Gauge-invariant response functions in algebraic Fermi liquids

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A method is developed that permits a simple evaluation of two-loop response functions for fermions coupled to a gauge field. We employ this method to study the gauge-invariant response functions in the algebraic Fermi liquid, a non-Fermi liquid state proposed to describe the pseudogap phase in the QED_3 theory of cuprate superconductors. The staggered spin susceptibility is found to exhibit a characteristic anomalous dimension exponent η_4, while other correlators show a behavior consistent with the conservation laws imposed by the symmetries of the underlying theory.

Low-energy effective theories of certain correlated electronic systems are known to be equivalent to (2+1)-dimensional quantum electrodynamics (QED_3).[1–6] The latter describes N species of massless "relativistic" Dirac fermions coupled to a massless gauge field a_μ. While the physics behind these different reincarnations of QED_3 varies from case to case, these theories are all of considerable general interest for the following reason: In the so-called symmetric phase of QED_3, long-range interactions mediated by the massless gauge field produce characteristic anomalous correlations between electrons which decay on long length and time scales as *nontrivial power laws*. This symmetric phase has been termed variously as the algebraic spin[5] or algebraic Fermi[6] liquid (AFL) and embodies a unique realization of a non-Fermi liquid state of electronic matter in 2 spatial dimensions.

The power law correlations are encoded in the fermion propagator of the theory, $G(x) = \langle \Psi(x)\bar{\Psi}(0)\rangle \sim x^{-(2+\eta)}$, where $\eta = (4/3\pi^2 N)(3\xi - 2)$ is the anomalous dimension exponent and $\xi \geq 0$ is a gauge fixing parameter. The above fermionic propagator is gauge dependent, and therefore it cannot represent the behavior of the physical electron in the underlying theory. Much effort has gone into constructing and calculating the proper gauge-invariant electron propagator[7–10] but the situation remains unclear. The most natural candidate, $\mathcal{G}(x-x') = \langle \exp(i\int_x^{x'} a_\mu ds_\mu)\Psi(x)\bar{\Psi}(x')\rangle$, suffers from severe ultraviolet divergences due to the line integral of the gauge field, and is meaningless in the absence of a physically motivated regularization scheme.[9] Attempts to evaluate $\mathcal{G}$ directly have yielded an unphysical negative anomalous dimension.[7,8,10]

In this paper we search for anomalous power law correlations in various physical response functions of QED_3 which represent susceptibilities and conductivities of the spin and charge degrees of freedom in the underlying theory. By construction these quantities describe gauge-invariant physical responses of the system to an external probe, and therefore no difficulties arise as to the interpretation of the results. We evaluate the requisite correlators to leading nontrivial order in the $1/N$ expansion.[11] The principal technical hurdle in any such calculation is the evaluation of the vertex correction which is essential to preserve gauge invariance.[12] In the

typical treatment[13,14] this leads to lengthy algebra due to overlapping singularities within Feynman diagrams. Here we devise a method for isolating the leading divergent behavior of such diagrams which reduces the task to computing the trace of small number of Dirac gamma matrices. We test this method against known results and derive additional results with essentially no extra effort. We formulate a simple general rule for determining whether or not a given correlator exhibits any anomalous dimension and discuss this result in terms of a theorem that prohibits the correlators of conserved operators from acquiring an anomalous dimension.[15,16]

In what follows we focus on the QED_3 theory of cuprate pseudogap as formulated in Refs. 6 and 9 but our technique remains applicable to any other reincarnation of QED_3.[3–5] In the former, QED_3 emerges as a low-energy effective theory for the nodal topological fermions $\Psi_l(x)$, in a phase-disordered d-wave superconductor (dSC). The Lagrangian density is

$$\mathcal{L}_D = \sum_{l=1}^{2} \bar{\Psi}_l(x)\gamma_\mu(i\partial_\mu - a_\mu)\Psi_l(x) + \frac{1}{2}K_\mu(\partial\times a)_\mu^2, \quad (1)$$

where the gauge field a_μ encodes the topological frustration encountered by fermions as they propagate through the "soup" of fluctuating unbound vortex–antivortex pairs. $x = (\tau, \mathbf{r})$ denotes the space-time coordinate and $\gamma_\mu(\mu = 0,1,2)$ are 4×4 gamma matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The bare dynamics of the gauge field is Maxwellian[6,9,17] with stiffness K_μ.

