Weak localization and integer quantum Hall effect in a periodic potential

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We consider magnetotransport in a disordered two-dimensional electron gas in the presence of a periodic modulation in one direction. Existing quasiclassical and quantum approaches to this problem account for Weiss oscillations in the resistivity tensor at moderate magnetic fields, as well as a strong modulation-induced modification of the Shubnikov–de Haas oscillations at higher magnetic fields. They do not account, however, for the operation at even higher magnetic fields of the integer quantum Hall effect, for which quantum interference processes are responsible. We introduce a field-theory approach, based on a nonlinear σ model, which encompasses naturally both the quasiclassical and quantum-mechanical approaches, as well as providing a consistent means of extending them to include quantum interference corrections. A perturbative renormalization-group analysis of the field-theory shows how weak localization corrections to the conductivity tensor may be described by a modification of the usual one-parameter scaling, such as to accommodate the anisotropy of the bare conductivity tensor. We also show how the two-parameter scaling, conjectured as a model for the quantum Hall effect in unmodulated systems, may be generalized similarly for the modulated system. Within this model we illustrate the operation of the quantum Hall effect in modulated systems for parameters that are realistic for current experiments.

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I. INTRODUCTION

The problem of electron motion in a disordered conductor in a periodic potential and strong magnetic field displays a rich combination of phenomena of both classical and quantum origin. The complexity arises from the appearance of two independent types of periodicities, of the potential and of the cyclotron orbits, which may interplay in complicated ways. Perhaps the most striking effect is known as Weiss oscillations,[1] whereby very large oscillations in the resistivity are induced by even a weak periodic potential at moderate magnetic fields. At different magnetic fields or temperatures, other types of contributions to the resistivity become significant. For example, at very small fields, a positive magnetoresistance results from a classical mechanism of channeling of orbits.[2–4] At larger magnetic fields, the resistivity develops Shubnikov–de Haas oscillations, which originate from the onset of Landau quantization and may still be affected by the periodic potential.[1,5–8] At even higher fields, the integer quantum Hall effect[9] (IQHE) becomes operative due to the contribution of quantum interference processes.

Much of the above phenomenology has been thoroughly analyzed theoretically, by a variety of techniques including quantum-mechanical approaches[5,6] involving diagrammatics[7,10,11] or solution of a quantum Boltzmann equation[8], and quasiclassical approaches.[2–4,12,13] The experimental realization of these systems is also well advanced[1,2,5,6,14,15] with precise confirmation of the theoretical predictions already possible. So far, however, a theory for the influence of quantum interference processes in a periodic potential and high magnetic field has been missing. Such processes lead to weak localization corrections to the conductivity which become significant at low temperatures (see, e.g., Refs. 16–19 for reviews). At high magnetic fields, they are also responsible for the operation of the IQHE,[9,20] whereby the Hall conductivity becomes quantized at low temperatures. At the same time, the observation of the IQHE in systems with a weak potential modulation is well within current experimental capability.[14,21]

This paper aims to fill this gap by showing how the standard theory for weak localization and the IQHE may be generalized so as to incorporate the presence of a periodic potential. We employ a field-theory approach based on a nonlinear σ model,[20,22–25] which is well established in the study of mesoscopic disordered conductors. In deriving the appropriate field theory, we are able to show how previous theoretical calculations of the conductivity, within both quasiclassical[12,4] and quantum-mechanical approaches,[7] may be recovered in a natural way within the field-theory formalism. This enables us to extend previous theoretical results in a consistent way so as to include the influence of quantum interference processes.

The experiments of Weiss *et al.*[1] employed weakly modulated two-dimensional (2D) electron systems of high mobility with a well-known period, $a\sim300$ nm, much less than the mean free path, $\ell\sim10$ μm. Such samples were engineered using a holographic modulation technique, based on the persistent photoconductivity effect in GaAs/Al_xGa_{1-x}As heterostructures at low temperatures ~4.2 K. The Weiss oscillations appear in only one component of the resistivity tensor, ρ_{xx}, when the modulation is in the x direction. Furthermore, they appear at magnetic fields B such that the cyclotron radius $R_c=v_F/\omega_c$ (where $\omega_c=eB/mc$ is the cyclotron frequency, $-e$ is the electron charge, and v_F is the Fermi velocity) satisfies the commensurability condition $2R_c=(n-1/4)a$, for integer n. Hence the oscillations are periodic in $1/B$. In addition they are relatively stable with respect to temperature, suggesting a quasiclassical origin.

At even higher magnetic fields, such that the cyclotron radius is much less than the period of modulation ($R_c \ll a$), the quasiclassical theory predicts that the ρ_{xx} component shows a large, nonoscillatory increase proportional to B^2, leading to a strong positive magnetoresistance. This result has been confirmed in experiments[14] for which the temperature was kept deliberately high so as to avoid the intervention of the IQHE.

A limitation of the quasiclassical approach is that it fails to account for the renormalization of single-particle properties, such as the density of states and scattering lifetime, by the strong magnetic field according to Landau quantization. Even in unmodulated samples, quantization leads to oscillations in the density of states with respect to magnetic fields, and hence to Shubnikov–de Haas oscillations in the resistivity. In samples with the periodic potential, the Shubnikov–de Haas oscillations start to appear at higher magnetic fields than the Weiss oscillations (for R_c an integer multiple of the Fermi wavelength $\ll a$). A quantum-mechanical approach that does allow for such quantization effects has been provided by Zhang and Gerhardts[7] (see also Peeters and Vasilopoulos[8]); it is a diagrammatic treatment that generalizes the approach of Ando and co-workers[26] to modulated samples. The quantum-mechanical approach is then capable of describing both Weiss oscillations and the Shubnikov–de Haas oscillations (also affected by the modulation) at higher magnetic fields.

In principle, the calculation of weak localization corrections in a strong magnetic field (unitary ensemble) is possible even in the presence of a periodic potential, by a generalization of the diagrammatic approach of Zhang and Gerhardts,[7] but the procedure would be complicated (although the simpler orthogonal case has been examined diagrammatically for a periodic magnetic field[27,28]). Instead, the calculation of high-order diagrams is more convenient in the field-theory formalism and, furthermore, with this method the possibility exists of calculating contributions of diagrams to all orders by the renormalization-group (RG) technique. We show how the field-theory takes the form of a nonlinear σ model with a topological term.[20,22–25] The effective Lagrangian is slightly nonstandard since it contains an anisotropy in the coefficients, corresponding to the difference between the longitudinal conductivities in the x and y directions, due to the periodic potential.

The effect of weak localization corrections to the conductivity for unmodulated samples is accounted for by a scaling (one-parameter scaling) of the conductivity with the system size. As a first step we derive the analog of the one-parameter scaling, for the conductivity[22–24,29] in the modulated system by means of a perturbative RG analysis of the effective Lagrangian. We then turn to the study of the IQHE, implementing a generalization of a two-parameter scaling, which has been conjectured[25,30,31] as a model for the IQHE in unmodulated systems. We examine how the resistivity tensor should be affected at low temperatures by the IQHE in modulated samples for parameters that are realisable in actual experiments. We see, for example, how the Hall conductivity becomes quantized under scaling at low temperatures, while, in the regions between the plateaus, the longitudinal conductivities develop peaks of differing heights according to the anisotropy in the x and y directions.

Model. In the following, we employ the Hamiltonian for the disordered conductor in a magnetic field and periodic potential in two dimensions:

$$H = H_0 - V(\mathbf{r}),$$

$$H_0 = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 + U_0 \cos(qx). \tag{1}$$

Here $\mathbf{A}$ is the vector potential, so that $\nabla \times \mathbf{A} = (0,0,B)$, where B is the perpendicular, uniform magnetic field. Also $U_0 \cos(qx)/(-e)$ is the periodic potential, which is taken to be weak ($U_0 \ll \epsilon_F$, where ϵ_F is the Fermi energy) and with modulation period $a = 2\pi/q$. V is the disorder potential, which we assume to be δ correlated in space, with the associated scattering time τ. We assume that the mean free path $\ell = v_F \tau$ greatly exceeds the modulation period a.

