\tilde{g}\Sigma_j(1+\Delta_j)]$. In a real cuprate superconductor $\Omega' \approx \Omega$ to less than 1%.

B. Estimate of model parameters

We assume that the N junctions in the cuprate superconductor consist of $N+1$ layers of CuO_2, spaced a distance d apart, and each of area S. The capacitance (in esu) is therefore $C \approx \epsilon S/(4\pi d)$, where ϵ is the relative dielectric constant of the material between the CuO_2 layers. A typical value of the c-axis resistivity for $BiSr_2Ca_2CuO_{8+x}$ is $\sim 10\ \Omega$ cm at a temperature T just above the superconducting transition temperature T_c,[5] corresponding to a shunt resistance $R_j = d\rho_{cj}/S$, where ρ_{cj} is the resistivity of the material in the jth junction.

We estimate I_{cj} using a modified zero-temperature Ambegaokar-Baratoff relation[10] $2eI_{cj}R_j = \alpha\Delta(0) = 2eI_{cj}d\rho_{cj}/S$, where $\Delta(0)$ is the $T=0$ superconducting energy gap. In the original Ambegaokar-Baratoff relation,[10] $\alpha = \pi/2$, but according to measurements for $BiSr_2Ca_2CuO_{8+x}$,[11] the I_cR product corresponds to $\alpha \sim 0.2$. From the above relations for ω_{pj} and $2eI_{cj}R_j$, the Josephson plasmon frequency is $\overline{\omega}_p^2 \approx 4\pi\alpha\Delta(0)/(\hbar\epsilon\rho_{cj})$. Using all these expressions for C, $2eI_{cj}R_j$, and $\overline{\omega}_p^2$, we find $Q_J^2 = \alpha\Delta(0)\rho_{cj}\epsilon/(4\pi\hbar)$. Thus, for a material with a given $\Delta(0)$, $Q_{Jj} \propto \sqrt{\rho_{c,j}}$.

Using these expressions for Q_{Jj} and ω_{pj}, we find $\omega_{pj}Q_{Jj} = \alpha\Delta(0)/\hbar$. This relation is useful because ρ_{cj} does not appear directly. Using the estimate $\Delta(0)/k_B = 400$ K and $\alpha = 0.2$, we get $\alpha\Delta(0)/\hbar \approx 10^{13}\ \mathrm{sec}^{-1}$, and hence $\omega_{pj}/(2\pi) \sim (1600/Q_{Jj})$ GHz.

Next, we estimate $\tilde{g}$ for a cuprate superconductor such as $BiSr_2Ca_2CuO_{8+x}$. We assume that $\mathbf{E}(\mathbf{x})$ is polarized perpendicular to the layers with constant magnitude E_0 throughout that layer. Then $g = 2\pi(2e)^2 d^2 E_0^2/(\Omega\hbar V)$ and hence $\tilde{g} = 2\pi E_J(2e)^2 d^2 E_0^2/[\hbar^2\Omega\overline{\omega}_p]$.

We calculate $\tilde{g}$ for $\Omega = \omega_p$. Combining the relation $E_J = \hbar I_c/(2e)$ with our expressions for C and $\overline{\omega}_p^2$, we obtain $\tilde{g} = \epsilon v E_0^2/2$, where $v = Sd$ is the junction volume. To make a rough estimate, we consider a rectangular cavity with dimensions L_x, L_y, and L_z, with $L_x \sim L_y \gg L_z$, and with $L_y > L_x$, containing material of dielectric constant ϵ_c and magnetic permeability μ_c. In this case, the lowest TE mode has frequency $\Omega = (c/\sqrt{\epsilon_c\mu_c})\sqrt{\pi^2/L_x^2 + \pi^2/L_y^2}$; the corresponding electric field is[12] $\mathbf{E}(x,z) = E_0\hat{\mathbf{z}}\sin(\pi x/L_x)\sin(\pi y/L_y)$, where we assume that $\hat{z}$ is parallel to the c axis. Requiring that $\int|\mathbf{E}(x,y,z)|^2 d^3x = 1$ gives $E_0 = 2/\sqrt{V}$, where $V = L_xL_yL_z$, and hence $\tilde{g} = 2\epsilon v/V$. The lowest cavity frequency Ω is approximately $c\sqrt{2}\pi/(L_x\sqrt{\epsilon_c\mu_c})$, which implies that $\tilde{g} \approx 4\epsilon\epsilon_c\mu_c S d\nu^2/(c^2 L_z)$, where $\nu = \Omega/(2\pi)$. Taking $\nu = 300$ GHz, $d = 1.5$ nm, $\epsilon = 10$, and arbitrarily choosing $L_z = 1\ \mu$m, $S = (15\ \mu\mathrm{m})^2$, and $\epsilon_c = \mu_c = 1$, we find $\tilde{g} \sim 2\times 10^{-5}$, a value which is obviously very sensitive to the parameter values. This type of coupling might be achievable in a geometry in which a thin film of superconductor is oriented within a cavity, such that its c axis is parallel to the small dimension of a cavity. Possibly the film might actually form one wall of this cavity, whose large dimensions extend parallel to the ab plane.