In the above theory [Eq. (1)], $\Psi_l(x)$ is a four component spinor describing the fermionic excitations at the lth pair of antipodal nodes of the underlying dSC. Its individual components are related to the original electron operators c_σ through the singular gauge transformation

$$c_\sigma(\mathbf{r}, \tau) = e^{i\varphi_\sigma(\mathbf{r}, \tau)}\psi_\sigma(\mathbf{r}, \tau), \quad (2)$$

detailed in Refs. 6 and 9. The purpose of this transformation is to "unwind" the phase $\varphi(\mathbf{r}, \tau) = \varphi_\uparrow(\mathbf{r}, \tau) + \varphi_\downarrow(\mathbf{r}, \tau)$ of the fluctuating SC order parameter $\Delta(\mathbf{r}, \tau) = \Delta_0 e^{i\varphi(\mathbf{r}, \tau)}$ in favor of coupling to the gauge field a_μ which is related to coarse-grained gradients of phases φ_σ. The difficulty in computing the gauge-invariant electron propagator $\mathcal{G}(x-x')$ stems from
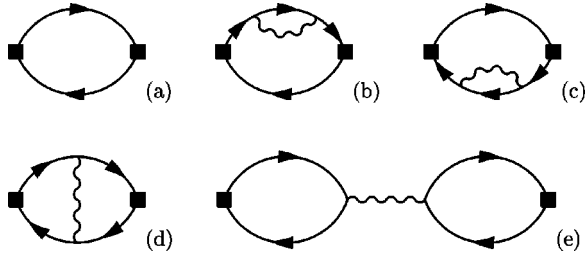
FIG. 1. Feynman diagrams for the susceptibility [Eq. (5)] to leading order in $1/N$. Solid lines represent the bare fermion propagator, wavy lines represent the screened gauge-field propagator, and the solid squares stand for vertex Γ_l.

the necessity to evaluate the averages of such phase factors. The latter are dominated by short length-scale physics that is not properly described by the effective theory [Eq. (1)].

Here we focus on the charge and spin density correlators which do not suffer from the above problem. Consider, in particular, the spin susceptibility,

$$\chi^s(\mathbf{q}, i\omega) = \int_0^\beta d\tau e^{i\omega\tau} \langle T_\tau S_{\mathbf{q}}^z(\tau) S_{-\mathbf{q}}^z(0)\rangle, \qquad (3)$$

with $\beta = 1/T$ and $S_{\mathbf{q}}^z = \int d^2r e^{i\mathbf{q}\cdot\mathbf{r}} \Sigma_\sigma \sigma c_\sigma^\dagger(\mathbf{r}) c_\sigma(\mathbf{r})$ the z-component of the electron spin operator at wave vector $\mathbf{q}$. The charge susceptibility χ^c is defined as in Eq. (3) but with $S_{\mathbf{q}}^z$ replaced by the charge density operator $\rho_{\mathbf{q}} = \int d^2r e^{i\mathbf{q}\cdot\mathbf{r}} \Sigma_\sigma c_\sigma^\dagger(\mathbf{r}) c_\sigma(\mathbf{r})$. Both spin and charge densities are gauge singlets with respect to a_μ, i.e., they do not pick up any phase factors under the transformation (2). Following Ref. 9 we adopt the representation for the gamma matrices given by $\gamma_\mu = \sigma_3 \otimes (\sigma_2, \sigma_1, -\sigma_3)$ with σ_i the Pauli matrices. We further define $\gamma_3 = \sigma_1 \otimes \mathbb{1}$ and $\gamma_5 = \sigma_2 \otimes \mathbb{1}$ and note that these anticommute with all γ_μ's. With these definitions we may express the density operators as

$$\left. \begin{array}{c} S^z(q) \\ \rho(q) \end{array} \right\} = \int_p \bar{\Psi}_l(p) \Gamma_l \Psi_l(q-p), \qquad (4)$$

where vertex Γ_l takes the form $\Gamma_l^s = (i\gamma_0, i\gamma_0)$ and $\Gamma_l^c = (-i\gamma_0\gamma_2, i\gamma_0\gamma_1)$ for spin and charge densities with momentum transfer close to $\mathbf{q} = (0,0)$, respectively, and $\Gamma_l^s = (\gamma_5, 0)$, $\Gamma_l^c = (-\gamma_0\gamma_5, 0)$ for momentum transfer near $\mathbf{q} = \mathbf{Q}_1 \approx (\pi, \pi)$. We have switched to Euclidean notation with 3-momenta $q = (q_0, \mathbf{q})$, $\int_p$ denotes $\int d^3p/(2\pi)^3$ and a summation over $l = 1,2$ is assumed. The susceptibilities are then given as

$$\chi(q) = \int_p \int_{p'} \langle \bar{\Psi}_l(p) \Gamma_l \Psi_l(p+q) \bar{\Psi}_l(p') \Gamma_l \Psi_l(p'-q)\rangle. \qquad (5)$$

