Fermions, strings, and gauge fields in lattice spin models

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We investigate the general properties of lattice spin models with emerging fermionic excitations. We argue that fermions always come in pairs and their creation operator always has a stringlike structure with the newly created particles appearing at the end points of the string. The physical implication of this structure is that the fermions always couple to a nontrivial gauge field. We present exactly soluble examples of this phenomenon in two and three dimensions. Our analysis is based on an algebraic formula that relates the statistics of a lattice particle to the properties of its hopping operators. This approach has the advantage in that it works in any number of dimensions—unlike the flux-binding picture developed in fractional quantum Hall theory.

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I. INTRODUCTION

For many years, it was thought that Fermi statistics were fundamental, in the sense that one could only obtain a theory with fermionic excitations by introducing them by hand (via anticommuting fields). Then, over the past two decades, this view began to change. A number of real world and theoretical examples showed that fermions and anyons could emerge as low-energy collective modes of purely bosonic systems. The first examples along these lines were the fractional quantum Hall (FQH) states.[1,2] Usually we think of the FQH states as examples of anyonic excitations emerging from interacting fermions. However, from a purely theoretical point of view, the same effects should occur in systems of interacting bosons in a magnetic field.[3] There were also indications of emerging fermions in the slave-boson approach to spin-$\frac{1}{2}$ systems[4–10] and in the study of resonating valence bond (RVB) states.[11–14] Unfortunately, the RVB picture and the slave-boson approach both rely on approximate or mean-field techniques to construct and analyze these exotic states. More recently, a number of researchers have introduced exactly soluble or quasiexactly soluble models with emerging fermions.[14–21] These models allow for a more well-controlled analysis, albeit, in very specific cases.

The mean-field approach and the exactly soluble examples both provide clues to the structure and basic properties of bosonic models with fermionic excitations. They indicate, among other things, that fermions never appear alone in lattice spin systems. Instead they always come together with a nontrivial gauge field and the emerging fermions are associated with the deconfined phase of the gauge field.[7,9,10] The deconfined phases always seem to contain a new kind of order—topological order,[10,22] and the emerging fermions and anyons are intimately related to the new order.

This was particularly apparent in the context of the slave-boson approach. In this technique, one expresses a spin-$\frac{1}{2}$ Hamiltonian in terms of fermion and gauge fields.[4,5] Clearly, the presence of fermion fields does not, by itself, imply the existence of fermionic quasiparticles—the fermions appear only when the gauge field is in the *deconfined* phase. References 6–10 present constructions of several deconfined phases where the fermion fields do describe well-defined quasiparticles. Depending on the properties of the deconfined phases, these quasiparticles can carry fractional statistics (for the chiral spin states) (Refs. 6–8) or Fermi statistics (for the Z_2 deconfined states).[9,10]

Although it was less evident initially, a similar picture emerged from the study of RVB states and the associated quantum dimer models. The authors of Refs. 11 and 12 originally proposed that fermions (spinons) could emerge from a nearest-neighbor dimer model on a square lattice. Later, it was realized that the fermions couple to a Z_2 gauge field, and the fermionic excitations only appear when the field is in the deconfined phase.[13] It turns out that the dimer liquid on a square lattice with only nearest-neighbor dimers is not in the deconfined phase (except at a critical point).[12,13] However, on a triangular lattice, the dimer liquid does have a Z_2 deconfined phase.[14] The results in Refs. 11 and 12 are valid in this case and fermionic quasiparticles do emerge in a dimer liquid on a triangular lattice.

In this paper, we attempt to clarify these observations and put them on firmer foundations. We give a general argument that shows that emerging fermions always occur together with a nontrivial deconfined gauge field. In addition, we derive an algebraic formula that allows one to calculate the statistics of a lattice particle from the properties of its hopping operators. We feel that this formula both elucidates the fundamental meaning of statistics in a lattice system and simplifies their computation. This approach has the added advantage in that it works in any number of dimensions—unlike the flux-binding picture[3] developed in FQH theory. We would like to point out that all the previously mentioned examples of emerging fermions are two-dimensional models. The emerging fermions in those models are related to the flux-binding picture in Ref. 3. Our algebraic approach allows us to establish the emergence of fermions in an exactly soluble three-dimensional (3D) bosonic model.

Our paper is organized as follows: in the first section, we give a definition of the statistics of a lattice particle. In the second section, we derive the algebraic statistics formula discussed above. In the third section, we apply the formula to the case of fermionic (or anyonic) excitations in a lattice spin system. The formula demands that the fermions always come in pairs, and that their pair-creation operator has a stringlike structure with the newly created particles appearing at the ends. We show that the strings represent gauge fluctuations, and the fermions carry the corresponding gauge charge. After presenting the general argument, we devote the last two sections to exactly soluble examples of this phenomenon in two and three dimensions.

The above result suggests a very interesting picture of emerging fermions. Fermions and gauge fields appear to be two sides of the same coin: in some sense, fermions are the ends of strings and gauge fields are the fluctuations of strings. Extended stringlike structures seem to be the key to understanding both Fermi statistics and gauge fields.

II. DEFINITION OF STATISTICS

What do we mean when we say that a particle is a "boson" or a "fermion?" The most common way to define statistics is to use the algebra of creation and annihilation operators. If these operators satisfy the bosonic algebra, the corresponding particles are said to be bosons; if they satisfy the fermionic algebra, the particles are said to be fermions.

In this paper, we will use another, equivalent, definition based on the path integral formulation of quantum mechanics. Given any m particle system with short-range interactions, we can write down the corresponding multiparticle action S. The most general action S has two components:

$$S = S_{\text{loc}} + S_{\text{top}}.$$

The first term, S_{loc}, can be any local expression. Typically, it's just the usual classical action. The second term, S_{top}, is less familiar. In general, it can be any expression which only depends on the topology (e.g. homotopy class) of paths. In 3 or more dimensions, the form of S_{top} is highly constrained. The amplitude for a closed path P must be of the form

$$e^{iS_{\text{top}}} = (\pm 1)^n,$$

where n is the number of particle exchanges that occur in P. The statistics of the particles are defined by the sign in this expression. If this sign is positive, we say the particles are "bosons", if the sign is negative, the particles are "fermions". In two dimensions, there are other, more complicated possibilities for S_{top}. These particles are called "anyons".

According to this definition, the statistics are completely determined by the topological term S_{top} in the action. One way to isolate this term is to compare the amplitude for two paths, which are the same locally (this will be made more precise in the next section), but differ in their global properties. In particular, we can compare the amplitude for a path, which exchanges two particles with another path, which is the same locally, but doesn't exchange the two particles. The difference in phase is then precisely $e^{iS_{\text{top}}} = (\pm 1)$, the statistical phase.

We now reformulate this (theoretical) test of statistics so that it can be applied to particles on a lattice. The prescription is as follows: we take a two-particle state $|r_1, s_1\rangle$ and consider a product of hopping amplitudes along a lattice path, which exchanges the particles,

$$\langle r_n, s_n|H|r_{n-1}, s_{n-1}\rangle\langle r_{n-1}, s_{n-1}|H|r_{n-2}, s_{n-2}\rangle \\ \ldots\langle r_3, s_3|H|r_2, s_2\rangle\langle r_2, s_2|H|r_1 s_1\rangle. \tag{1}$$

Each hopping amplitude in this expression involves one of the two particles moving to a neighboring site while the other particle remains fixed. At the end, the two particles have exchanged places: $r_n = s_1$, $s_n = r_1$.

Now we construct another path, which is the same locally, but doesn't exchange the two particles, and take a product of hopping amplitudes along this path. The claim is that the difference between the phases of these two expressions is precisely the statistical phase of the particles. That is, the particles are bosons, fermions, or anyons depending on whether the phase difference is $+1$, -1, or something else.