The estimates given above assume that the cavity electric field can penetrate without restriction between the CuO_2 planes. In reality, this penetration is restricted by screening. The relevant screening length is the Josephson penetration depth, denoted λ_J, which characterizes the spatial variation of fields within the c-axis Josephson junctions. This length has been experimentally estimated as $15\ \mu$m in $BiSr_2Ca_2CuO_{8+x}$ for $T \ll T_c$,[13] and presumably substantially larger at higher T. These values are comparable to the linear dimensions we assumed for our CuO_2 planes for $\tilde{g} = 2\times 10^{-5}$. Hence, this value of $\tilde{g}$ may be experimentally achievable. For larger linear dimensions, our "small junction" model would need to be generalized, as discussed below (Sec. IV).

III. NUMERICAL RESULTS

To illustrate these predictions, we have solved Eqs. (1) and (2) numerically for a range of parameters. As in Ref. 3, this was accomplished by rewriting these equations as a set of coupled first-order differential equations:

$$\dot{\gamma}_j = \tilde{n}_j - 2\tilde{\Omega}\tilde{a}_I, \tag{4}$$

$$\dot{\tilde{n}}_j = \frac{I}{I_c(1+\Delta_j)} - \frac{\tilde{n}_j}{Q_{Jj}} - \sin(\gamma_j) + 2\frac{\tilde{\Omega}}{Q_{Jj}}\tilde{a}_I, \tag{5}$$

$$\dot{\tilde{a}}_R = \tilde{\Omega}\tilde{a}_I, \tag{6}$$

$$\dot{a}_I = -\tilde{\Omega}\tilde{a}_R - 2\tilde{\Omega}\tilde{g}\frac{\tilde{a}_I}{Q_{Jj}}\sum_j (1+\Delta_j) + \tilde{g}\sum_j (1+\Delta_j)\sin(\gamma_j)$$

$$-N\tilde{g}\frac{I}{I_c} + \frac{\tilde{g}}{Q_{Jj}}\sum_j (1+\Delta_j)\tilde{n}_j. \tag{7}$$

Here $\tilde{\Omega} = \Omega/\overline{\omega}_p$, and $\tilde{n}_j$ is the scaled number variable.[3] We solve these equations using a constant-time-step fourth-order Runge-Kutta procedure with a time step of 0.001. We begin the simulation by initializing the parameters $\tilde{a}_R$, $\tilde{a}_I$, and I/I_c, and the $\tilde{n}_j$'s to zero, while the γ_j's are initialized at independent random values uniformly distributed between 0 and 2π. For a given I/I_c, the differential equations were then integrated, and the voltages averaged, over a dimensionless time interval of $\tau = 5\times 10^3$. The ratio I/I_c was then increased or decreased by an amount 0.01 and the set of equations was solved again.

When we solve these equations using $Q_{Jj} \sim 100$, as expected for $BiSr_2Ca_2CuO_{8+x}$, we have not as yet found nu-