We evaluate $\chi(q)$ at $T=0$ by formally considering $N/2$ identical copies of fields Ψ_1 and Ψ_2. Then to leading order in $1/N$ expansion[11] we require four diagrams depicted in Figs. 1(a)–(d). The wavy line represents the gauge field propagator that becomes universal at long wavelengths due to the screening by topological fermions,[6]

$$D_{\mu\nu}(q) = \frac{8}{qN}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}(1-\xi)\right), \qquad (6)$$

in the sense that K_μ drops out and only reappears as an upper cutoff of the theory, $\Lambda \sim 1/K$.

The bare bubble [Fig. 1(a)] reads

$$\chi_0(q) = \int_p \text{Tr}[G_0(p)\Gamma_l G_0(p-q)\Gamma_l], \qquad (7)$$

where $G_0(p) = \not{p}/p^2 \equiv p_\mu \gamma_\mu / p^2$ is the free Dirac propagator, and can be evaluated by standard methods (see e.g. Appendix B of Ref. 9). One obtains

$$\chi_0(q) = -\text{Tr}[\gamma_\mu \Gamma_l \gamma_\nu \Gamma_l] \frac{q}{64}\left(\delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right). \qquad (8)$$

We observe that the structure of the bare bubble susceptibility is determined entirely by the commutation relation of the vertex with the γ matrices. We shall see that this property remains true for the higher order diagrams as well.

Diagrams (b), (c), and (d) represent the leading $1/N$ corrections due to the gauge field and contain all the interesting physics. We note that their sum is *gauge invariant*. This follows since they represent correlators of gauge invariant density operators. One can explicitly verify that this is so by employing the Ward identity $G_0^{-1}(p+k) - G_0^{-1}(k) = p_\mu \gamma_\mu$. We are thus free to fix the gauge and in what follows we adopt the Feynman gauge ($\xi = 1$) in which $D_{\mu\nu}(q) = (8/N)\delta_{\mu\nu}/q$. Diagrams (b) and (c) contain the self-energy insertion and read

$$\chi_{(b+c)}(q) = 2\int_p \text{Tr}[G_0(p)\Gamma_l G_0(p-q)\Gamma_l G_0(p)\Sigma(p)], \qquad (9)$$

with $\Sigma(p) = \int_k D_{\mu\nu}(k)\gamma_\mu G_0(p-k)\gamma_\nu = -\eta_F \not{p} \ln(\Lambda/p)$, and

$$\eta_F = \frac{4}{3\pi^2 N} \qquad (10)$$

the fermion anomalous dimension in the Feynman gauge.[9] Inserting $\Sigma(p)$ into Eq. (9) and combining with Eq. (8), one can express the divergent contribution as

$$\chi_{(b+c)}(q) \simeq 2\eta_F \chi_0(q) \ln(\Lambda/q). \qquad (11)$$

Diagram (d) contains the vertex correction and reads

$$\chi_{(d)}(q) = \int_k \int_p \text{Tr}([G_0(p-q)\Gamma_l G_0(p)]\gamma_\mu[G_0(p-k) \\ \times \Gamma_l G_0(p-k-q)]\gamma_\nu)D_{\mu\nu}(k). \qquad (12)$$

As mentioned above the direct evaluation of this type of integral presents a significant challenge.[13,14] We are interested in the leading $q\to 0$ behavior of $\chi(q)$. In this limit the p integral has singularities as $p\to 0$ and $p\to k$, corresponding to divergences in the first and second terms in the square brackets, respectively. The main contribution to the integral therefore comes from the vicinity of these two points, and we

may evaluate it by expanding the regular part of the integrand at the singular point. Retaining only the leading term thus gives

$$\chi_{(d)}(q) \simeq \int_k \int_p \text{Tr}([G_0(p-q)\Gamma_l G_0(p)]$$

$$\times \gamma_\mu [G_0(-k)\Gamma_l G_0(-k-q)]\gamma_\nu) D_{\mu\nu}(k)$$

$$+ \int_k \int_p \text{Tr}([G_0(k-q)\Gamma_l G_0(k)]\gamma_\mu$$

$$\times [G_0(p-k)\Gamma_l G_0(p-k-q)]\gamma_\nu) D_{\mu\nu}(k).$$
(13)