One way to see this is to remember the derivation of the path-integral formulation of quantum mechanics. According to the standard derivation, the amplitude e^{iS} for a two-particle path $|r(t), s(t)\rangle$ is given by a product

$$e^{iS} \propto \langle r(t_n), s(t_n)|e^{-iH\Delta t}|r(t_{n-1}), s(t_{n-1})\rangle \\ \ldots\langle r(t_3), s(t_3)|e^{-iH\Delta t}|r(t_2), s(t_2)\rangle \\ \times\langle r(t_2), s(t_2)|e^{-iH\Delta t}|r(t_1), s(t_1)\rangle$$

in the limit that $\Delta t = t_n - t_{n-1} \to 0$.

Now, in the discrete (lattice) case, the above expression can be further simplified. Since $\Delta t \to 0$, we can rewrite it as

$$e^{iS} \propto \langle r(t_n), s(t_n)|(1 - iH\Delta t)|r(t_{n-1}), s(t_{n-1})\rangle \\ \ldots\langle r(t_3), s(t_3)|(1 - iH\Delta t)|r(t_2), s(t_2)\rangle \\ \times\langle r(t_2), s(t_2)|(1 - iH\Delta t)|r(t_1), s(t_1)\rangle.$$

Successive states $|r(t_i), s(t_i)\rangle$, $|r(t_{i+1}), s(t_{i+1})\rangle$ must either be identical or must differ by a single-particle hop. The matrix elements between identical states don't contribute to the phase. Thus, we can drop them without affecting our result. We are left with

$$e^{iS} \propto \langle r(t_k'), s(t_k')|(-iH\Delta t)|r(t_{k-1}'), s(t_{k-1}')\rangle \\ \ldots\langle r(t_3'), s(t_3')|(-iH\Delta t)|r(t_2'), s(t_2')\rangle \\ \times\langle r(t_2'), s(t_2')|(-iH\Delta t)|r(t_1'), s(t_1')\rangle,$$

where the t_k''s are all distinct.

Now, as we discussed earlier, the statistical phase can be obtained by comparing the phase of this product with another product, which is the same locally, but doesn't exchange the two particles. When we make this comparison, the phase factors of $-i$ drop out (since they contribute equally to the two products). Thus, it suffices to compare products of the form

$$\langle r(t_k'), s(t_k')|H|r(t_{k-1}'), s(t_{k-1}')\rangle \\ \ldots\langle r(t_3'), s(t_3')|H|r(t_2'), s(t_2')\rangle \\ \times\langle r(t_2'), s(t_2')|H|r(t_1'), s(t_1')\rangle$$

This is precisely the expression in Eq. (1).

III. STATISTICS AND THE HOPPING OPERATOR ALGEBRA

In this section we derive a simple algebraic formula for the statistics of a particle hopping on a lattice. This formula is completely general and holds irrespective of whether the particles are fundamental or are low-energy excitations of an underlying condensed-matter system (e.g., quasiparticles).

We begin with a Hilbert space that describes n hard-core particles hopping on a d-dimensional lattice. The states can be labeled by listing the positions of the n particles: $|i_1,i_2,\ldots,i_n\rangle$. The particles are identical so the states $|i_1,i_2,\ldots,i_n\rangle$ do not depend on the order of $i_1,i_2,\ldots,i_n$. For example,

$$|i_1,i_2,\ldots,i_n\rangle=|i_2,i_1,\ldots,i_n\rangle.$$

A typical Hamiltonian for this system is of the form

$$H=\sum_{\langle ij\rangle}(t_{ij}+t_{ji}),$$

where t_{ij} are "hopping operators" with the property that

$$t_{ij}|j,i_1,\ldots,i_{n-1}\rangle\propto|i,i_1,\ldots,i_{n-1}\rangle. \tag{2}$$

We assume that the hopping is local, i.e.,

$$[t_{ij},t_{kl}]=0 \tag{3}$$

if i, j, k, and l are all different.

Our goal is to compute the statistics of the particles described by this hopping Hamiltonian H. In the following, we show that the statistical angle can be derived from the simple algebraic properties of the hopping operators. More precisely, we show that the particles obey statistics $e^{i\theta}$ if

$$t_{il}t_{ki}t_{ij}=e^{i\theta}t_{ij}t_{ki}t_{il} \tag{4}$$

for any three hopping operators t_{ij}, t_{ki}, and t_{il}, where j, k, and l are (distinct) neighbors of i (ordered in the clockwise direction in the case of two dimensions). The orientation convention in two dimensions is necessary for the anyonic case.

The simplest examples for which we can apply this formula are the cases of noninteracting hard-core bosons or fermions. In these cases the n-particle Hamiltonian can be written as

$$H=-t\sum_{\langle ij\rangle}(c_i^\dagger c_j+c_j^\dagger c_i),$$

where the c_i's are the boson or fermion annihilation operators. The hopping operators are just $t_{ij}=-tc_i^\dagger c_j$. A little algebra confirms that the boson and fermion hopping operators do indeed satisfy Eq. (4) with $e^{i\theta}=+1$ and -1, respectively.

We now give a general derivation of the formula. We begin with the state $|i,j,\ldots\rangle$, which contains particles at sites i,j and other particles far away.

Imagine that we exchange the two particles at i,j using an appropriate product of hopping operators. One expression for the final "swapped" state is

$$|i,j,\text{swapped}\rangle=(t_{jj'})(t_{j'p}\ldots t_{qr})(t_{il}t_{lm}\ldots t_{nj'})\times(t_{j'j})(t_{rs}\ldots t_{ti})|i,j,\ldots\rangle.$$

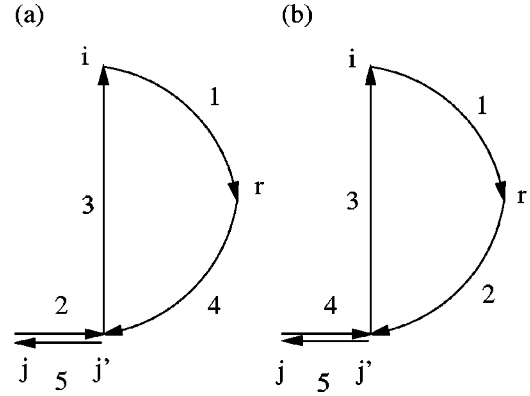

FIG. 1. (a) The path of the particles in the construction of the swapped state $|i,j,\text{swapped}\rangle$. The numbers label the order in which the paths are traversed. (b) The path of the particles in the unswapped state $|i,j,\text{unswapped}\rangle$. Notice that the two states only differ in the ordering of the paths. However, in one case the particles switch places, while in the other, they do not.

Here, we've grouped the hopping operators into 5 terms. Each of these terms moves a particle along some path on the lattice. The cumulative effect of all 5 terms is to exchange the particles [see Fig. 1(a)].

Similarly, we can construct a path that doesn't swap the two particles. One expression for the final "unswapped" state is

$$|i,j,\text{unswapped}\rangle=(t_{jj'})(t_{j'j})(t_{il}t_{lm}\ldots t_{nj'})(t_{j'p}\ldots t_{qr})(t_{rs}\ldots t_{ti})|i,j,\ldots\rangle.$$

Once again, we've grouped the hopping operators into 5 terms. In this case, the cumulative effect of these terms is simply to move the two particles along independent loops without any swapping taking place [see Fig. 1(b)].

