Role of impurities in stabilizing quantum Hall effect plateaus

Alejandro Cabo Montes de Oca and Danny Martinez-Pedrera
Instituto de Cibernetica, Matematica y Fisica, Calle E 309, Vedado, Ciudad Habana, Cuba
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It is shown how the electromagnetic response of two-dimensional electron gas under quantum Hall effect regime transforms the sample impurities and defects in charge reservoirs that stabilize the Hall-conductivity plateaus. The results determine the basic dynamical origin of the singular properties of localization under the occurrence of the quantum Hall effect obtained in the pioneering works of Laughlin and of Joynt and Prange by means of a gauge invariance argument and a purely electronic analysis, respectively. The common intuitive picture of electrons moving along the equipotential lines gets an analytical realization.

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I. INTRODUCTION

Since its discovery in 1980, the integer quantum Hall effect (QHE) has been the subject of vast active research.[1,2] It is widely accepted that the explanation of this rather remarkable effect, particularly the existence of Hall-conductivity plateaus, is intimately connected with localization of electrons by sample impurities.[2–4]

In the case of a perfect noninteracting two-dimensional electron gas (2DEG) at sufficiently low temperature and with an exactly integral number n of Landau levels filled, the Hall conductivity $\sigma_{yx}=n_e ec/B=ne^2/h$, and this is a consequence of the well-known fact that each Landau level has a degeneracy eB/hc per unit area.[5] For the Hall conductivity to show these values (with integral n) for wide ranges of the quantity n_e/B and not only for the integral filling magnetic field values $B_n=n_e hc/ne$, which is indeed the essential feature of the QHE, there must be some sort of charge reservoir in order to adjust the number of electrons in extended current-carrying states to that required by a quantized conductivity.[6] The charge reservoir, it has been argued, is provided by sample impurities or imperfections, and the mechanism that of localization of electrons by the associated random potential. It has been shown that as long as the Fermi energy remains within a region of localized states, the electrons in extended states carry the right current for the Hall conductivity to be quantized.[3]

However, it is also recognized that a complete microscopic theory from which the properties of the effect could be deductively obtained is still missing.[5,7] Particularly, the situation regarding the role of impurities is not yet completely clear. The present work is intended to show how the electromagnetic response of the system, described by the Chern-Simons topological action as predicted by the field theory considered in Refs. 8, 9, and 10 effectively transforms the sample impurities in charge reservoirs, leading to a picture from which some basic properties of QHE could be derived. Moreover, for some ranges of variation of applied magnetic field the system is capable of adjusting the electron density in most of the sample points to those values required for satisfying the integral filling of Landau levels locally. This picture gives a fundamental explanation of the results of Refs. 11 and 3, highlighting the gauge invariance and topological character of the effect. The results also give support to the applicability of the field description of integer QHE presented in Ref. 12 where the mechanism of chemical potential stabilization in a gap remained unclear.

To simplify the discussion the analysis has been carried on a multi-quantum-well structure (superlattice),[13,14] where the field distribution of interest can be obtained analytically. The paper is organized as follows. In Sec. II we present the effective Maxwell equations and the corresponding Chern-Simons effective action. In Sec. III we describe our model for impurities in the superlattice and obtain the linear response associated with them. Here it is also detailed our fundamental argument, the way in which the system of impurities can act as a charge reservoir. Finally, in Sec. IV, it is discussed the stability of the proposed picture for QHE regime and the emergence of the Hall-conductivity plateaus is seen. Appendix A gives a very brief introduction to the effective action in field theory and shows its interpretation in terms of energy density, specifically when we are interested in deviations from a background field. Appendix B briefly develops an alternative phenomenological program for obtaining the basic effective Maxwell equations. Appendix C obtains a contribution to the energy density that is used in Sec. IV.