The plan for the remainder of the paper is as follows. In Sec. II we describe the field-theory approach and derive the effective Lagrangian for the system. In Sec. III we show how the field-theory provides the scaling of the conductivity tensor under changes of length scale due to the contribution of quantum interference processes, and hence a description of the IQHE in these samples. Section IV concludes with a summary and discussion.

II. FIELD-THEORY APPROACH

In this section we describe a field-theoretical approach to the description of a disordered conductor in the presence of a periodic potential and a strong magnetic field. The approach is based on a diffusive nonlinear σ model, which will serve as a tool for the analysis of quantum interference effects in the remainder of this paper. In Sec. II A we discuss this model and the relation to previous semiclassical[12] and quantum-mechanical[7] approaches. Technical details of the derivation are presented in Sec. II B.

A. Description of field-theory approach

The field-theory apparatus that we employ is by now well established in the study of spectral and wave-function statistics of disordered conductors. It is based on a functional integral expression of electron Green functions in the presence of disorder, and in the diffusive regime takes the form of a nonlinear σ model in terms of a Q matrix.[22–24] In two dimensions, the field-theory technique provides a convenient means of calculating weak localization corrections to the conductivity, a task that may become very cumbersome by the more conventional technique of diagrammatics.[32,29] The field-theory may also provide a resummation of diagrams to all orders by a renormalization-group procedure.[22–24,29]

We show below that the effective Lagrangian for the disordered conductor in the presence of a periodic potential and strong magnetic field takes the form

$$\mathcal{L}[Q]=\frac{1}{16}\int dr\, \mathrm{Str}\{\sigma_{xx}^0(\nabla_x Q)^2+\sigma_{yy}^0(\nabla_y Q)^2 -\sigma_{xy}^0\tau_3 Q[\nabla_x Q,\nabla_y Q]\}, \qquad (2)$$

where σ_{ij}^0 is the classical conductivity tensor of the system, in units of e^2/h. The 8×8 supermatrix field $Q(\mathbf{r})$ satisfies the nonlinear constraint $Q(\mathbf{r})^2=1$. The supertrace operation is defined in Ref. 24.

In the absence of a strong magnetic field or modulation, a renormalization-group analysis of the first two terms in Eq. (2) leads to the well-known one-parameter scaling[22–24,29] of the conductivity with system size due to weak localization.

The final term in Eq. (2) is known as a topological term and appears in the theory of the IQHE proposed by Pruisken and collaborators.[20,25] The influence of the extra term does not appear within perturbation theory, but becomes evident only through a nonperturbative analysis. Such an analysis has been conjectured[25,30,31] in the form of a two-parameter renormalization-group procedure. The two parameters are the longitudinal and Hall conductivities that follow coupled scaling equations with respect to changes of length scale.

While the validity of the two-parameter scaling has been vigorously debated (see, e.g., Refs. 33,34), it has remained a valuable guide to experimental[35] and numerical[36] data for a number of years. In the following, we take a pragmatic approach by not contesting the validity of the two-parameter scaling and its relation to the proposed field-theory. Instead, we assume its validity and explore how it may be generalized to take account of the periodic potential.

The main difference of the Lagrangian (2) from the unmodulated case is the anisotropy in the diffusion coefficients for the x and y directions. The Lagrangian, therefore, contains three, rather than two, parameters, which scale together under changes of length scale. A simple scale transformation, however, maps the Lagrangian to an isotropic version for which the usual two-parameter scaling may be applied.

This Lagrangian applies in the diffusive limit, that is, to configurations of the $Q(\mathbf{r})$ field that vary on scales much longer than the mean free path. This allows the calculation of the contribution of low-momentum relaxational modes to weak localization corrections to the conductivity. Although momentum relaxation is diffusive on large length scales, electron motion on the scale of the periodic potential is ballistic, since the modulation period is much less than the mean free path. To arrive at the above Lagrangian, it is therefore necessary to integrate over degrees of freedom corresponding to length much smaller than the mean free path.

One way to do so is to start from a description of ballistic electron motion on length scales much less than the mean free path. Such a description is provided by the "ballistic σ model"[37,38] which, as the name suggests, generalizes the diffusive σ model to the ballistic regime. Starting from this model, we show in Sec. II B how the contribution from short-length scales may be integrated out. The result is the Lagrangian of a diffusive σ model that describes the interaction of diffusion modes on scales longer than the mean free path.

When derived in this way, the effective Lagrangian takes the form of Eq. (2), but with the σ_{ij}^0 coefficients replaced by $\sigma_{ij}^{\mathrm{qc}}(h/e^2)$. Here σ_{ij}^{qc} correspond precisely to the components of the quasiclassical conductivity tensor as derived from the Boltzmann equation for the modulated system.[12] One finds that, in the quasiclassical approach, only the xx component of the quasiclassical resistivity tensor is affected by the modulation, all other components being the same as in the unmodulated case, so that $\rho_{xx}^{\mathrm{qc}}=\rho_0=(2e^2\nu D)^{-1}$ and $\rho_{xy}^{\mathrm{qc}}=-\rho_{yx}^{\mathrm{qc}}=\omega_c\tau\rho_0$ (here $D=v_F^2\tau/d$ is the diffusion coefficient in dimension d). For details of the solution of the Boltzmann equation, we refer the reader to Beenakker.[12] Here we only remark that for moderate magnetic fields, weak enough that the cyclotron radius is much larger than the modulation period ($R_c\gg a$), but strong enough that $\omega_c\tau \gg 1$, the quasiclassical result simplifies to

$$\frac{\rho_{xx}^{\mathrm{qc}}}{\rho_0}=1+\frac{1}{2\pi}\left(\frac{U_0}{\epsilon_F}\right)^2(\omega_c\tau)^2 R_c q\cos^2\left(R_c q-\frac{\pi}{4}\right)\quad (R_c q\gg 1). \qquad (3)$$

Equation (3) demonstrates the oscillatory dependence of the resistivity known as Weiss oscillations. For strong magnetic fields, such that $R_c\ll a$, a very large oscillatory increase in ρ_{xx} proportional to B^2 is predicted and has been observed experimentally, see, e.g., Geim *et al.*[14]

However, for high magnetic fields ($\omega_c\tau\gg 1$), the quasiclassical results are reliable only at sufficiently high temperatures ($k_BT\gg\hbar/\tau$), while at lower temperatures the effects of quantization on single-particle properties such as the density of states must be taken into account by a quantum-mechanical approach. These effects are not taken into account in the derivation via the ballistic σ model, since here the magnetic field is treated only as a weak perturbation. Indeed, once temperatures are low enough for weak localization corrections to be significant ($k_BT\sim\hbar/\tau$), then quantization is already well established (as $k_BT\ll\hbar\omega_c$, since $\omega_c\tau \gg 1$). Hence this method of deriving the Lagrangian is unfortunately of no use in describing the effects of quantum interference processes at high magnetic fields.

An improvement on this situation may be found by proceeding instead along a second route to derive the effective Lagrangian, following more closely the original lines of Pruisken.[20] He showed how to derive a σ model in the presence of a strong magnetic field, including the effects of quantization. The Lagrangian is applicable at high values of the Landau-level index n, such that the fluctuations of the density of states may be neglected. We show below in Sec. II B how this method may be adapted to describe a disordered conductor in a strong magnetic field and a periodic potential. This route bypasses the intermediate step of a ballistic σ model, instead computing more directly the final diffusive Lagrangian.