Intuitively, we expect that the phase difference between the swapped and unswapped states is related to the statistical phase of the particles. In fact, the special way we've constructed our states allows us to make a much stronger statement. Notice that the two states involve the same product of hopping operators. The only difference is the order of these operators. This means that, in the swapped and unswapped states, the two particles trace out the same total path in configuration space. This is important because it implies that any phase which comes from gauge fields or other Berry phases contributes equally to the swapped and unswapped states. The phase difference between the swapped and unswapped states is therefore exactly equal to the statistical phase of the particles.

This intuitive argument can be made rigorous using the discussion in the previous section. Indeed, it's not hard to see that the two space-time paths traced out by the swapped and unswapped states are exactly the same locally—they only differ by a reordering. Furthermore, the phase difference between the swapped and unswapped states can be written as a phase difference between two expressions of the form (1). Thus, taking the phase difference between the swapped and unswapped states is completely equivalent to the procedure derived in the previous section.

We now compute this phase difference. We use the assumed algebraic relation

$$t_{j'p}t_{nj'}t_{j'j}=e^{i\theta}t_{j'j}t_{nj'}t_{j'p}.$$

Applying this relation to the swapped state and reordering the hopping operators using locality, we find

$$|i,j,\text{swapped}\rangle=e^{i\theta}|i,j,\text{unswapped}\rangle.$$

The phase difference is $e^{i\theta}$, so the particles obey statistics $e^{i\theta}$ as claimed. A similar formula can be derived for the relative statistics of distinguishable particles. This formula, together with its derivation, is given in Appendix A [see Eq. (A1)].

IV. FERMIONS AND STRINGS

In this section we consider the properties of a lattice spin system with fermionic or anyonic excitations. We argue that the excitations are always created in pairs, and the creation operator for a pair of particles has a stringlike structure, with the new particles located at the ends. One interpretation of this is that fermions never appear alone—they always come with some kind of gauge field.

We now make these statements more precise. Suppose we have a lattice spin system with fermionic excitations (the anyonic case is completely analogous). The total Hilbert space of the lattice spin model is a direct product of local Hilbert spaces $\mathcal{H}_I$, each associated with an individual lattice site I. We expect that the total Hilbert space contains a low-energy subspace spanned by n fermion states. The corresponding low-energy effective Hamiltonian is given by

$$H_{\text{eff}}=PHP,$$

where P is a projection operator onto the n fermion subspace. Typically, H_{eff} can be written as a sum,

$$H_{\text{eff}}=\sum_{\langle ij\rangle}(t_{ij}+t_{ji}),$$

where the t_{ij} are hopping operators. We expect that the t_{ij} are local in the underlying spin degrees of freedom. That is, they act only within the local Hilbert spaces near sites i,j.[30] We would like point out that the lattice on which the fermion hops (labeled i) may not be the same as the lattice formed by the local Hilbert spaces (labeled I).

Now, consider the stringlike product of hopping operators

$$W_{ir}=t_{ij}t_{jk}t_{kl}\ldots t_{pq}t_{qr}.$$

This operator destroys a fermion at site r and creates one at site i:

$$W_{ir}|r\ldots\rangle\propto|i\ldots\rangle,$$

where $|r\ldots\rangle,|i\ldots\rangle$ are low-energy states with fermions at r, i (and possibly other fermions far away). Furthermore, this operator clearly has a stringlike structure: it is made up of a stringlike product of operators, each of which is local in the underlying spin degrees of freedom. The only issue is that this string operator might be trivial. That is, W_{ir} might act trivially on all the intermediate spins at $j, k,\ldots,q$ and only have a nontrivial effect near i and r. In that case, W_{ir} isn't really a string at all.

This is a legitimate concern, since the string operator typically is trivial in the case of *bosonic* quasiparticles. Consider, for example, a lattice of noninteracting spins in a magnetic field: $H=-B\Sigma_i\sigma_i^3$. The ground state has $\sigma_i^3=1$ for all i. The excitations are obtained by flipping one spin: $\sigma_j^3=-1$ for some j. If we perturb the Hamiltonian by a term $t\Sigma_{\langle ij\rangle}\sigma_i^1\sigma_j^1$, these excitations acquire dynamics. The corresponding hopping operators are then $t_{ij}=t/2\sigma_i^1\sigma_j^1$ and the string operator is

$$W_{ir}=t_{ij}t_{jk}\ldots t_{qr}\propto(\sigma_i^1\sigma_j^1)(\sigma_j^1\sigma_k^1)\ldots(\sigma_q^1\sigma_r^1)=\sigma_i^1\sigma_r^1.$$

The operator W_{ir} clearly creates a particle at position i and destroys one at position r. However, as we see above, it is trivial and has no stringlike structure. The particle creation and annihilation operators are completely local.

Our claim is that this can never happen in the case of emerging fermions. That is, the string operator W_{ir} can never be written as a product

$$W_{ir}=A_iB_r,$$

where A_i, B_r are operators that act only within the local Hilbert spaces near sites i,r. Physically, this means that the fermion creation and annihilation operators are never local in the underlying bosonic degrees of freedom. Instead, they naturally appear in pairs, with a stringlike structure connecting them.[31]

Proof: We wish to use the algebraic formula (4) from the previous section. That formula relied on the locality of the hopping operators (3). Here, we would like to use a weaker locality condition

$$[t_{ij},t_{kl}]=0 \tag{5}$$

if i,j are *far* from k,l. One can make a similar argument using this weaker assumption. One arrives at a slightly weaker algebraic relation

$$W_{il}W_{ki}W_{ij}=-W_{ij}W_{ki}W_{il}$$

if i, l, and k are sufficiently far from each other. Since $W_{ki}W_{ij}=W_{kj}$, we can rewrite this as

$$W_{il}W_{kj}=-W_{ij}W_{kl}. \tag{6}$$

Condition (6) is a direct consequence of the fermionic nature of the quasiparticles. We will now show that it implies that the strings are nontrivial. Indeed, if the W_{ir} could be written as $W_{ir}=A_iB_r$, then

$$W_{il}W_{kj}=A_iB_lA_kB_j,$$

$$W_{ij}W_{kl}=A_iB_jA_kB_l.$$

But the A's and B's are local operators, so they commute with each other when they are well separated. We can therefore rearrange the operators in these two equations to obtain

$$W_{il}W_{kj}=W_{ij}W_{kl}.$$

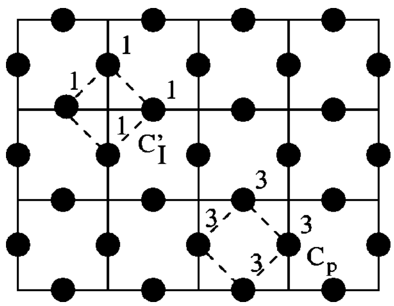

FIG. 2. Schematic diagram depicting Kitaev's model. Each spin is drawn as a dot. The paths $C'_{\mathbf{I}}$ and $C_{\mathbf{p}}$ are drawn as dotted diamonds and are labeled with 1's or 3's according to whether the corresponding term in the Hamiltonian involves σ^1's or σ^3's.

This directly contradicts the fermion condition (6). We conclude that the string operator is always nontrivial.

The presence of this nontrivial string indicates that fermions always appear together with some kind of gauge field. One way to see this is to consider a closed loop of hopping operators $t_{ij}t_{jk}\ldots t_{pq}t_{qi}$. This string can be interpreted as a Wilson loop operator, since its phase is precisely the accumulated phase of the particle when it traverses a loop. The fact that it is nontrivial (that is, not equal to the identity operator) means that the particle is coupled to a nontrivial gauge field.

V. A 2D EXAMPLE

In this section, we present an exactly soluble lattice spin model with fermionic excitations. The exactly soluble model provides a concrete realization of the string picture discussed above. The emerging fermions turn out to be coupled to a Z_2 gauge field.