II. EFFECTIVE MAXWELL EQUATIONS

The electromagnetic response $a_\mu(x)=(i\phi,\mathbf{a})$ represents a linear disturbance of the electromagnetic potential associated with the presence of the constant magnetic field B_n at which the filling factor $\nu=n_e hc/eB$ has precisely the value n. In Ref. 8 it was found the effective Maxwell equations satisfied by the electromagnetic response of a superlattice in QHE regime from a calculation of the first quantum corrections to the effective action of a 2DEG in the presence of a magnetic field B_n within the long wavelength approximation $\lambda \gg r_0 = \sqrt{\hbar c/eB_n}$. The equations appropriate to a layered multi-quantum-well structure were obtained by simply adding the equations for each plane. These equations are

$$\partial^2 a_\mu(x)+i\frac{4\pi\sigma_H}{ca}\epsilon^{\mu\alpha\sigma\nu}n_\alpha\partial_\sigma a_\nu(x)+4\pi\frac{\chi_e}{a}[P_{\mu\nu}u_\alpha u_\beta + u_\mu u_\nu P_{\alpha\beta}-(u_\mu P_{\nu\alpha}+u_\nu P_{\mu\alpha})u_\beta]\partial_\alpha\partial_\beta a_\nu(x)=0, \quad (1)$$

which have been written in Lorentz-covariant form for evidencing the Chern-Simons terms, the ones proportional to σ_H. In these equations, a is the distance between successive 2DEG in the superlattice, σ_H and χ_e are respectively the Hall conductivity and the dielectric susceptibility of a single 2DEG at filling factor $\nu=n$, $P_{\mu\nu}$ is the projection tensor on 2DEG plane, u_μ is the four-velocity of the superlattice, and n_μ is a unit spacelike four-vector normal to the 2DEG plane. In a reference system where the sample is at rest and with three axis perpendicular to the 2DEG plane, all those are given by

$$\sigma_H=n\frac{e^2}{h},$$

$$\chi_e=n\frac{e^2}{h}\frac{mc}{eB_n},$$

$$P_{\mu\nu}=\text{diag}(0,1,1,0), \tag{2}$$

$$u_\mu=(1,0)=(1,0,0,0),$$

$$n_\mu=(0,\text{n})=(0,0,0,1).$$

Taking $a_3=0$ these effective Maxwell equations can be alternatively derived by extremizing with respect to $a_\mu(x)$ the effective action per unit length in x_3 direction

$$\tilde{\Gamma}[a_\mu]=\int d^2x dt\left(\frac{1}{8\pi}\epsilon E^2-\frac{1}{8\pi}B^2+\frac{i\sigma_H}{4ca}\epsilon^{\alpha\mu\nu}a_\alpha F_{\mu\nu}\right), \tag{3}$$

where $\epsilon=1+4\pi\chi_e/a$ and we recall that $B=(\nabla\times\mathbf{a})\cdot\mathbf{n}$ adds up to the integral filling field B_n. The third term in the integrand of Eq. (3) is recognized as the Chern-Simons topological action and produces the Chern-Simons terms in Eq. (1), which describes an effective current-density

$$\mathbf{J}^{\text{Hall}}=\frac{\sigma_H}{a}\mathbf{E}\times\mathbf{n}. \tag{4}$$

This expression reflects the existence of a local Hall conductivity and from it can be deduced the quantization of the Hall conductivity regardless the particular shape of the sample. The Chern-Simons terms can be alternatively obtained accepting the existence of a local Hall conductivity given by the last expression written and using the gauge invariance properties. This phenomenological program was developed in Ref. 15 and it is sketched in Appendix B for completion.

The effective action (3) corresponds to a situation in which there is a background field B_n, as is described within the formalism presented in Appendix A. As is shown there, it is proportional to the energy density required to produce a static deformation of the background field.

It can be underlined that this effective action embodies the structure of the electronic spectrum in the magnetic field, which is incorporated in the propagator for evaluating the relevant Feynman diagrams determining its form.[8] Thus, it should not be surprising that the energetics of the 2DEG system is reflected in it.