Performing a variable shift $p \to p+k$ in the second term we may simplify the above expression as

$$\chi_{(d)}(q) = 2\text{Tr}\left(\Omega_l(q) \int_p [G_0(p-q)\Gamma_l G_0(p)] \right), \quad (14)$$

with $\Omega_l(q) = \int_k \gamma_\mu [G_0(k)\Gamma_l G_0(k-q)]\gamma_\nu D_{\mu\nu}(k)$. The last integral is again easy to evaluate and gives

$$\Omega_l(q) = [\gamma_\mu \gamma_\nu \Gamma_l \gamma_\nu \gamma_\mu] \eta_F \ln(\Lambda/q). \quad (15)$$

We now notice that for vertices composed of products of gamma matrices (such as those entering the spin and charge densities defined above) it holds that

$$\gamma_\mu \gamma_\nu \Gamma_l \gamma_\nu \gamma_\mu = \lambda \Gamma_l, \quad (16)$$

where λ is a number. In particular we shall encounter two types of vertices. Type-I vertices commute or anticommute with all γ_μ's (e.g., $\Gamma_l = \mathbb{1}, \gamma_3, \gamma_5, \gamma_3\gamma_5$) and in this case $\lambda = 9$. Type-II vertices anticommute with one or two of γ_μ's and commute with the rest (e.g., $\Gamma_l = \gamma_0, \gamma_0\gamma_1, \dots$). In this case $\lambda = 1$. With this insight Eq. (14) becomes

$$\chi_{(d)}(q) = 2\eta_F \lambda \ln(\Lambda/q) \int_p \text{Tr}[G_0(p)\Gamma_l G_0(p-q)\Gamma_l].$$

Combining this result with Eqs. (11) and (7) we can write the result for the full susceptibility to $1/N$ order:

$$\chi(q) = \chi_0(q)[1 - 2\eta_F(1-\lambda)\ln(\Lambda/q)]. \quad (17)$$

This correction may be interpreted as the leading term of a power law,[11] so that we have

$$\chi(q) \sim \chi_0(q)\left(\frac{\Lambda}{q}\right)^{\eta_4}, \quad \eta_4 = \frac{8(\lambda-1)}{3\pi^2 N}. \quad (18)$$

The anomalous dimension exponent η_4 is entirely determined by the algebraic properties of the vertex Γ_l through Eq. (16). In particular for a type-I vertex ($\lambda = 9$),

$$\eta_4 = \frac{64}{3\pi^2 N}, \quad (19)$$

while for a type-II vertex, $\eta_4 = 0$.

Returning back to physics we see that the vertex for staggered spin susceptibility is type I and will therefore exhibit

nontrivial anomalous dimension exponent [Eq. (19)], in agreement with the result of Ref. 14 obtained by laborious direct evaluation of the vertex correction. The uniform spin susceptibility has a type-II vertex and therefore does not acquire anomalous dimension from diagrams (b)-(d), again in agreement with Ref. 14. In addition both charge susceptibilities are type II and will not exhibit anomalous dimension.

Finally we note that if Γ_l coincides with one of the γ_μ's then diagrams of the type shown in Fig. 1(e) are nonvanishing and must be included in the leading order. One can show that the complete resummation of diagrams (e) modifies the $\chi_0(q) \sim q$ behavior in Eq. (8) to q^2. This can be viewed as another type of anomalous dimension due to the coupling to the gauge field, and we shall discuss it more fully elsewhere.

The formalism we have developed allows us to study spin and charge conductivities in the AFL. These can be calculated through the Kubo formula as $\sigma_{ij}(\omega) = -\text{Im}\Pi_{ij}^{\text{ret}}(\omega)/\omega$, where $\Pi_{ij}^{\text{ret}}(\omega) = \Pi_{ij}(i\omega \to \omega + i\delta)$ with

$$\Pi_{ij}(i\omega) = -\int_0^\beta d\tau e^{i\omega\tau}\langle T_\tau j_i(\tau)j_j(0)\rangle \quad (20)$$

the current-current correlation function. Indices $i,j = 1,2$ label the spatial components of the 3-current j_μ. The spin current is given as $j_\mu^s(x) = \bar{\Psi}_l(x)\gamma_\mu\Psi_l(x)$, while the electric current is given as $j_\mu^e(x) = (\bar{\Psi}_1\gamma_0\gamma_2\Psi_1 - \bar{\Psi}_2\gamma_0\gamma_1\Psi_2, -\bar{\Psi}_1\gamma_1\gamma_2\Psi_1, -\Psi_2\gamma_1\gamma_2\Psi_2)$. To leading $1/N$ order the computation of correlator (20) involves the same diagrams depicted in Fig. 1 and we can therefore simply adopt the result obtained for the susceptibility in Eq. (18). We see that the vertices involved in j_μ^s and j_μ^e are all type II, and therefore neither spin nor charge conductivity will exhibit anomalous dimension $\sim 1/N$ arising from correlations.