In this spin-$\frac{1}{2}$ system, proposed by Kitaev,[15] the spins live on the links of a square lattice (Fig. 2). The Hamiltonian is

$$H=-U\sum_{\mathbf{I}}\left(\prod_{C'_{\mathbf{I}}}\sigma^1_{\mathbf{j}}\right)-g\sum_{\mathbf{p}}\left(\prod_{C_{\mathbf{p}}}\sigma^3_{\mathbf{j}}\right). \tag{7}$$

Here $\mathbf{i}$ labels the links, $\mathbf{I}$ the sites, and $\mathbf{p}$ the plaquettes of the square lattice. Also, $C'_{\mathbf{I}}$ denotes the loop connecting the four spins adjacent to site $\mathbf{I}$, while $C_{\mathbf{p}}$ the loop connecting the four spins adjacent to plaquette $\mathbf{p}$ (see Fig. 2).

This model is exactly soluble since all the terms in the Hamiltonian commute with each other. The ground state satisfies

$$\prod_{C'_{\mathbf{I}}}\sigma^1_{\mathbf{j}}=1$$

for all sites I and

$$\prod_{C_{\mathbf{p}}}\sigma^3_{\mathbf{j}}=1$$

for all plaquettes $\mathbf{p}$. There are two types of (localized) excited states. We can have a site at which

$$\prod_{C'_{\mathbf{I}}}\sigma^1_{\mathbf{j}}=-1$$

or we can have a plaquette at which

$$\prod_{C_{\mathbf{p}}}\sigma^3_{\mathbf{j}}=-1.$$

We call the first type of excitation a "charge" and the second type of excitation a "flux." Static charge and flux configurations are exact eigenstates of the above Hamiltonian. Thus, the charge and flux quasiparticles have no dynamics.

This lack of dynamics is a special feature of the above model. However, we are interested in the properties of a *generic* Hamiltonian in the same quantum phase. Thus, we need to perturb the system and analyze the resulting dynamics. The simplest nontrivial perturbation is

$$H'=H+J_1\sum_{\mathbf{i}}\sigma^1_{\mathbf{i}}+J_3\sum_{\mathbf{i}}\sigma^3_{\mathbf{i}}. \tag{8}$$

It's not hard to see that the first term allows the fluxes to hop from plaquettes to adjacent plaquettes, while the second term allows the charges to hop from sites to neighboring sites.

We now calculate the statistics of the fluxes. To do this, we restrict our Hamiltonian to the low-energy subspace with n fluxes and zero charges. Within this subspace, our Hamiltonian reduces to

$$H'_{\text{eff}}=J_1\sum_{\mathbf{i}}\sigma^1_{\mathbf{i}}.$$

To make contact with our previous formalism, we write this as

$$H'_{\text{eff}}=\sum_{\langle\mathbf{pq}\rangle}(t_{\mathbf{pq}}+t_{\mathbf{qp}}),$$

where the sum is taken over adjacent plaquettes $\mathbf{p},\mathbf{q}$; $t_{\mathbf{pq}}$ is defined by $t_{\mathbf{pq}}=J_1/2\sigma^1_{\mathbf{i}}$; and $\mathbf{i}$ is the link joining $\mathbf{p}$ and $\mathbf{q}$.

To calculate the statistics, we need to compare $t_{\mathbf{pq}}t_{\mathbf{rp}}t_{\mathbf{ps}}$ with $t_{\mathbf{ps}}t_{\mathbf{rp}}t_{\mathbf{pq}}$. It's obvious from the definition that all the hopping operators $t_{\mathbf{pq}}$ commute with one another. Therefore,

$$t_{\mathbf{pq}}t_{\mathbf{rp}}t_{\mathbf{ps}}=t_{\mathbf{ps}}t_{\mathbf{rp}}t_{\mathbf{pq}}.$$

We conclude that the fluxes are bosons. In the same way, one can show that the charges are also bosons.

Next, we consider the bound state of a flux and a charge. That is, we consider excitations with a flux through some plaquette $\mathbf{p}$ and also a charge at one of the sites $\mathbf{I}$ adjacent to $\mathbf{p}$.

These bound states are not actually stable for the above Hamiltonian (8)—the charge and flux will separate from one another over time. However, one can imagine modifying the Hamiltonian so that charges and fluxes prefer to be adjacent to each other. In this case, the bound state is a true quasiparticle.

Suppose we've made such a modification. We can then consider the statistics of the bound state. If we restrict our

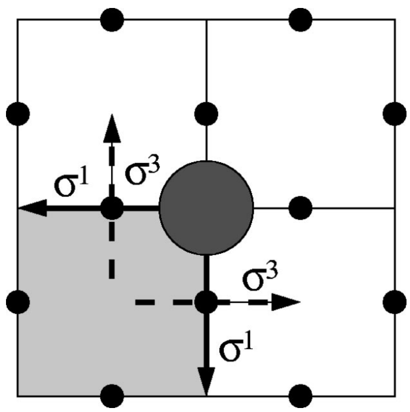

FIG. 3. A bound state of a charge, denoted by a dark circle, and a flux, denoted by a shaded square. The arrows illustrate the four ways the bound state can hop. The two charge hops are denoted by solid arrows and have hopping operator σ^3. The flux hops are denoted by dotted arrows and have hopping operator σ^1. Notice that each charge hopping operator anticommutes with a corresponding flux hopping operator, but everything else commutes.

Hamiltonian (8) to the low-energy subspace with n bound states, then our Hamiltonian reduces to

$$H'_{\text{eff}}=J_1\sum_{\mathbf{i}}\ \sigma^1_{\mathbf{i}}+J_3\sum_{\mathbf{i}}\ \sigma^3_{\mathbf{i}}.$$

To understand the effect of these terms, we imagine that we have a bound state with a flux at $\mathbf{p}$ and a charge at $\mathbf{I}$. It's not hard to see that the first term allows the flux to hop to the two neighboring plaquettes, which are also adjacent to $\mathbf{I}$. Similarly, the second term allows the charge to hop to the two neighboring sites, adjacent to $\mathbf{p}$. All other hopping destroys the bound state and is therefore forbidden for energetic reasons. Thus, the perturbation gives rise to four types of hopping operators—two corresponding to fluxes and two to charges (see Fig. 3).

Formally, we can write our low-energy Hamiltonian as

$$H'_{\text{eff}}=\sum_{\langle(\mathbf{p},\mathbf{I})(\mathbf{q},\mathbf{J})\rangle}[t_{(\mathbf{p},\mathbf{I})(\mathbf{q},\mathbf{J})}+t_{(\mathbf{q},\mathbf{J})(\mathbf{p},\mathbf{I})}],$$

where the hopping operators are defined by

$$t_{(\mathbf{p},\mathbf{I})(\mathbf{q},\mathbf{J})}=\frac{J_1}{2}\sigma^1_{\mathbf{i}}\ \text{ or }\ \frac{J_3}{2}\sigma^3_{\mathbf{i}},$$

depending on whether $(\mathbf{p},\mathbf{I})$, $(\mathbf{q},\mathbf{J})$ differ by a flux hop or a charge hop. In the first case, $\mathbf{i}$ is defined to be the link joining $\mathbf{p}$ and $\mathbf{q}$, while in the second case, $\mathbf{i}$ is the link joining $\mathbf{I}$ and $\mathbf{J}$.

To calculate the statistics we need to compare a product of the form $t_1t_2t_3$ with the product $t_3t_2t_1$, where the t_k are hopping operators involving a single bound state at position $(\mathbf{p},\mathbf{I})$. (We could write out these expressions precisely, but they are lengthy and not very enlightening.)