III. IMPURITY MODEL AND FIELD SOLUTION

In the present work, the impurity in superlattice sample will be modelled as a cylindrical hole of radius η having the axis normal to the 2DEG. Then we search for stationary axially-symmetric solutions to the effective Maxwell equations, which in this case reduces to consider the following equations for the scalar potential $\phi(r)$ and the azimuthal component of the vector potential $a_\theta(r)$:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)=-\frac{4\pi\sigma_H}{ca}\frac{1}{\epsilon}\frac{d}{dr}(ra_\theta),$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{da_\theta}{dr}\right)-\frac{a_\theta}{r^2}=\frac{4\pi\sigma_H}{ca}\frac{d\phi}{dr}.$$

These equations were investigated in Ref. 15. A solution, finite for $r\to\infty$, is given by

$$\phi(r)=\frac{C}{\sqrt{\epsilon}}K_0(kr),$$

$$a_\theta(r)=C\left(-\frac{1}{kr}+K_1(kr)\right), \tag{5}$$

where C is an overall constant factor to be determined, $K_0(x)$ and $K_1(x)$ are MacDonald functions of 0th and first order, respectively, and $k=4\pi\sigma_H/ca\sqrt{\epsilon}$. From these expressions, the electromagnetic-response fields of the superlattice are readily obtained as

$$E(r)=\frac{kC}{\sqrt{\epsilon}}K_1(kr),\ B(r)=-kCK_0(kr), \tag{6}$$

where E is the radial component of the electric field and the magnetic field is normal to the 2DEG's. Note that the magnetic field is proportional to the electric potential. These fields decay exponentially for $r\to\infty$, being k^{-1} a measure of its effective extension, and will be adopted as valid for the region external to the cylinder $r\geqslant\eta$. Within the interior, for completely defining the impurity model, the electric potential will be assumed constant and the magnetic intensity will also be assumed to be a constant vector along the axis orthogonal to the superlattice planes.

Using the expression (6) for $B(r)$, the magnetic flux Φ associated with the impurity can be calculated to be

$$\Phi=\int\mathbf{B}\cdot\mathbf{n}dS=\int_\eta^\infty B(r)2\pi r dr=-2\pi C\eta K_1(k\eta),$$

which in the limit $\eta\to 0$

$$\lim_{\eta\to 0}\Phi=-2\pi\frac{C}{k}. \tag{7}$$

The edge current which flows through the boundary of the cylinder can be simply evaluated from the condition that in conjunction with the continuous distributed Hall currents it would produce the just determined total magnetic flux.

Let us now consider the edge electric charge. Applying Gauss' law to the cylinder, the free edge charge q per unit length in x_3 direction in the impurity can be obtained as

$$q=\frac{1}{2}\eta\epsilon E(\eta^{+})=\frac{1}{2}C\sqrt{\epsilon}k\eta K_1(k\eta)$$

and

$$\lim_{\eta\to 0} q=\frac{1}{2}C\sqrt{\epsilon}. \tag{8}$$

After substituting the solution (5) in the expression (3) for the effective action we get

$$\Gamma=\frac{C^2}{4}K_0(k\eta)\int dt. \tag{9}$$

Using the connection between the effective action and the energy density shown in Appendix A, considering the contribution to the energy of the free edge charge, and using Eqs. (5), (8), and (9), we obtain for the mean energy density $\mathcal{U}_{\mathrm{imp}}$ associated with an impurity the expression

$$\mathcal{U}_{\mathrm{imp}}=-\frac{\Gamma}{A\int dt}+\frac{q\phi(\eta^{+})}{A}=\frac{C^2}{4A}K_0(k\eta), \tag{10}$$

where A is the sectional area of the sample. As has been previously commented, this is the energy required to produce a static deformation (5) of the integral filling homogeneous field B_n.