While the resulting Lagrangian takes again the form displayed in Eq. (2), now the σ_{ij}^0 coefficients correspond precisely to $\sigma_{ij}^{\mathrm{qu}}h/e^2$, where $\sigma_{ij}^{\mathrm{qu}}$ are the components of the conductivity tensor calculated in the fully quantum-mechanical approach of Ref. 7. Since the effects of quantization are included, this approach correctly describes certain features that are observed in experiment but are beyond the quasiclassical

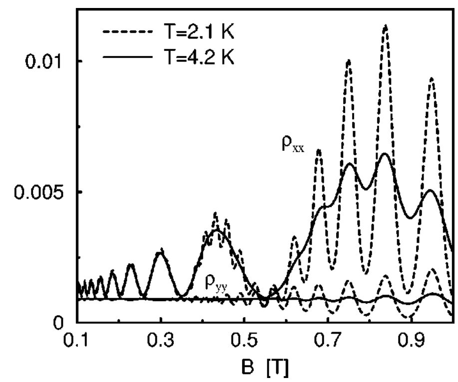

FIG. 1. Resistivity components as a function of magnetic field B for two different temperatures using the approximation scheme of Peeters and Vasilopoulos (Ref. 8). Corrections to the Hall resistivity are small in this regime (Refs. 7,8). Parameters are $a=200$ nm, $U_0=0.3$ meV, $n_{el}=3.4\times10^{11}$ cm^{-2}, $\hbar/\tau=0.011$ meV.

approach. Some typical results for the resisitivity components (ρ_{ij}^{qu}) calculated within the quantum-mechanical approach are shown in Fig. 1.

As well as the appearance of Weiss oscillations in ρ_{xx}^{qu} at moderate magnetic fields (up to $\sim 0.4T$ in Fig. 1), we see that both ρ_{xx}^{qu} and ρ_{yy}^{qu} show strong, in-phase Shubnikov–de Haas oscillations at higher magnetic fields. These in-phase oscillations are essentially a consequence of the Shubnikov–de Haas oscillations of the density of states. At weaker fields, for which the Weiss oscillations in ρ_{xx}^{qu} are visible, weaker, out-of-phase oscillations in ρ_{yy}^{qu} also appear. Again, the latter oscillations in ρ_{yy}^{qu} are beyond the quasiclassical approach.

The quasiclassical result (3) for the change in ρ_{xx}^{qc} (for $\omega_c\tau\gg1$ and $R_cq\gg1$) may be reproduced by the quantum-mechanical approach, as long as the temperature is high enough for the thermal broadening to be greater than the separation of the Landau levels, $k_BT\gg\hbar\omega_c$ (see, e.g., Refs. 7,8). As the temperature is lowered, however, the region of applicability of the quasiclassical formula (3) shrinks to progressively lower magnetic fields: The sinusoidal Weiss oscillations start to interfere at low temperatures with the Shubnikov–de Haas oscillations. Indeed, at vanishingly low temperatures, the quasiclassical result becomes unjustified for all magnetic fields.

The Lagrangian (2), supplied with the values of the conductivities determined by the quantum-mechanical approach, may then be used for a reliable description of quantum interference processes at high magnetic fields, and hence the operation of the IQHE.

B. Derivation of the Field Theory

In this section we describe in more detail the derivation of the effective Lagrangian (2). In the field-theory approach, it is necessary to average the functional integral expression for products of Green functions over the disorder configurations. To perform the disorder average, several possible techniques exist, including the use of either supersymmetry or the replica trick. Although we may use either of these two approaches, we choose here the supersymmetry approach, since it is free of certain technical problems regarding nonperturbative calculations that appear for the replica approach. The starting point for both routes is a functional integral representation of the partition function, Z, in terms of "superfields" $\Psi,\bar{\Psi}$:

$$Z=\int D[V]P[V]D[\Psi,\bar{\Psi}]\exp-\mathcal{L}(\Psi,\bar{\Psi},V), \qquad (4)$$

$$\mathcal{L}=i\int dr\bar{\Psi}(\mathbf{r})(E-H-i\delta\Lambda)\Psi(\mathbf{r}). \qquad (5)$$

Here the superfields $\Psi,\bar{\Psi}$ contain eight components, corresponding to fermion/boson, retarded/advanced and time-reversal sectors.[24] Also $\Lambda=\mathrm{diag}(1,-1)$ in advanced-retarded space space, δ is a positive infinitesimal and H is the Hamiltonian (1) up to the substitution $\mathbf{A}\rightarrow\tau_3\mathbf{A}$, where $\tau_3=\mathrm{diag}(1,-1)$ in time-reversal space. The short-range random potential $V(\mathbf{r})$ is taken to be Gaussian distributed, according to

$$P[V(\mathbf{r})]=\frac{1}{\mathcal{Z}}\exp\left\{-\frac{\pi\nu\tau}{\hbar}\int drV(\mathbf{r})^2\right\},$$

where $\mathcal{Z}$ provides the normalization.

Ballistic σ model. The first route is to derive a ballistic σ model from the Lagrangian in Eq. (5). Andreev *et al.*[38] have suggested a derivation of the ballistic σ model by means of an energy averaging procedure, leading to a Lagrangian of the same form as that originally proposed by Muzykantskii and Khmelnitskii.[37] Following the lines of Andreev *et al.*,[38] one treats the disorder field V as a perturbation in Eq. (5). Performing an energy averaging on the clean Hamiltonian leads to a term in the Lagrangian that is nonlocal and quartic in the Ψ, $\bar{\Psi}$ fields. The quartic term is then decoupled by a Hubbard-Stratonovich field, $Q(\mathbf{r}_1,\mathbf{r}_2)$, and the Ψ, $\bar{\Psi}$ fields are integrated out.

The resulting Lagrangian is then simplified by subjecting it to a saddle-point analysis, and performing a gradient expansion around the saddle point. Note that, since at this stage we are neglecting the effects of quantization on single-particle properties, we do not include the renormalization of this saddle-point solution by the magnetic field, instead we treat the magnetic field as a perturbation.

Following a Wigner transform, and after applying a semiclassical[39] approximation, the Q matrix becomes a function of position, $\mathbf{r}$, and the direction of momentum, $\mathbf{n}$, where $\mathbf{n}$ is a unit vector. We do not repeat these steps here but refer to Refs. 37,38 for further details. The resulting Lagrangian (ballistic σ model) corresponding to the Hamiltonian (1) is

$$\mathcal{L}[Q_{\mathbf{n}}]=\frac{\pi\nu\hbar}{4}\int dr\,\mathrm{Str}\left\{-\frac{1}{2\tau}\int\frac{dn}{2\pi}\frac{dn'}{2\pi}Q_{\mathbf{n}}(\mathbf{r})Q_{\mathbf{n}'}(\mathbf{r})\right.$$
$$\left.-2\int\frac{dn}{2\pi}\Lambda\tau_3T_{\mathbf{n}}^{-1}\mathcal{L}^0(\mathbf{n},\mathbf{r})T_{\mathbf{n}}\right\}. \qquad (6)$$

In this equation $\mathcal{L}^0$ is the Louiville operator defined by

$$\mathcal{L}^0 = v(\mathbf{r})\mathbf{n}\cdot\boldsymbol{\nabla} + \omega_c\partial_\phi - \sin\phi v'(x)\partial_\phi, \tag{7}$$

where ϕ is the polar angle of $\mathbf{n}$, ν is the density of states per spin direction at the Fermi level, and

$$v(\mathbf{r}) = v_F\left(1 - \frac{U_0}{E_F}\cos(qx)\right)^{1/2}$$

is the electron velocity at the Fermi surface. The supermatrix $Q_\mathbf{n}(\mathbf{r})$ satisfies the nonlinear constraint $Q_\mathbf{n}(\mathbf{r})^2 = 1$, as a result of which it may be parametrized by

$$Q_\mathbf{n}(\mathbf{r}) = T_\mathbf{n}(\mathbf{r})\Lambda T_\mathbf{n}^{-1}(\mathbf{r}).$$

Due to the presence of the magnetic field, the $Q_\mathbf{n}$ field satisfies unitary symmetry, that is, $[Q_\mathbf{n},\tau_3]=0$. The $Q_\mathbf{n}(\mathbf{r})$ field also satisfies the further standard symmetries $\overline{Q_\mathbf{n}} = Q_\mathbf{n} = KQ_\mathbf{n}^\dagger K$, where $\overline{Q_\mathbf{n}} \equiv CQ_\mathbf{n}^T C^T$. The matrices C and K, as well as supertrace operation, are defined in Ref. 24.