Now, as we discussed above, there are four different ways that a bound state at $(\mathbf{p},\mathbf{I})$ can hop: two charge hops and two flux hops. Each of the flux hopping operators can be paired with a corresponding charge operator that involves the same link $\mathbf{i}$. It's easy to see that each of the flux operators anticommutes with the corresponding charge operator (since one involves a σ^1, while the other involves a σ^3). However, everything else commutes (see Fig. 3).

With these facts in mind, we can compare $t_1t_2t_3$ with $t_3t_2t_1$. By our discussion above, any set of three hopping operators involving $(\mathbf{p},\mathbf{I})$ must contain exactly two that anticommute. Thus, exactly two of t_1, t_2, and t_3 anticommute. This implies that

$$t_1t_2t_3=-t_3t_2t_1.$$

We conclude that the bound states are fermions. Of course, this result is not that surprising once we notice that the charges and fluxes have relative statistics $e^{i\phi_{\text{rel}}}=-1$.

We can easily check that the charges and fluxes have relative statistics -1. Normally, we would just use the formula (A1) derived in the appendix. However, this relation assumes that the two types of particles both hop on the same lattice. In our case, the charges hop on the square lattice, while the fluxes hop on the dual lattice.

It's not hard to see that in our case the formula (A1) needs to be modified to

$$t_{\mathbf{IJ}}t_{\mathbf{pq}}=e^{i\phi_{\text{rel}}}t_{\mathbf{pq}}t_{\mathbf{IJ}},$$

where $\mathbf{I},\mathbf{J}$ are neighboring sites, $\mathbf{p},\mathbf{q}$ are neighboring plaquettes, and the links connecting $\mathbf{I},\mathbf{J}$ and $\mathbf{p},\mathbf{q}$ are the same. From our previous calculations we know that $t_{\mathbf{pq}}=J_1/2\sigma^1_{\mathbf{i}}$ and $t_{\mathbf{IJ}}=J_3/2\sigma^3_{\mathbf{i}}$. Therefore, $t_{\mathbf{pq}}$, $t_{\mathbf{IJ}}$ anticommute, and the relative statistics are $e^{i\phi_{\text{rel}}}=-1$.

We now see why the two particles were called "charges" and "fluxes"—they have the same statistics and relative statistics as Z_2 charges and fluxes. It turns out that this connection with Z_2 gauge theory extends beyond the low-energy regime—in fact, it extends all the way to the lattice scale. One can show that the Kitaev model is exactly equivalent to standard Z_2 gauge theory coupled to a Z_2 Higgs field.

We argued earlier that whenever fermions or anyons occur in a bosonic system, they are always created in pairs, and the pair-creation operator has a stringlike structure. The above exactly soluble model (7) provides a good example of this phenomenon.

We begin with the charges. We can construct the string operators associated with these particles by taking products of their hopping operators along some path $P=\mathbf{I}_1\ldots\mathbf{I}_n$ on the lattice. We find

$$W(P)=t_{\mathbf{I}_1\mathbf{I}_2}t_{\mathbf{I}_2\mathbf{I}_3}\ldots t_{\mathbf{I}_{n-2}\mathbf{I}_{n-1}}t_{\mathbf{I}_{n-1}\mathbf{I}_n}\propto\sigma^3_{\mathbf{i}_1}\sigma^3_{\mathbf{i}_2}\ldots\sigma^3_{\mathbf{i}_{n-2}}\sigma^3_{\mathbf{i}_{n-1}}, \tag{9}$$

where $\mathbf{i}_1\ldots\mathbf{i}_{n-1}$ are the links along the path (see Fig. 4). Notice that $W(P)$ is nontrivial; it acts nontrivially on the spin degrees of freedom along the string P.

It is also a pair-creation operator: If we apply $W(P)$ to the ground state, the resulting state is an exact eigenstate with two charges, one located at each end point of P. One way to see this is to notice that $W(P)$ commutes with everything in

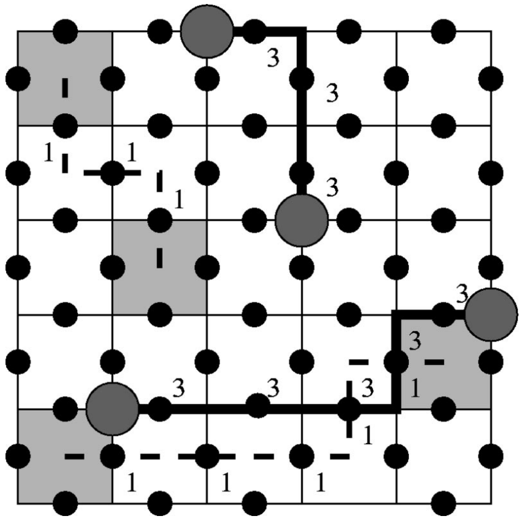

FIG. 4. Three examples of stringlike creation operators. The string operators are drawn together with the particles they create at their end points. The thick solid line (drawn on the lattice) is an example of a charge string operator. The dotted line (drawn on the dual lattice) is an example of a flux string operator. At the bottom, we give an example of a string operator for the bound state of a charge and flux. It is essentially a combination of a charge string and a flux string.

the Hamiltonian except $\Pi_{C'_{\mathbf{I}_1}}\sigma^1_{\mathbf{j}}$ and $\Pi_{C'_{\mathbf{I}_n}}\sigma^1_{\mathbf{j}}$. The string anticommutes with these two operators, so when we apply it to the ground state, we get a state with two charges located at $\mathbf{I}_1,\mathbf{I}_n$.

The case of the flux quasiparticles is very similar. We take the product of the flux hopping operators along some path $P'=\mathbf{p}_1\ldots\mathbf{p}_n$ on the dual lattice. We find

$$W'(P')=t_{\mathbf{p}_1\mathbf{p}_2}t_{\mathbf{p}_2\mathbf{p}_3}\ldots t_{\mathbf{p}_{n-2}\mathbf{p}_{n-1}}t_{\mathbf{p}_{n-1}\mathbf{p}_n}\propto\sigma^1_{\mathbf{i}_1}\sigma^1_{\mathbf{i}_2}\ldots\sigma^1_{\mathbf{i}_{n-2}}\sigma^1_{\mathbf{i}_{n-1}}, \tag{10}$$

where $\mathbf{i}_1\ldots\mathbf{i}_{n-1}$ are the links along the path (see Fig. 4). Just as before, one can show that the string $W'(P')$ is a creation operator that creates two fluxes at the end points of P'.

Finally, consider the case of the bound state of the charge and the flux. The hopping operator for the bound state is a combination of the charge and flux hopping operators, so the string turns out to be a combination of the charge and flux strings. Let $P=(\mathbf{p}_1,\mathbf{I}_1)\ldots(\mathbf{p}_n,\mathbf{I}_n)$ be a path in bound-state configuration space. Then the associated bound-state string operator is

$$W(P)\propto\prod_k \sigma^{a_k}_{\mathbf{i}_k}, \tag{11}$$

where $\mathbf{i}_1\ldots\mathbf{i}_{n-1}$ are the links along the path and $a_k=1$ or 3 depending on whether $(\mathbf{p}_k,\mathbf{I}_k)$ differs by a flux hop or a charge hop, respectively (see Fig. 4). Once again, one can show that the string $W(P)$ is a creation operator for a pair of bound states located at the ends of P.

In each of these examples, the creation operator for fermions or anyons has an extended stringlike structure. It is important to note that the position of this string is completely unobservable. That is, in each case, the excited state $W(P)|0\rangle$ is independent of the position of P: it only depends on the position of the *end points* of P.