We may conclude that of all quantities considered only the staggered spin susceptibility will bear the unique signature of the AFL in that its frequency, momentum and temperature dependence will be controlled by the anomalous dimension exponent η_4 given in Eq. (19). The other quantities will behave essentially as if the theory was uncorrelated.

The fact that some quantities remain essentially unaffected by strong long range correlations can be traced to a field theoretic theorem[15,16] which states that the correlator of conserved currents (i.e., currents satisfying $\partial_\mu j_\mu = 0$) must exhibit a scaling dimension which agrees with its engineering dimension. Physically this means that some quantities are constrained by their conservation laws to such a degree that correlations cannot alter their long-distance scaling behavior. As an example consider the spin current j_μ^s. It is a conserved current $\partial_\mu j_\mu^s = 0$ guaranteed by the gauge invariance of the theory [i.e. invariance under local symmetry $\Psi_l(x) \to e^{i\theta(x)}\Psi_l(x)$]. As we have verified by explicit calculation neither spin conductivity nor uniform spin susceptibility exhibit anomalous dimension ($\sim 1/N$) beyond the $\chi_0 \sim q^2$ behavior mentioned above, in accord with the theorem. Understanding the behavior of the charge conductivity is less straightforward, as we do not expect the quasiparticle current to be conserved in a (phase-disordered) superconductor. In-

deed there is no symmetry that would guarantee j_μ^e conservation. However, the theory is known to possess a ''chiral'' symmetry $(\Psi_l \to e^{i\gamma_3\gamma_5\phi_l}\Psi_l)$[18] which produces two conserved currents: $j_\mu^{(l)} = \bar{\Psi}_l\gamma_\mu(i\gamma_3\gamma_5)\Psi_l$ (no sum over l). By virtue of the identity $\gamma_0\gamma_3\gamma_5 = -\gamma_1\gamma_2$ these are related to the spatial components of j_μ^e, explaining the lack of anomalous dimension in the electrical conductivity.

In the 4×4 representation of Dirac gamma matrices there are four type-I vertices listed below Eq. (16). Thus there should be three other physical observables which exhibit anomalous scaling dimensions. What are these? In turns out that γ_3 corresponds to the staggered spin susceptibility at the wave vector $\mathbf{Q}_{\bar{1}} = -\mathbf{Q}_1$, while $\mathbb{1}$ and $\gamma_3\gamma_5$ correspond to susceptibilities to the formation of subdominant (phase-incoherent) SC order in s and p channels, respectively. These quantities are members of the QED_3 chiral manifold,[17,18] which is the manifold of broken-symmetry states occurring for $N < N_c \approx 32/\pi^2 = 3.24$ at zero temperature. This phenomenon is known as spontaneous chiral symmetry breaking.[19,20] The anomalous dimension exponent η_4 that we find in the above susceptibilities appears to anticipate the transition into the chiral symmetry broken phase. Indeed if we combine $\chi_0(q)$ from Eq. (8) with Eq. (18) we find $\chi(q) \sim q^{1-\eta_4}$, implying divergent $q \to 0$ susceptibility for $N < 64/3\pi^2 = \frac{2}{3}N_c$.

In the context of the QED_3 theory of cuprates[6] we expect the experimental manifestations of the nontrivial anomalous dimension to appear in the quantities inhabiting the QED_3 chiral manifold, most prominently the staggered spin susceptibility that is directly measurable by neutron scattering. In addition, a similar nontrivial response should obtain in the charge channel near $(0,\pi)$ as a consequence of enlarged chiral manifold discussed in Ref. 18. This particular response is unique to the present theory, and will not be present in the SU(2) theory of Ref. 14.

We conclude by observing that the physics of the algebraic Fermi liquid bears a distinct similarity to that of one-dimensional Luttinger liquids. While the single particle properties are distinctly non-Fermi-liquid like, only certain measurable physical responses exhibit the unique fingerprint of the AFL in the form of anomalous scaling dimension. The method developed here allows us to easily identify these quantities, and helps in our search for situations where other quantities might develop anomalous behavior.

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