The Lagrangian (6) is in its current form too complicated for our purposes: it describes fluctuations of $Q_\mathbf{n}(\mathbf{r})$ on length scales smaller than the mean free path (and longer than the Fermi wavelength) and for any dependence on $\mathbf{n}$. At the same time, electron motion on length scales much longer than the mean free path is diffusive. The strategy therefore is to integrate out the modes corresponding to fluctuations on length scales smaller than the mean free path, to produce a Lagrangian that describes the interaction of diffusion modes on larger length scales.

To perform the integration, we isolate the the fluctuations of lowest mass (termed massless), corresponding to matrices $Q_\mathbf{n}(\mathbf{r})$ that are independent of $\mathbf{n}$ and vary more slowly than the mean free path ℓ. We then integrate out all other fluctuations that preserve $Q_\mathbf{n}^2 = 1$. A similar procedure has been carried out by Wölfle and Bhatt[40] to derive the diffusive Lagrangian for a disordered conductor with anisotropic masses in the x and y directions (see also Refs. 41,42 for further examples).

To perform integration over the weakly massive modes, we write

$$Q_\mathbf{n} = U(\mathbf{r})Q_\mathbf{n}^0(\mathbf{r})\overline{U}(\mathbf{r}).$$

Here the $U(\mathbf{r})$ matrix, obeying the symmetries $\overline{U}U = 1$ and $\overline{U} = KU^\dagger K$, represents the massless fluctuations, while $Q_\mathbf{n}^0(\mathbf{r})$ contains the weakly massive fluctuations. The matrix $Q_\mathbf{n}(\mathbf{r})$ may, in turn, be parametrized by its generator $P_\mathbf{n}(\mathbf{r})$, for example, by[24]

$$Q_\mathbf{n}^0 = \Lambda\left(\frac{1+iP_\mathbf{n}}{1-iP_\mathbf{n}}\right).$$

Here $P_\mathbf{n}$ is off-diagonal in retarded-advanced space and satisfies the symmetry $P_\mathbf{n} = -\overline{P_\mathbf{n}}$. Since the fluctuations represented by the $Q_\mathbf{n}$ matrix are weakly massive, it is sufficient to treat them within a Gaussian approximation: $Q_\mathbf{n} = \Lambda(1 + 2iP_n - 2P_n^2 \dots)$. Inserting into the Lagrangian (6), we find

$$\mathcal{L} = \frac{\pi\nu\hbar}{2}\int d\mathbf{r}\,\mathrm{Str}\Big\{-\int\frac{dn}{2\pi}\Lambda\tau_3P_\mathbf{n}\mathcal{L}^0(\mathbf{r},\mathbf{n})P_\mathbf{n} + \frac{1}{\tau}\int\frac{dn}{2\pi}\times\frac{dn'}{2\pi}(P_\mathbf{n}^2 - P_\mathbf{n}P_{\mathbf{n}'}) - 2i\int\frac{dn}{2\pi}\Lambda\tau_3P_\mathbf{n}v(\mathbf{r})\mathbf{n}\cdot\boldsymbol{\Phi}^\perp\Big\},$$

where $\boldsymbol{\Phi} \equiv \overline{U}\boldsymbol{\nabla}U$, $\boldsymbol{\Phi}^\perp = 1/2[\boldsymbol{\Phi},\Lambda]\Lambda$. The next step is to integrate over matrices $P_\mathbf{n}$. This amounts to a set of Gaussian integrals with a linear term in $P_\mathbf{n}$, and hence may be performed by means of a shift of $P_\mathbf{n}$.

The resulting Lagrangian has the form

$$\mathcal{L}[Q] = \frac{h}{16e^2}\int d\mathbf{r}\;\mathrm{Str}\{\sigma_{xx}^{\mathrm{qc}}(\boldsymbol{\nabla}_xQ)^2 + \sigma_{yy}^{\mathrm{qc}}(\boldsymbol{\nabla}_yQ)^2 - \sigma_{xy}^{\mathrm{qc}}\tau_3Q[\boldsymbol{\nabla}_xQ,\boldsymbol{\nabla}_yQ]\}. \tag{8}$$

It is written in terms of a matrix $Q(\mathbf{r}) \equiv U(\mathbf{r})\Lambda\overline{U}(\mathbf{r})$ that depends only on position and varies on length scales much longer than the mean free path (and hence the modulation period, a). The $Q(\mathbf{r})$ field also satisfies the nonlinear constraint $Q(\mathbf{r})^2 = 1$, as well as the symmetries $Q = \overline{Q} = KQ^\dagger K$. We see that the Lagrangian takes the usual form of a diffusive σ model with a topological term,[20] the only nonstandard feature being the anisotropy of $\sigma_{xx}^{\mathrm{qc}} \neq \sigma_{yy}^{\mathrm{qc}}$.

The coefficients $\sigma_{ij}^{\mathrm{qc}}$ are defined by

$$\sigma_{ij}^{\mathrm{qc}} = 2e^2\nu\left\langle\int d\mathbf{r}'\frac{dn}{2\pi}\frac{dn'}{2\pi}v(\mathbf{r})v(\mathbf{r}')n_in_j'\Gamma(\mathbf{n},\mathbf{n}',\mathbf{r},\mathbf{r}')\right\rangle, \tag{9}$$

where the averaging in Eq. (9) is over one period of the modulation a [this averaging arises naturally in the Lagrangian (2) since the $\boldsymbol{\Phi}$ matrix varies slowly on this scale]. Also, $\Gamma(\mathbf{n},\mathbf{n}',\mathbf{r},\mathbf{r}')$ is defined by the Boltzmann-like equation

$$[\mathcal{L}^0(\mathbf{r},\mathbf{n}) - \hat{C}]\Gamma(\mathbf{r},\mathbf{r}',\mathbf{n},\mathbf{n}') \equiv \delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{n}-\mathbf{n}'). \tag{10}$$

Here, the collision operator is given as

$$\hat{C}\{F(\phi)\} \equiv -\frac{1}{\tau}\left(F(\phi) - \int\frac{d\phi'}{2\pi}F(\phi')\right).$$

As we will show now, the $\sigma_{ij}^{\mathrm{qc}}$ coefficients in the Lagrangian (8) are the components of the quasiclassical conductivity tensor as derived by Beenakker.[12] In his approach the Boltzmann equation is expressed[12] (see also Refs. 2–4,13) in terms of the distribution function $\mathbf{F}(\mathbf{r},\mathbf{n})$

$$(\mathcal{L}^0 - \hat{C})F(\mathbf{r},\mathbf{n}) = -ev(\mathbf{r})\mathbf{E}\cdot\mathbf{n}, \tag{11}$$

and the current $\mathbf{J}(\mathbf{r})$ is given by

$$J^i(\mathbf{r}) = -2e\nu\langle F(\mathbf{r},\mathbf{n})v(\mathbf{r})n^i\rangle, \tag{12}$$

where the averaging is over both the velocity direction $\mathbf{n}$ and one period of the modulation, a. The relation $J^i = \sigma_{ij}^{\mathrm{qc}}E_j$ defines the quasiclassical conductivity tensor $\sigma_{ij}^{\mathrm{qc}}$, whose inversion gives the quasiclassical resistivity tensor, ρ_{ij}^{qc}.

Comparing Eqs. (10) and (11), we see that $\Gamma(\mathbf{n},\mathbf{n}',\mathbf{r},\mathbf{r}')$ is related to the distribution function $F(\mathbf{r},\mathbf{n})$ by

$$F(\mathbf{r},\mathbf{n}) = -e\int dr'\frac{dn'}{2\pi}v(\mathbf{r}')\mathbf{E}\cdot\mathbf{n}'\Gamma(\mathbf{r},\mathbf{r}',\mathbf{n},\mathbf{n}').$$

Comparing also Eqs. (9) and (12), we see that the $\sigma_{ij}^{\mathrm{qc}}$ coefficients in the Lagrangian (8) calculated by this method coincide precisely with the components of the quasiclassical conductivity as derived[12] from the Boltzmann equation (11). In this way we see that we have rederived the quasiclassical results within the field-theory formalism.