VI. EXACTLY SOLUBLE 3D MODEL

In two-dimensional systems, one can create a fermion by binding a Z_2 vortex to a Z_2 charge. This is how we obtained the fermion in the above spin-$\frac{1}{2}$ model on the square lattice. Since both the Z_2 vortex and Z_2 charge appear as the ends of open strings, the fermions also appear as the ends of strings. However, in three dimensions, we cannot change a boson into a fermion by attaching a π flux. Thus one may wonder if fermions still appear as the ends of strings in $(3+1)$ dimensions. In this section, we study an exactly soluble spin-$\frac{3}{2}$ model on a cubic lattice. We will show that the creation operator for fermions does indeed have a stringlike structure. This example demonstrates that the string picture for fermions is more general then the flux-charge picture.

Our model has four states for each site of a cubic lattice. Thus we call it a spin-$\frac{3}{2}$ model. Let γ^{ab}, $a,b\in\{x,\bar{x},y,\bar{y},z,\bar{z}\}$, $a\neq b$, be 4×4 Hermitian matrices that satisfy

$$\gamma^{ab}=-\gamma^{ba}=(\gamma^{ab})^\dagger,$$

$$[\gamma^{ab},\gamma^{cd}]=0,\quad \text{if } a,b,c,d \text{ are all different},$$

$$\gamma^{ab}\gamma^{bc}=i\gamma^{ac},\quad a\neq c,$$

$$(\gamma^{ab})^2=1. \tag{12}$$

A solution to the above algebra can be constructed by taking pairwise products of Majorana fermion operators λ^a, $a\in\{x,\bar{x},y,\bar{y},z,\bar{z}\}$:

$$\gamma^{ab}=i\lambda^a\lambda^b,$$

$$\{\lambda^a,\lambda^b\}=2\delta_{ab}. \tag{13}$$

The six Majorana fermion operators naturally require a space of dimension $2^{6/2}=8$, but if we restrict them to the space $\Pi_a\lambda^a=1$, we obtain the desired 4×4 Hermitian matrices. Alternatively, a more concrete description of the γ^{ab} is given in Appendix B, where we express the γ^{ab} in terms of Dirac matrices.

In terms of $\gamma^{ab}_{\mathbf{i}}$, the exactly soluble spin-$\frac{3}{2}$ Hamiltonian can be written as

$$H=-g\sum_{\mathbf{p}} F_{\mathbf{p}}, \tag{14}$$

where $\mathbf{p}$ labels all the square plaquettes in the cubic lattice and the $F_{\mathbf{p}}$ are "flux" operators defined by

$$F_{\mathbf{p}}=\begin{cases}\gamma^{yx}_{\mathbf{i}}\gamma^{\bar{x}\bar{y}}_{\mathbf{i}+\hat{\mathbf{x}}}\gamma^{\bar{y}\bar{x}}_{\mathbf{i}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}\gamma^{x\bar{y}}_{\mathbf{i}+\hat{\mathbf{y}}} & \text{or}\\ \gamma^{zy}_{\mathbf{i}}\gamma^{\bar{y}\bar{z}}_{\mathbf{i}+\hat{\mathbf{y}}}\gamma^{\bar{z}\bar{y}}_{\mathbf{i}+\hat{\mathbf{y}}+\hat{\mathbf{z}}}\gamma^{y\bar{z}}_{\mathbf{i}+\hat{\mathbf{z}}} & \text{or}\\ \gamma^{xz}_{\mathbf{i}}\gamma^{\bar{z}\bar{x}}_{\mathbf{i}+\hat{\mathbf{z}}}\gamma^{\bar{x}\bar{z}}_{\mathbf{i}+\hat{\mathbf{z}}+\hat{\mathbf{x}}}\gamma^{z\bar{y}}_{\mathbf{i}+\hat{\mathbf{x}}}\end{cases} \tag{15}$$

depending on the orientation of the plaquette $\mathbf{p}$. Just as in the Kitaev model,[15] all the $F_{\mathbf{p}}$ commute with each other and all the $F_{\mathbf{p}}$ have only two eigenvalues: ± 1. Thus, we can solve the Hamiltonian by simultaneously diagonalizing all the $F_{\mathbf{p}}$. If $|\{f_{\mathbf{p}}\}\rangle$ is a common eigenstate of $F_{\mathbf{p}}$ with $F_{\mathbf{p}}|\{f_{\mathbf{p}}\}\rangle = f_{\mathbf{p}}|\{f_{\mathbf{p}}\}\rangle$, $f_{\mathbf{p}} = \pm 1$, then it is also an energy eigenstate with energy

$$E(\{f_{\mathbf{p}}\}) = -g\sum_{\mathbf{p}} f_{\mathbf{p}}.$$

The one subtlety is that the $f_{\mathbf{p}}$ are not all independent. If C is the surface of a unit cube, then we have the operator identity

$$\prod_{\mathbf{p}\in C} F_{\mathbf{p}} = 1.$$

It follows that

$$\prod_{\mathbf{p}\in C} f_{\mathbf{p}} = 1$$

for all cubes C. This constraint means that the spectrum of our model is identical to a Z_2 gauge theory on a cubic lattice. The ground state can be thought of as a state with no flux: $f_{\mathbf{p}} = 1$ for all $\mathbf{p}$. Similarly, the elementary excitations are small flux loops where

$$f_{\mathbf{p}} = -1$$

for the four plaquettes $\mathbf{p}$ adjacent to some link $\langle \mathbf{ij}\rangle$. We can think of these excitations as quasiparticles that live on the *links* of the cubic lattice.

We would like to compute the statistics of these excitations. It is tempting to assume that they are bosons since the model is almost the same as a Z_2 gauge theory. However, as we will show, this superficial similarity is misleading: the flux loops are actually fermions.

As in the Kitaev model, the quasiparticles are exact eigenstates of our Hamiltonian. Thus, they have no dynamics and it is difficult to compute their statistics. However, this is a special feature of our model. We need to perturb the theory to understand generic states in the same quantum phase. The simplest perturbation is

$$H' = H + t\sum_{\mathbf{i},a,b} \gamma_{\mathbf{i}}^{ab}.$$

To compute the statistics, we need to restrict ourselves to the low-energy subspace with n quasiparticles. In this subspace, H' reduces to

$$H'_{\text{eff}} = t\sum_{\mathbf{i},a,b} \gamma_{\mathbf{i}}^{ab}.$$

The effect of each term $\gamma_{\mathbf{i}}^{ab}$ is to allow the quasiparticles to hop between the two links $\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{a}})\rangle$ and $\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{b}})\rangle$ adjacent to site $\mathbf{i}$. Here $\hat{\mathbf{a}} = \hat{\mathbf{x}}$ if $a = x$, $\hat{\mathbf{a}} = -\hat{\mathbf{x}}$ if $a = \bar{x}$, etc. (See Fig. 5.) Thus, our Hamiltonian can be written in the standard hopping form:

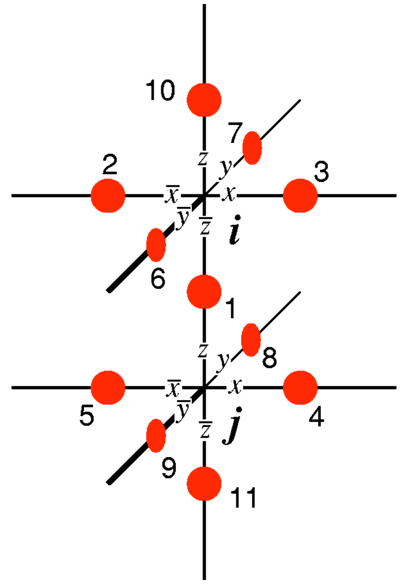

FIG. 5. A fermion on link 1 can hop onto ten different links: 2–11. The hopping $1\rightarrow 2$ is generated by $\gamma_{\mathbf{i}}^{\bar{x}\bar{z}}$, the hopping $1\rightarrow 10$ by $\gamma_{\mathbf{i}}^{z\bar{z}}$, and the hopping $1\rightarrow 4$ by $\gamma_{\mathbf{j}}^{xz}$.

$$H'_{\text{eff}} = \sum_{\mathbf{i},a,b} [t_{\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{a}})\rangle\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{b}})\rangle} + t_{\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{b}})\rangle\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{a}})\rangle}],$$

where $t_{\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{a}})\rangle\langle \mathbf{i}(\mathbf{i}+\hat{\mathbf{b}})\rangle} = t/2\gamma_{\mathbf{i}}^{ab}$.