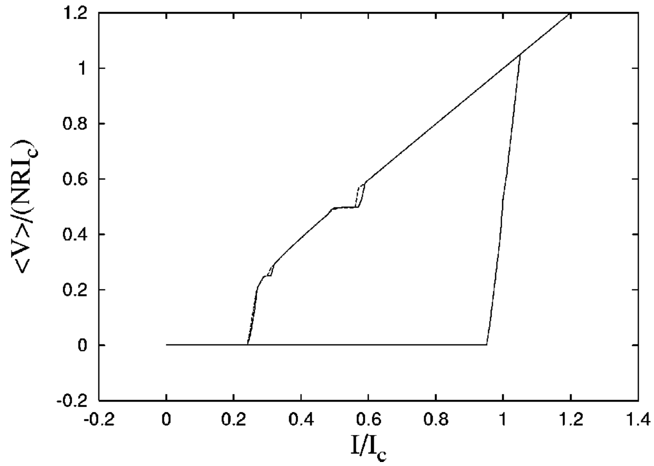

FIG. 1. Full curve: calculated IV characteristics for an array with $\Gamma=0.0$ (no damping from cavity walls), $N=40$, $\tilde{g}=1.75\times10^{-4}$, $Q_J=5$, $\tilde{\Omega}=2.5$, and disorder parameter $\Delta=0.05$. Parameters are defined in the text. Dashed curve: same as full curve but with cavity damping parameter $\Gamma=0.01$.

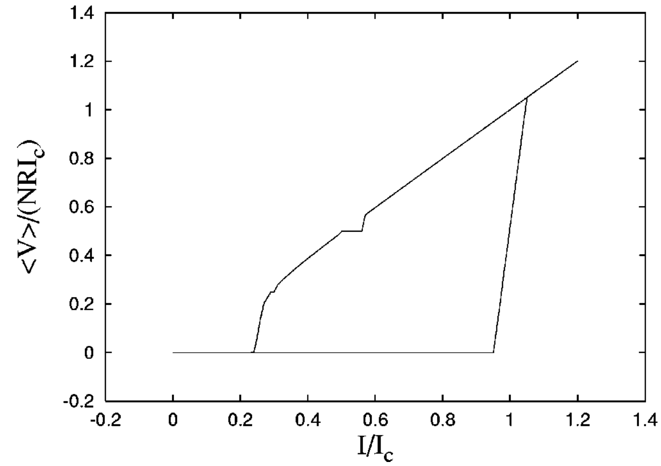

FIG. 2. Same as full curve in Fig. 1 but with $N=100$ and $\tilde{g}=1.75\times10^{-5}$.

merical evidence of SIRS's. We have therefore rerun our calculations using other model parameters which may be suitable for a cuprate superconductor slightly less anisotropic than $BiSr_2Ca_2CuO_{8+x}$. We find that, for all values of Q_{Jj} between 1 and 5, such SIRS's are easily detectable in our simulations.

Typical numerical results are shown as the full curve in Fig. 1. In these calculations, we have chosen all the $Q_{Jj}=5$. The other parameters are $\tilde{\Omega}=2.5$, $N=40$, and $\tilde{g}=1.75\times10^{-4}$; the disorder parameters Δ_j's were chosen as independent random numbers uniformly distributed between -0.05 and $+0.05$. Two SIRS's are visible, one at $\langle V\rangle/(NRI_c)=1/2$ and one at 1/4. The step at 1/2 is wider than that at 1/4, possibly because there are more junctions phase locked on the higher-voltage step: all the junctions are phase locked on the 1/2 step, while only half are locked on the 1/4 step. The lower voltage (with the large jump near $I/I_c=1$) is obtained when the current is swept in an increasing direction, while the upper voltage curve corresponds to a decreasing current sweep.

The result of the full curve in Fig. 1 corresponds to no damping of the cavity mode itself. To see the influence of cavity damping, we included a cavity damping term to Eq. (7) by arbitrarily adding the term $-\Gamma\tilde{a}_I$ to the right-hand side, Γ being the cavity damping parameter. For sufficiently large Γ, we find that the SIRS's are damped out, while for $\Gamma\leq0.05$, they are still visible. For sufficiently small Γ, the SIRS's are little changed from those of Fig. 1. We illustrate this similarity in the dashed curve of Fig. 1, which shows the IV characteristics for $\Gamma=0.01$; the other parameters are unchanged from the full curve.

A typical result for a weaker coupling is shown in Fig. 2, which shows the IV characteristics for $N=100$, $\tilde{g}=1.75\times10^{-5}$, and $\Gamma=0.0$; the other parameters are the same as in Fig. 1. For $N=100$, we have found both the 1/2 and the 1/4 steps for $\tilde{g}=1.75\times10^{-4}$ and 1.75×10^{-5}, while for 1.75×10^{-6} (not shown) we have seen thus far only the upper step. Thus, even for a coupling as weak as 1.75×10^{-6}, these SIRS's should show up for a suitable cuprate superconductor in an appropriate cavity. We have also carried out calculations with 200 junctions in an undamped cavity, with other parameters the same as those of Fig. 2; we obtained results similar to those for 100 junctions for all three values of $\tilde{g}$.