As discussed above, the inadequacy of the Lagrangian (8) is that it neglects the renormalization of the single-particle properties by the strong magnetic field. At high magnetic fields ($\omega_c\tau \gg 1$), once the temperature is low enough for the weak localization corrections to be significant ($k_B T \sim \hbar/\tau$), the quasiclassical results for the conductivity[12] have already become unreliable due to their neglect of quantization. In order to include these effects, and hence account for weak localization effects reliably, we need to follow a different route to derive the Lagrangian. Such a route for unmodulated systems has been provided by Pruisken,[20] whose method may be adapted (as we show here) to include a periodic potential. This method includes the renormalization of the saddle-point equation for the Q matrix by the strong magnetic field. It does not require a derivation of a ballistic σ model as an intermediate step, but provides more directly the final form of the Lagrangian. While the first route contains certain parallels with the quasiclassical approach,[12] the second route contains closer parallels with the quantum-mechanical approach.[7]

Generalization of Pruisken derivation. We now present the second route to the derivation of the effective Lagrangian (2), starting again from form (5) for the Lagrangian. The approach now is to average over the short-range disorder. This produces a term in the Lagrangian that is quartic in the Ψ, $\bar{\Psi}$ fields. The quartic term may then be decoupled by a (now local) Hubbard-Stratonovich field $Q(\mathbf{r})$. After integrating out the Ψ, $\bar{\Psi}$ fields, one finds

$$\mathcal{L}[Q] = \int dr\left\{\frac{\hbar\pi\nu}{8\tau}\mathrm{Str}\,Q^2 - \frac{1}{2}\mathrm{Str}\ln\left[-i[\pi_\mu\pi^\mu - E + U_0\cos(qx)] + \delta\Lambda + \frac{\hbar Q}{2\tau}\right]\right\}, \tag{13}$$

where $\pi_\mu \equiv -i\hbar\nabla_\mu - e\tau_3 A_\mu/c$. Here Q satisfies the symmetry $Q=\bar{Q}$.

The Lagrangian (13), in principle, provides an exact description of the system, although in its current form it is too general to be useful. Instead, one proceeds by finding the saddle-point value of Q that minimizes the Lagrangian, and by performing a gradient expansion about this minimum. The saddle-point equation may be written as

$$Q = \frac{i}{\pi\nu}\langle\mathbf{r}|\left(\pi_\mu\pi^\mu + U_0\cos(qx) - E + i\delta\Lambda + \frac{i\hbar Q}{2\tau}\right)^{-1}|\mathbf{r}\rangle. \tag{14}$$

In order to find the matrix inverse of the operator in Eq. (14), we make use of the eigenvalues and eigenfunctions of the operator $H_0 = \pi_\mu\pi^\mu + U_0\cos(qx)$. Using the Landau gauge $\mathbf{A}=(0,Bx,0)$, one may write the eigenfunctions of H_0 in the form $\psi_{k,n}(x,y) = L_y^{-1/2}\exp(i\tau_3 ky)\phi_{n,x_0}(x)$, where L_y is a normalization length. The center coordinate $x_0 = l_B^2 k$ remains a good quantum number despite modulation, where $l_B = (\hbar c/eB)^{1/2}$ is the magnetic length. The $\phi_{n,x_0}(x)$ are eigenfunctions of the Hamiltonian

$$H_{x0} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_c^2(x-x_0)^2 + U_0\cos(qx). \tag{15}$$

In the absence of modulation ($U_0=0$), Eq. (15) represents the Hamiltonian of a harmonic oscillator. We proceed by a first-order perturbative expansion in modulation U_0. This turns out to be a very good approximation for typical parameters that we consider, such as a weak periodic potential. It is very difficult to improve on this approximation analytically, although exact numerical diagonalizations have been performed for the density of states.[5,7] Within the perturbative expansion, the eigenvalues $\epsilon_n(x_0) = \epsilon_n(x_0+a)$ of the Hamiltonian (15) are given by

$$\epsilon_n(x_0) \simeq E_n(x_0) \equiv E_n + u_n\cos(qx_0), \tag{16}$$

where $E_n = \hbar\omega_c(n+1/2)$ are the unperturbed Landau energies, and $u_n = U_0\exp(-1/2X)L_n(X)$, where $X = q^2 l_B^2/2$ and L_n are the Laguerre polynomials.[43]

We see that the modulation lifts the degeneracy of the Landau levels, and discrete levels are broadened into bands, whose width depends on the band index n in an oscillatory manner (due to the behavior of the Laguerre polynomials at large n). It is this oscillatory dependence that leads to the Weiss oscillations in the resistivity. The validity of the perturbation theory depends only on the smallness of the u_n parameter, which is assured for large values of the Landau-level index n. We refer the reader to, e.g., Refs. 5–8 for further details of this perturbative expansion.

We also make the assumption that in this basis, the saddle-point value of Q is independent of the Landau indices and x_0: this approximation is analogous to the C-number approximation CNA introduced by Zhang and Gerhardts[7] for the self-energy matrix in the presence of modulation, and is valid as long as the magnetic field is not too high. In order to go beyond the CNA, one would need to generalize the saddle-point Q to include a matrix structure in the space of Landau indices and a dependence on x_0: such a task is of interest for future work but is beyond the scope of this paper.

The saddle-point equation (14) may now be written as

$$\frac{i\hbar Q}{2\tau} = \Gamma_0^2\sum_n \frac{1}{a}\int_0^a dx_0 \frac{1}{E - E_n(x_0) - i\delta\Lambda - \frac{i\hbar Q}{2\tau}}, \tag{17}$$

where Γ_0 is the width of the Landau level in the absence of modulation and

$$\Gamma_0^2 = \frac{1}{2\pi}\hbar\omega_c\frac{\hbar}{\tau}. \tag{18}$$

The saddle-point equation (17) coincides with the self-energy equation derived in the self-consistent Born approximation (SCBA) by Zhang and Gerhardts,[7] under the replacement $i\hbar Q^{1,2}/(2\tau)\rightarrow\Sigma^{R,A}$, where indices refer to advanced/retarded space and Σ denotes the self-energy. The solution for Q is then of the form $iQ=e_0+i\Lambda\rho_0$, where ρ_0 is proportional to the density of states. In the absence of modulation, the density of states reduces to that determined by Ando and co-workers[26] within the SCBA (as shown by Pruisken[20]).

Having identified the saddle-point value Q, we proceed by performing a gradient expansion of the Lagrangian (13) around the saddle point. We follow very closely the calculation of Pruisken,[20] although we use the supersymmetric rather than the replica formulation. We use the representation

$$Q(\mathbf{r})=T(\mathbf{r})P(\mathbf{r})T^{-1}(\mathbf{r}),$$

where the P fields are diagonal in retarded/advanced space and represent the massive modes. The procedure is to integrate over P and T separately. To do so, we split the fields of integration,

$$\int DQ=\int DP\int DT\exp\{\mathrm{Str}\ln(I[P])\},$$

in the process acquiring the associated Jacobian $I[P]$ (as discussed in Ref. 20). $\mathcal{L}$ becomes

$$\mathcal{L}[P,T]=\int dr\left\{-\mathrm{Str}\ln(I[P])+\frac{\hbar\pi\nu}{8\tau}\mathrm{Str}P^2-\frac{1}{2}\mathrm{Str}\ln\left[i\left(E-T^{-1}\pi_\mu\pi^\mu T-U_0\cos(qx)-\frac{i\hbar P}{2\tau}\right)\right]\right\}.$$