To calculate the statistics we need to compare a product of the form $t_1 t_2 t_3$ with the product $t_3 t_2 t_1$, where the t_k are hopping operators involving a single link $\langle \mathbf{ij}\rangle$. (As in the Kitaev model, we could write out these expressions explicitly, but the results are not very enlightening.)

Note that a quasiparticle on the link $\langle \mathbf{ij}\rangle$ can hop to any of the ten neighboring links, five of which are adjacent to site $\mathbf{i}$, and five to site $\mathbf{j}$ (see Fig. 5). Using the algebra (12) or the Majorana fermion representation, we find that the five hopping operators associated with site $\mathbf{i}$ all anticommute with each other and similarly for site $\mathbf{j}$. On the other hand, each of the operators associated with $\mathbf{i}$ commutes with each of the operators associated with $\mathbf{j}$.

With these facts in mind, we can now compare $t_1 t_2 t_3$ with $t_3 t_2 t_1$. There are essentially two cases: either all three of the t_k's are associated with a single site $\mathbf{i}$ or $\mathbf{j}$, or two involve one site and one involves the other. In the first case, all the t_k's anticommute; in the second case, a pair of the t_k's anticommute and everything else commutes. In either case, we have

$$t_1 t_2 t_3 = -t_3 t_2 t_1.$$

We conclude that the quasiparticles are indeed fermions.

Next, we construct the associated string operator. Taking a product of hopping operators along a path (of links) $P = \langle \mathbf{i}_1\mathbf{i}_2\rangle, \langle \mathbf{i}_2\mathbf{i}_3\rangle \ldots \langle \mathbf{i}_{n-1}\mathbf{i}_n\rangle$ gives

$$W(P) \propto \gamma_{\mathbf{i}_1}^{a_1 b_1}\gamma_{\mathbf{i}_2}^{a_2 b_2}\ldots\gamma_{\mathbf{i}_n}^{a_n b_n}, \tag{16}$$

where b_m, a_{m+1} is the pair of indices associated with the link $\langle \mathbf{i}_m \mathbf{i}_{m+1}\rangle$. (See Fig. 6.) Notice that the string operator commutes with the Hamiltonian, except at the ends. Thus, when

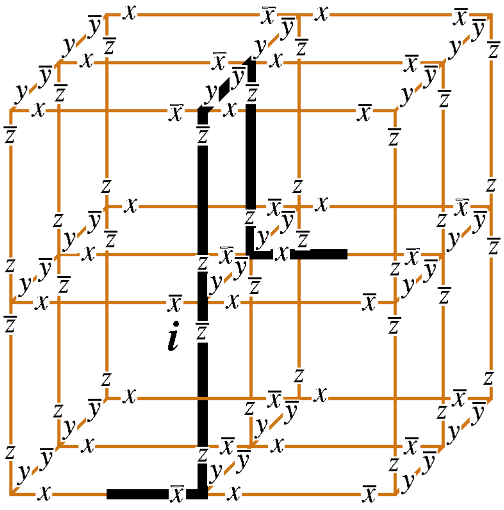

FIG. 6. An example of a string in the 3D model. This string corresponds to the operator $W(P)\propto \gamma^{\bar{x}z}_{\mathbf{i}-\hat{\mathbf{z}}}\gamma^{\bar{z}z}_{\mathbf{i}}\gamma^{\bar{z}y}_{\mathbf{i}+\hat{\mathbf{z}}}\gamma^{\bar{y}z}_{\mathbf{i}+\hat{\mathbf{z}}+\hat{\mathbf{y}}}\gamma^{\bar{z}x}_{\mathbf{i}+\hat{\mathbf{y}}}$.

we apply $W(P)$ to the ground state, the only effect is to flip the signs of $f_{\mathbf{p}}$ for the plaquettes $\mathbf{p}$ adjacent to the links $\langle \mathbf{i}_1\mathbf{i}_2\rangle, \langle \mathbf{i}_{n-1}\mathbf{i}_n\rangle$. This means that the open string operator creates a pair of fermions at its two ends.

VII. CONCLUSION

In this paper we have derived a general relation (4) between the statistics of a lattice particle and the algebra of its hopping operators. This relation allows us to analyze emerging fermions in three dimensions. We have also shown that there is a close connection between strings and fermions. The statistical algebra (4) is fundamental to this connection since it allows us to determine the statistics of the ends of strings from the structure of the string operator.

It is interesting to put this string picture of emerging fermions in context. Indeed, it is well-known that gauge theories and strings are closely related—the strings correspond to electric flux lines in the associated gauge theory.[23–29] Thus, it appears that the fundamental concepts of Fermi statistics, gauge theory, and strings are all connected. They are all just different aspects of a kind of order—topological order—in bosonic lattice systems.

ACKNOWLEDGMENTS

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APPENDIX A: RELATIVE STATISTICS

In this section, we derive an algebraic formula, analogous to Eq. (4), for the relative statistics of distinguishable particles. We begin with a Hilbert space that describes two types of hard-core particles hopping on a 2D lattice. For concreteness, say that there are m particles of type 1 and n particles of type 2. The states can be labeled by listing the positions of the two types of particles: $|i_1,\ldots,i_m;\ j_1,\ldots,j_n\rangle$.

A typical Hamiltonian for this system is of the form

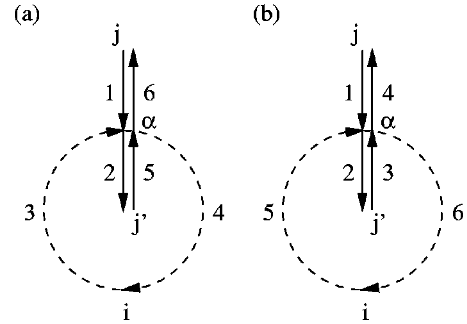

FIG. 7. (a) The path of the particles in the construction of the wound state $|i;j,\text{wound}\rangle$. The dotted line is the path of particle 1, and the solid line is the path of particle 2. The numbers label the order in which the paths are traversed. (b) The path of the particles in the unwound state $|i;j,\text{unwound}\rangle$. Notice that the two states only differ in the ordering of the paths. However, in one case the particles wind around each other, while in the other, they do not.

$$H=\sum_{\langle ij\rangle}(t^1_{ij}+t^1_{ji}+t^2_{ij}+t^2_{ji}),$$

where t^1_{ij},t^2_{ij} are hopping operators for the two types of particles.

We wish to calculate the relative statistics of the two types of particles. As before, the statistics are related to the simple algebraic properties of the hopping operators. Specifically, we will show that the particles have relative statistics $e^{i\phi}$ if

$$(t^2_{ip}t^2_{pj})(t^1_{kp}t^1_{pl})=e^{i\phi}(t^1_{kp}t^1_{pl})(t^2_{ip}t^2_{pj}). \tag{A1}$$

Here, i, k, j, and l are (distinct) neighbors of p oriented in the clockwise direction.

We begin with the state $|i;j\ldots\rangle$ with a type-1 particle at i, a type-2 particle at j, and other particles that are far away.