The choice of parameters in Figs. 1 and 2 is based on arguments of the preceding section. According to those arguments, a value of $Q_J\sim5$ would correspond to a cuprate superconductor less anisotropic than $BiSr_2Ca_2CuO_{8+x}$, but still in the underdamped regime where the superconductor should behave like a stack of underdamped Josephson junctions. Although even a larger Q_J should produce SIRS's, we have not yet seen them in our simulations.

IV. DISCUSSION

There is a remarkable formal similarity between Eqs. (1) and (2) governing the coupled Josephson-cavity system and those recently proposed by Helm *et al.*[8] to describe the interaction between an intrinsic Josephson junction and an optical phonon in a cuprate superconductor. In fact, our Eqs. (1) and (2) would be identical in form to Eq. (5) and (6) of Ref. 8 if there is only a single junction and the optical phonon in the model of Ref. 8 is assumed undamped. The precise mapping for a single junction is $\tilde{a}_R=-K\dot{p}$, where $\dot{p}$ is a polarization current appearing in the model of Ref. 8. The proportionality constant K depends on the choice of time units in the two sets of differential equations.

The interaction between a Josephson junction and an optical phonon has been observed experimentally and the predicted IV structure has been experimentally confirmed.[8] Thus, the SIRS's predicted by our model for the Josephson-cavity system are the analogs of the IV subgap structure calculated for $BiSr_2Ca_2CuO_{8+x}$ with a Josephson optical phonon interaction. The most important difference is that the cavity interaction tends to cause the junctions to phase lock, because each intrinsic junction interacts with the *same* cavity mode, whereas in Ref. 8, the optical phonons in different junctions are independent; so no locking is predicted.

If all the intrinsic junctions were to interact with the *same* vibrational mode, then these junctions might also be induced to phase lock, as with the electromagnetic cavity mode. This type of locking may be conceivable using a suitable micromechanical resonator. Such resonators can now be fabricated with fundamental vibrational modes in the range of 0.01–1 GHz (Refs. 14 and 15), and interact with small underdamped Josephson junctions according to the same Hamiltonian as that discussed here for electromagnetic cavities.[16] Hence, a similar type of phase locking is possible, given a suitable experimental geometry.

Finally, we briefly discuss the small junction approximation used here. Specifically, we have characterized each layer by a single phase and have neglected the spatial variation of that phase in the ab plane. In a more realistic model, this variation would be included, as would the coupling between the intrinsic junctions induced by that phase variation. As has been shown by Sakai *et al.*,[17] the junction dynamics are then described by a set of equations for coupled weakly damped sine-Gordon solitons. If, however, both the linear dimensions of the junctions in the ab plane and the thickness of the junction in the c direction are small compared to λ_J, then their equations reduce to the ones used here. Thus, the present approach should be accurate for disklike cuprate samples (‘‘mesas’’) of linear dimensions in the ab plane smaller than $\lambda_J \sim 15\ \mu$m for $T \ll T_c$.[13] For much larger mesas, the present model should be generalized to the case of coupled solitons interacting with a resonant cavity.

To summarize, we have proposed that when a very anisotropic cuprate superconductor (which behaves like a stack of underdamped Josephson junctions) is placed in a suitable single mode resonant cavity, the stack should be able to phase lock and to exhibit self-induced resonant voltage steps (SIRS’s) at a frequency related to the cavity frequency. In support of this suggestion we have provided numerical illustrations of these steps for suitable parameters. We also provide estimates showing that these parameters may be achievable in a physically realizable cuprate superconductor in a realistic cavity.

If this suggestion is verified, it might have a range of intriguing consequences. For example, an array locked on a SIRS would radiate coherently into the cavity.[1–3] Thus, a suitable cuprate superconductor might be usable in this way as a source of coherent microwave radiation. This possibility would be particularly exciting because the source would not be an artificially nanostructured material but one within the equilibrium phase diagram of a multicomponent system.

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