To integrate over the P modes, we split $\mathscr{L}$ into two parts:

$$\mathcal{L}_0[P]=\int dr\left\{-\mathrm{Str}\ln(I[P])+\frac{\hbar\pi\nu}{8\tau}\mathrm{Str}P^2-\frac{1}{2}\mathrm{Str}\ln\left[E-\pi_\mu\pi^\mu-U_0\cos(qx)-\frac{i\hbar P}{2\tau}\right]\right\},$$

$$\delta\mathcal{L}[P,T]=\mathcal{L}[P,T]-\mathcal{L}_0[P].$$

Integration over the P fields then proceeds by cumulant-expanding $\delta\mathcal{L}[P,T]$ with respect to $\mathcal{L}_0[P]$. In turn, $\delta\mathcal{L}[P,T]$ is computed by a gradient expansion of the Str ln term up to second order in the combination $D_\mu\equiv T^{-1}\boldsymbol{\nabla}_\mu T$. The propagators for the latter expansion are of the form

$$g(\mathbf{r},\mathbf{r}')=\langle\mathbf{r}|\left(E-\pi_\mu\pi^\pi-U_0\cos(qx)-\frac{i\hbar P}{2\tau}(\mathbf{r})\right)^{-1}|\mathbf{r}'\rangle,$$

weighted with respect to $\mathcal{L}_0[P]$. A typical second-order term is

$$\mathcal{L}_{\mathrm{typ}}=\int dr\int dr'\,\mathrm{Str}[D_\mu(\mathbf{r})][D_\nu(\mathbf{r}')]\times\langle g(\mathbf{r}',\mathbf{r})\pi^\mu g(\mathbf{r},\mathbf{r}')\pi^\nu\rangle,$$

where the averaging is over P with respect to $\mathcal{L}_0$. At this point, we now exploit the assumption that the $T(\mathbf{r})$ matrices vary in space more slowly than the modulation. Thus the averages of the products of the Green functions are short ranged with respect to the T matrices, and are translationally invariant after averaging over one cycle of the modulation. This allows us to perform a gradient expansion in the propagator averages:

$$\begin{aligned}\langle g(\mathbf{r}',\mathbf{r})\pi^\mu g(\mathbf{r},\mathbf{r}')\pi^\nu\rangle &=K(\mathbf{r}-\mathbf{r}')\\ &=K^{(0)}\delta(\mathbf{r}-\mathbf{r}')+K^{(2)}\delta(\mathbf{r}-\mathbf{r}')\boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu+\cdots,\end{aligned}$$

here the averaging includes that over a cycle of the modulation. A series of Ward identities may now be used as in Ref. 20 to simplify the resulting expressions up to second order in D_μ. The final Lagrangian is then of the form displayed in Eq. (2), where $Q=T\Lambda T^{-1}$.

Again, we have the standard form of a nonlinear σ model with a topological term, although with the anisotropy $\sigma^0_{xx}\neq\sigma^0_{yy}$. Again, Q satisfies $Q^2=1$ and the symmetries $Q=\overline{Q}=KQ^\dagger K$. Also the bare coefficients σ^0_{ij} are the components of the conductivity tensor at zero temperature in units of e^2/h, according to the formulas

$$\begin{aligned}\sigma^0_{xx,yy}&=-\frac{\hbar^2}{m}\int dr'\langle\pi_{x,y}[g^1(\mathbf{r},\mathbf{r}')-g^2(\mathbf{r},\mathbf{r}')]\times\pi_{x,y}[g^1(\mathbf{r}',\mathbf{r})-g^2(\mathbf{r}',\mathbf{r})]\rangle,\\ \sigma^0_{xy}&=\sigma^{0,I}_{xy}+\sigma^{0,II}_{xy},\\ \sigma^{0,II}_{xy}&=\frac{i\hbar c}{e}\left\{\int_{-\infty}^{E}dE^1\frac{\partial}{\partial B^1}\langle g^1(\mathbf{r},\mathbf{r})\rangle-\int_{-\infty}^{E}dE^2\frac{\partial}{\partial B^2}\langle g^2(\mathbf{r},\mathbf{r})\rangle\right\},\\ \sigma^{0,I}_{xy}&=\frac{\hbar^2}{m}\int dr'\{\langle\pi_x g^1(\mathbf{r},\mathbf{r}')\pi_y g^2(\mathbf{r}',\mathbf{r})\rangle-\langle\pi_y g^1(\mathbf{r},\mathbf{r}')\pi_x g^2(\mathbf{r}',\mathbf{r})\rangle\},\end{aligned}\tag{19}$$

here the averaging is over P as well as over one cycle of the modulation with respect to $\mathbf{r}$. The propagator averages may now be evaluated by their saddle-point values with respect to P.

The bare coefficients σ^0_{ij} are the components of the conductivity tensor, in units of e^2/h, calculated in the SCBA at zero temperature. In the absence of modulation, the expressions (19) for the bare conductivity then reduce to those computed in the SCBA by Ando and co-workers.[26] In the presence of modulation, they reduce to the conductivity tensor computed in the SCBA by Zhang and Gerhardts:[7] $\sigma^0_{ij}=\sigma^{\mathrm{qu}}_{ij}$ (the equivalence of may be seen by following the working of Streda[44]). Therefore, in contrast to those derived via the ballistic σ model, the bare conductivities in the Lagrangian (2) represent the values calculated by a fully quantum-mechanical approach.

III. WEAK LOCALIZATION AND QUANTUM HALL EFFECT

Having derived the final form of the Lagrangian (2), we are now in a position to use it to calculate weak localization corrections to the conductivity tensor. The effect of the corrections may be expressed as a scaling of the conductivity tensor under changes of length scale. A perturbative treatment of the Lagrangian (2) is valid in the limit of large longitudinal conductivity ($\sigma_{xx}^0 \gg 1$). Within perturbation theory, however, the Hall conductivity is not renormalized by weak localization corrections. Instead, the effect of the topological term in the Lagrangian may only be made apparent through a nonperturbative analysis. Such an analysis has been conjectured[25,30,31] in the form of a two-parameter renormalization-group procedure. The two parameters are the longitudinal and Hall conductivities that follow coupled scaling equations with respect to changes of length scale. The derivation of the scaling equations presented in Refs. 25,31 is based on approximating configurations of the field-theory parameter as a gas of instantons. While the validity of this derivation is being vigorously debated (see, e.g., Refs. 33,34), it has remained a valuable guide to experimental[35] and numerical[36] data for a number of years. In the following, we take a pragmatic approach by not contesting the validity of the two-parameter scaling and its relation to the proposed field-theory. Instead, we assume its validity and explore how it may be generalized to take account of the periodic potential.

A. Weak localization

First we subject the Lagrangian to a perturbative RG analysis. The perturbation theory is valid in the limit of $\sigma_{xx}^0 \gg 1$. The topological term in the Lagrangian does not contribute at any perturbative order and hence σ_{xy} is unrenormalised. For an unmodulated system, $\sigma_{xx} = \sigma_{yy}$ and the perturbative RG procedure has been explained in detail, e.g., in Ref. 24.

In the presence of modulation, the σ model is slightly nonstandard due to the anisotropy of the longitudinal conductivities, $\sigma_{xx} \neq \sigma_{yy}$. However, quantum interference processes in a diffusive, anisotropic conductor have been considered before, for example by Wölfle and Bhatt,[40] where the origin of the anisotropy was envisaged as due to a difference in effective electron masses in the two directions. In the latter paper, a diffusive σ model was derived with the same form as Eq. (2), although without the topological term.