Imagine that we wind particle 1 around particle 2 using an appropriate product of hopping operators. One expression for the final "wound" state is

$$|i;j,\text{wound}\rangle=(t^2_{j\alpha})(t^2_{\alpha j'})(t^1_{il}\ldots t^1_{m\alpha})\times(t^1_{\alpha n}\ldots t^1_{pq}t^1_{qi})(t^2_{j'\alpha})(t^2_{\alpha j})|i;j\ldots\rangle.$$

Here, we've grouped the hopping operators into 6 terms. The cumulative effect of these terms is to wind one particle around the other [see Fig. 7(a)]. Similarly, we can construct a state where the two particles don't wind around each other [see Fig. 7(b)]:

$$|i;j,\text{unwound}\rangle=(t^1_{il}\ldots t^1_{m\alpha})(t^1_{\alpha n}\ldots t^1_{pq}t^1_{qi})(t^2_{j\alpha})\times(t^2_{\alpha j'})(t^2_{j'\alpha})(t^2_{\alpha j})|i;j\ldots\rangle.$$

As before, the two states involve the same path in configuration space. Thus, the phase difference between the states is precisely the relative statistics of the particles. This can be made rigorous using an argument similar to the exchange statistics case.

We now calculate this phase difference. We use the assumed algebraic relation

$$(t^2_{j\alpha}t^2_{\alpha j'})(t^1_{m\alpha}t^1_{\alpha n}) = e^{i\phi}(t^1_{m\alpha}t^1_{\alpha n})(t^2_{j\alpha}t^2_{\alpha j'}).$$

Applying this relation, and reordering the hopping operators using locality, we find

$$|i;j,\mathrm{wound}\rangle = e^{i\phi}|i;j,\mathrm{unwound}\rangle.$$

This establishes the desired result: the particles have relative statistics $e^{i\phi}$.

A good consistency check for this formula can be obtained by considering the relative statistics of *two particles* of the *same type*. In this case, we expect the angle for the relative statistics to be exactly twice the angle for the exchange statistics:

$$e^{i\phi_{\mathrm{rel}}} = e^{2i\theta_{\mathrm{stat}}}.$$

One way to see this is that exchanging the particles twice in the same direction is topologically equivalent to winding one particle around the other.

This result, which has a topological character to it, can be derived algebraically from our two formulas. We start with the expression $(t_{ip}t_{pj})(t_{kp}t_{pl})$. Applying the exchange statistics formula, Eq. (4), we find

$$(t_{ip}t_{pj})(t_{kp}t_{pl}) = t_{ip}(t_{pj}t_{kp}t_{pl}) = t_{ip}(e^{i\theta_{\mathrm{stat}}}t_{pl}t_{kp}t_{pj}).$$

Applying the formula again gives

$$t_{ip}(t_{pl}t_{kp}t_{pj}) = (t_{ip}t_{pl}t_{kp})t_{pj} = (e^{i\theta_{\mathrm{stat}}}t_{kp}t_{pl}t_{ip})t_{pj}.$$

Combining these two equations gives

$$(t_{ip}t_{pj})(t_{kp}t_{pl}) = e^{2i\theta_{\mathrm{stat}}}(t_{kp}t_{pl})(t_{ip}t_{pj}).$$

Comparing this with the relative statistics formula, Eq. (A1), we see that $e^{i\phi_{\mathrm{rel}}} = e^{2i\theta_{\mathrm{stat}}}$, as claimed.

APPENDIX B: DIRAC MATRIX REPRESENTATION OF γ^{ab}

In the following, we represent γ^{ab}'s in terms of the Dirac matrices. Note that Eq. (12) implies that

$$\{\gamma^{ac},\gamma^{bc}\} = 2\,\delta_{ab}.$$

Thus γ^{xz}, $\gamma^{\bar{x}z}$, γ^{yz}, and $\gamma^{\bar{y}z}$ satisfy the algebra of Dirac matrices. Introducing four Dirac matrices

$$\gamma^{x} = \sigma^1\otimes\sigma^1, \quad \gamma^{\bar{x}} = \sigma^2\otimes\sigma^1,$$

$$\gamma^{y} = \sigma^3\otimes\sigma^1, \quad \gamma^{\bar{y}} = \sigma^0\otimes\sigma^2,$$

we can write

$$\gamma^{az} = \gamma^{a}, \quad a = x,\bar{x},y,\bar{y}. \tag{B1}$$

Similarly, $\gamma^{x\bar{z}}$, $\gamma^{\bar{x}\bar{z}}$, $\gamma^{y\bar{z}}$, and $\gamma^{\bar{y}\bar{z}}$ satisfy the algebra of Dirac matrices. We can express those operators as

$$\gamma^{a\bar{z}} = i\gamma^{a}\gamma^{5}, \quad a = x,\bar{x},y,\bar{y}, \tag{B2}$$

where $\gamma^5 = \gamma^x\gamma^{\bar{x}}\gamma^y\gamma^{\bar{y}}$. Finally, for $a,b = x,\bar{x},y,\bar{y}$, we have

$$\gamma^{ab} = i\gamma^{a}\gamma^{b}. \tag{B3}$$

In this way, we can express all the γ^{ab}, $a,b \in \{x,\bar{x},y,\bar{y},z,\bar{z}\}$, $a\neq b$, in terms of Dirac matrices.

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[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[2] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[3] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[4] G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Commun. **63**, 973 (1987).
[5] G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, 580 (1988).
[6] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[7] X.-G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11 413 (1989).
[8] D. Kheshchenko and P. Wiegmann, Mod. Phys. Lett. B **3**, 1383 (1989).
[9] N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991).
[10] X.-G. Wen, Phys. Rev. B **44**, 2664 (1991).
[11] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987).
[12] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
[13] N. Read and B. Chakraborty, Phys. Rev. B **40**, 7133 (1989).
[14] R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
[15] A. Y. Kitaev, quant-ph/9707021 (unpublished).
[16] S. Sachdev and K. Park, Ann. Phys. (N.Y.) **298**, 58 (2002).
[17] L. Balents, M. P. A. Fisher, and S. M. Girvin, Phys. Rev. B **65**, 224412 (2002).
[18] O. I. Motrunich and T. Senthil, Phys. Rev. Lett. **89**, 277004 (2002).
[19] L. B. Ioffe, M. V. Feigel'man, A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, Nature (London) **415**, 503 (2002).
[20] X.-G. Wen, Phys. Rev. Lett. **90**, 016803 (2003).
[21] O. I. Motrunich, Phys. Rev. B **67**, 115108 (2003).
[22] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[23] F. Wegner, J. Math. Phys. **12**, 2259 (1971).
[24] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[25] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
[26] T. Banks, R. Myerson, and J. B. Kogut, Nucl. Phys. B **129**, 493 (1977).
[27] A. M. Polyakov, Phys. Lett. **82B**, 247 (1979).
[28] A. M. Polyakov, Nucl. Phys. (Proc. Suppl.) **68A**, 1 (1998).
[29] X.-G. Wen, cond-mat/0210040, Phys. Rev. B (to be published).

[30] More precisely, the t_{ij} are *local at low energies*: the t_{ij} can be written as $t_{ij} = P\tilde{t}_{ij}P$, where $\tilde{t}_{ij}$ is local and commutes with the projection operator P.
[31] A more precise statement of our claim is that W_{ir} cannot be written as a product $P_sA_iB_rP_s$, where P_s is the projection operator onto the n fermion subspace with no particles at any of the intermediate sites $j,k,\ldots,q$, and A_i, B_r are local operators commuting with P. This statement is slightly stronger than the one made in the text, but the same proof applies.