When $\sigma_{xx} \neq \sigma_{yy}$, the two parameters, σ_{xx} and σ_{yy}, follow coupled flow equations under changes of length scale. While it is straightforward to derive and solve the two flow equations, it is also instructive to follow a different strategy whereby we perform a scale transformation after which the Lagrangian (2) maps to an isotropic form: we scale

$$x' = x(\sigma_{yy}^0/\sigma_{xx}^0)^{1/4}, \quad y' = y(\sigma_{xx}^0/\sigma_{yy}^0)^{1/4}, \tag{20}$$

under which the Lagrangian (2) transforms to an isotropic σ model,

$$\mathcal{L}[Q] = \frac{1}{16}\int dr' \,\mathrm{Str}\{\tilde{\sigma}^0(\nabla Q)^2 - \sigma_{xy}^0 \tau_3 Q[\nabla_{x'} Q, \nabla_{y'} Q]\}, \tag{21}$$

where $\tilde{\sigma}^0 \equiv \sqrt{\sigma_{xx}^0 \sigma_{yy}^0}$. The perturbative scaling for $\tilde{\sigma}$ may now be derived in the standard way:[24] the flow equation is (to leading order)

$$\frac{d\tilde{\sigma}}{d\ln L} = -\frac{1}{2\pi^2\tilde{\sigma}}.$$

By integrating the flow equation from microscopic to macroscopic length scales (of the order of the system size L), we find the conductivity

$$\tilde{\sigma}(L) = \tilde{\sigma}^0\left(1 - \frac{1}{\pi^2}\frac{1}{(\tilde{\sigma}^0)^2}\ln\frac{L}{\ell}\right)^{1/2}. \tag{22}$$

Using the fact that the ratios $\sigma_{xx,yy}/\tilde{\sigma}$ are invariant under the scaling, we may recover $\sigma_{xx,yy}(L)$ from $\tilde{\sigma}(L)$ as follows:

$$\sigma_{xx}(L) = \tilde{\sigma}(L)\sqrt{\sigma_{xx}^0/\sigma_{yy}^0},$$
$$\sigma_{yy}(L) = \tilde{\sigma}(L)\sqrt{\sigma_{yy}^0/\sigma_{xx}^0}. \tag{23}$$

We remark that the perturbative scaling derived from a field-theory as above in the presence of anistropy in the conductivities in the x and y directions (due to an anisotropy in the electron mass) has been confirmed by direct diagrammatics (in the orthogonal ensemble) by Wölfle and Bhatt.[40]

Equation (22) shows how weak localization corrections affect the longitudinal conductivity within perturbation theory. We can see how the conventional scaling applies to $\tilde{\sigma}$ in the presence of the periodic potential. By inputting the SCBA values for σ_{xx}^0 and σ_{yy}^0 weak localization corrections are obtained; according to Eq. (22) they depend on *both* bare values σ_{xx}^0 and σ_{yy}^0. Corrections to σ_{xx}, for example, are influenced by the strong oscillatory behavior of σ_{yy}^0. Upon inverting the conductivity tensor to find the resistivity tensor, the weak localization corrections will therefore give rise to additional oscillatory corrections to the resistivity tensor as a function of the magnetic field. As mentioned earlier, strong Weiss oscillations in ρ_{xx} are accompanied by weak out-of-phase oscillations in ρ_{yy} as long as quantum interference effects may be neglected. It is interesting to notice that weak localization corrections to ρ_{yy} oscillate in phase with the (dominant) oscillations in ρ_{xx}.

In practice,[35] rather than changing the system size L, one varies the temperature to change the effective system size, which is given by the diffusion length, $[\hbar D/(k_B T)]^{1/2}$. The perturbative result (22) is only valid when the corrections are much smaller than the bare conductivities, and hence may be difficult to verify experimentally. A potentially more promis-

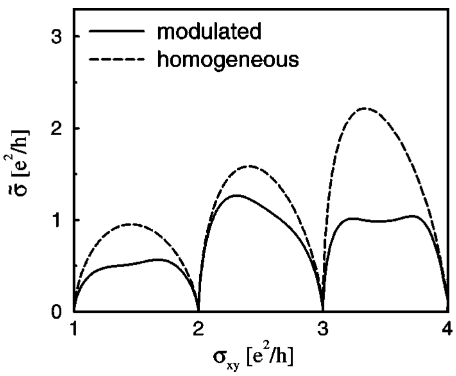

FIG. 2. Plot of bare conductivities, providing a starting point for scaling, for a modulated vs homogeneous system. Parameters: $a=140$ nm, $U_0=0.2$ meV, $\Gamma_0=0.048\sqrt{B[T]}$ meV, and $n_{el}=1\times10^{11}$ cm^2.

ing strategy is to illustrate the effect of weak localization corrections when their contribution is relatively large, as is the case for the IQHE.

B. Quantum Hall effect

We now generalize from the perturbative analysis to discuss the nonperturbative scaling that would follow from the Lagrangian (2). Again, we are able to make the scale transformation (20) as above, which maps the Lagrangian to the isotropic version, Eq. (21). The coupled flow of the two coupling constants, $\tilde{\sigma}$ and σ_{xy}, may written in a general form,

$$\frac{d\tilde{\sigma}}{d\ln L}=\tilde{\beta}(\tilde{\sigma},\sigma_{xy})\quad \frac{d\sigma_{xy}}{d\ln L}=\beta_{xy}(\tilde{\sigma},\sigma_{xy}). \tag{24}$$

The above beta functions, $\tilde{\beta}$ and β_{xy}, would take precisely the same form as the beta-functions that describe the flow of σ_{xx} and σ_{xy} in the equivalent, unmodulated system.

The starting point of the flow is determined by the bare conductivity tensor $\tilde{\sigma}_{ij}$. In general, the coupled flow equations (24) for $\tilde{\sigma}$ and σ_{xy} need to be integrated up to length

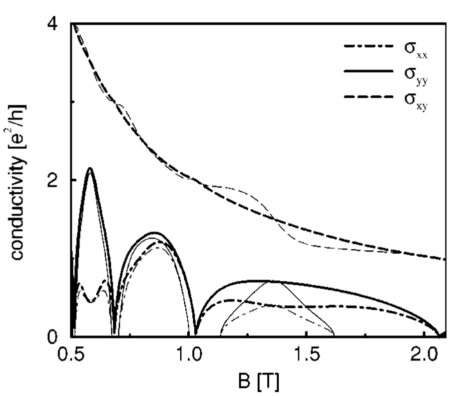

FIG. 3. Evolution of the conductivity tensor for the modulated system with system size L under the two-parameter scaling. The bold curves are the bare conductivities ($L=L_0$), and the thin curves are the scaled conductivities for $\ln(L/L_0)=2$. Parameters as in Fig. 2.

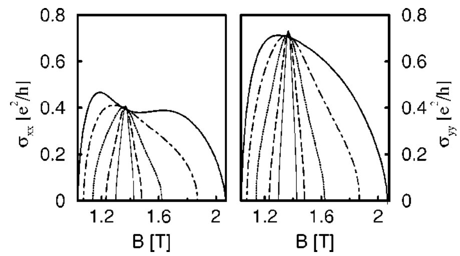

FIG. 4. Focus on the evolution of the longitudinal conductivities with system size L under the two-parameter scaling. Parameters as in Fig. 3, and $\ln(L/L_0)=0-4$, from the broader to the narrower peaks.

scales of the system size. The values of $\sigma_{xx}(L)$ and $\sigma_{yy}(L)$ may then be recovered from $\tilde{\sigma}(L)$ by Eqs. (23). As a consequence of Eq. (23), we see that the ratio $\sigma_{xx}(L)/\sigma_{yy}(L)$ remains constant under the scaling. This conclusion is furthermore independent of the form of the beta-functions, $\tilde{\beta}$ and β_{xy}, in Eq. (24).

In the absence of modulation, the SCBA values[26] of the bare conductivities correspond to an approximate semicircular dependence of σ^0_{xx} on σ^0_{xy}. In the presence of modulation, the dependence of $\tilde{\sigma}^0$ on σ^0_{xy} is modified from the semicircle in a complicated way; a typical dependence, calculated using the scheme proposed in Ref. 7, is shown in Fig. 2.

The corresponding dependency of the bare conductivity tensor on magnetic field for the same parameters is part of Fig. 3 below (thick curves). We remark that for the peaks centered around $B=1.5$ T one may already expect vertex corrections to become effective, although they are not included in the CNA approach that we have employed. Such corrections have the general tendency[10,11] to enhance σ^0_{yy} in comparison to σ^0_{xx}.

In order to provide an illustration of the IQHE in the modulated sample, it is necessary to assume a particular form of the scaling equations (24). The following scaling equations[25,31,45,36] were derived (originally in the replica formulation) within a dilute gas approximation of the instantonic configurations of the Q matrix from the Lagrangian (2):

$$\frac{d\tilde{\sigma}}{d\ln L}=-\frac{1}{2\pi^2\tilde{\sigma}}-\tilde{\sigma}^3D_c\cos(2\pi\sigma_{xy})\exp(-2\pi\tilde{\sigma}),$$

$$\frac{d\sigma_{xy}}{d\ln L}=-\tilde{\sigma}^3D_c\sin(2\pi\sigma_{xy})\exp(-2\pi\tilde{\sigma}). \tag{25}$$

The dimensionless constant D_c is of order unity and is related to the density of instantons. It may be seen immediately from the form of Eq. (25), that along the lines $\sigma_{xy}=(n+1/2)$, where n is an integer, the Hall conductance is unrenormalized. It may also been seen that the points $(\sigma_{xy},\tilde{\sigma})=(n,0)$, for integer n, are (attractive) fixed points. Upon scaling the system from microscopic to macroscopic length scales, the coordinates $(\sigma_{xy},\tilde{\sigma})$ scale from the bare nonuniversal values towards the quantized values $(n,0)$, for integer n. This tendency reflects the quantization of the Hall conductivity under scaling at low temperatures.

As mentioned before, the validity of the approximations underlying Eq. (25) has long been the subject of debate[34,33]. Keeping this in mind we use these equations to provide an illustration of the operation of the IQHE in the modulated system in Figs. 3 and 4.

Figure 3 shows the evolution of the conductivity for the modulated system under the two-parameter scaling. We see how the Hall conductivity becomes quantized under scaling in the modulated system. Between the plateaus in the Hall conductivity, the longitudinal conductivities develop peaks under scaling of differing heights, due to the anisotropy in the system. Figure 4 shows in more detail how these peaks in the longitudinal conductivities develop, while the ratio between the two conductivities remains constant under scaling.

We remark that the behavior described above should be within current experimental capabilities: for example, Geim *et al.*[14] have studied the high magnetic field ($R_c \ll a$) regime, although they kept the temperature high to avoid the effect of quantum interference processes. Bagwell *et al.*[21] have also studied the IQHE in modulated samples; in their work, however, they have not focused on the regime of a weak periodic potential or effects that are independent of the device boundaries.

IV. SUMMARY AND DISCUSSION

In this paper we have considered transport properties of a disordered conductor in a periodic potential and strong magnetic field. We focused on the contribution of quantum interference processes, whose influence at high fields is missing from previous approaches despite being responsible for a whole class of phenomena. For example, they lead to weak localization corrections to the conductivity and the operation of the IQHE at high magnetic fields. To this end, we introduced a field-theory approach, which is well established in the study of unmodulated disordered conductors.

The effective Lagrangian of the field-theory takes the form of a nonlinear σ model, which describes the interaction of diffusion modes on large length scales. The presence of the strong magnetic field leads to an extra, topological term in the Lagrangian. The form of the Lagrangian is the same as for unmodulated systems, except for an anisotropy in the coefficients corresponding to the bare longitudinal conductivities in the x and y directions.

We provided two different routes to deriving the Lagrangian. The first route was via a so-called "ballistic" σ model, and demonstrated how the results of the quasiclassical approach[12] may be recovered within the field-theory formalism. The drawback of this route, in common with the quasiclassical approximation, is that it neglects the renormalization of single-particle properies by Landau quantization. Consequently the resulting Lagrangian was too approximate to be useful in determining weak localization properties at low temperatures. The second route improved on this situation by including the effects of Landau quantization. It bypasses a derivation of a ballistic σ model and instead follows more closely the original derivation of Pruisken[20] for unmodulated systems. Indeed, while the first route contained more parallels with the quasiclassical approach, the second route contained more parallels with the quantum-mechanical approach[7] for the bare conductivity tensor.

Having derived the effective Lagrangian, we showed how it leads to the scaling of the conductivity tensor under changes of length scale (and hence temperature). A perturbative renormalization-group analysis of the free energy leads to a generalization, to modulated systems, of one-parameter scaling for the longitudinal conductivities. Perturbatively, the Hall conductivity is unrenormalized. In the regime of Weiss oscillations,[1] weak localization corrections give rise to an additional oscillatory dependence of the longitudinal conductivities as a function of the magnetic field. Due to their smallness, these corrections may, however, be hard to detect experimentally.

In order to describe the IQHE, whereby the Hall conductivity becomes quantized at low temperatures, a nonperturbative analysis of the Lagrangian is necessary. This is provided by the conjecture of the two-parameter scaling for unmodulated systems. Assuming the validity of this conjecture and the underlying instanton gas approximation, we have shown how the two-parameter scaling may be generalized to the case of the modulated system. This has allowed us to illustrate the evolution of the resistivity tensor under the IQHE for parameters that are realistic for experiments. We find, for example, that the ratio of the two longitudinal components of the conductivity remains constant under scaling (an observation that does not itself depend on the assumptions used to derive the flow equations). While the Hall conductivity still becomes quantized under scaling in the modulated system, between the plateaus the longitudinal conductivities develop peaks of differing heights due to the anisotropy in the system.

There are several directions in which our analysis may be generalized. A simple generalization is to consider a periodic magnetic field, rather than potential, a situation that leads to a similar phenomenology.[46,4,47] The derivation of the field-theory for this case follows very similar lines to the case of a periodic potential, with similar results.

Another direction in which the analysis may be extended straightforwardly is to the study of the low magnetic field regime, in which a positive magnetoresistance has been observed[1,2] and explained within a quasiclassical approach.[2–4] As long as $\omega_c\tau \ll 1$, one may derive the field-theory for this regime according to the first route of Sec. II, neglecting quantization; the coefficients of the field-theory then coincide with the components of the conductivity tensor derived within the Boltzmann approach. The weak localization to the conductivity (or related quantities such as the dephasing time[28]) may then be calculated using the Lagrangian for either the orthogonal or the unitary ensemble, or in the crossover between the pure symmetry classes.

A less straightforward generalization is to improve on the CNA approximation of the quantum-mechanical approach and its analog in the field-theory formalism. According to this approximation, the self-energy matrix (and the saddle-point solution of the Q matrix) are assumed to have a trivial structure in the space of Landau indices and coordinate x_0. While this simplifies the analysis considerably, we have seen already how this approximation breaks down for very high magnetic fields, such that the cyclotron radius is much less than the period of modulation, $R_c \ll a$. Improvement on the CNA is necessary, not only for such high fields, but also to study models of disorder with long-range correlations, for which vertex corrections are not negligible. Such models have been analyzed in the quasiclassical approach[13] and shown to represent experimental data more closely in certain respects.

Very recently it has been shown how to improve on the CNA (Refs. 10,11) within the quantum-mechanical approach, so as to analyze models at high magnetic fields or with long-range disorder correlations. In the field-theory formalism, improvement on the CNA approximation would require enlarging the space of the Q matrix even further to include the additional matrix structure in the space of Landau indices, as well as a dependence on x_0. Inclusion of such a structure within the Q field is to our knowledge a novel direction to pursue, although this task is left as a future project.

A further area to explore is the case of a strong periodic potential, for which the first-order perturbative expansion in the potential that we use is no longer valid. While an analytical approach would be very difficult, a numerical treatment would be better suited to this regime. For a strong enough potential that U_0 is of the order of ω_c, the smearing of the Landau levels is so great that they can no longer be individually resolved even at low temperatures. This leads to a quenching of the Shubnikov–de Haas oscillations (as noted by Beton *et al.*[2]) and hence one would expect the quantum Hall effect to be destroyed.

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