As the next step in the definition of the model we will consider N impurities such as that described above spread over the sample area to a mean separation $\xi\equiv\sqrt{A/N}$. The holes will be assumed to be small for having a total area much smaller than the sample surface. Hence, the electromagnetic mean field associated with the inspected many-electron state should be approximately given by superposing the fields of all N impurities. An illustrative picture of the spatial dependence on the 2DEG planes of the electric potential is shown in Fig. 1. At this point it should be noticed that the field solving the effective Maxwell equations exactly satisfies the following properties: (a) the Hall current always flows along the equipotential level curves and (b) the normal magnetic field to the planes and the charge densities at any point out of the cores of the impurities also exactly satisfies the integral filling condition as it was discussed in Ref. 15. Therefore, it can be concluded that the electric potential and the Hall currents of the considered many-electron state analytically furnish the usual intuitive picture for the conduction in QHE and also gives ground for early proposed percolation models.[2,7]

Let us resume the picture we intend to support in the argue to be done below. Whenever the external magnetic field B_{ext} slightly deviates from one of the values required by integral filling of Landau levels, each impurity accumulates a free edge charge q such that the electron density at any internal point in the sample volume adjusts to that required to satisfy the integral filling condition at the precise local magnetic field value. For the sample type we have been considering the electromagnetic response field is exponentially damped away from the impurities. Therefore, the excess of magnetic flux over that corresponding to integral filling flux density B_n will tend to concentrate more around the impurities. If the distance between them turns to be greater than the penetration length k^{-1} a sort of Meissner effect occurs. Then the difference between the external flux and the integral filling one will be expelled from large volumes and concentrated around the defects.[8] The defect properties will resemble in such regimes the vortices in type II superconductors. This should not be necessarily the case. It should also be stressed that although in real samples the defects are expected to be distributed in uncorrelated positions on each plane of the superlattice, it is natural to suppose that the magnetic flux lines of the real defects will follow trajectories that traverse the sample in x_3 direction with some but small distortion (similar to the real Abrikosov vortices in nonideal samples) respect to the perfectly straight configurations we are considering.

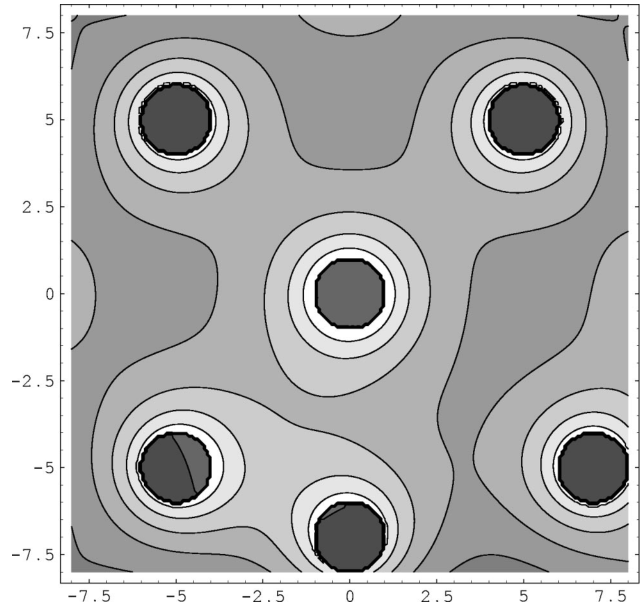

FIG. 1. Contour plot illustrating the equipotential curves of the superposed fields for various impurities. The Hall currents flow along them, and while some lines are closed, other ones connect different spatial regions reflecting the presence of extended states.

In order to justify the above picture it should be shown that the system energy is lower than the value corresponding to the completely homogeneous field configuration. Let us consider this problem in what follows.

IV. STABILITY AND QHE PLATEAUS

From the condition of equality of total magnetic flux in and out of the sample

$$B_{\mathrm{ext}}A=B_nA+N\Phi,$$

and from expression (7), the constant C is determined to be

$$C=-\frac{B_{\rm ext}-B_n}{2\pi}k\xi^2.$$

Substituting this result in expression (10) and considering there are N impurities, it is obtained the energy density in QHE regime

$$\mathcal{U}_{\rm QHE}(B_{\rm ext})=N\mathcal{U}_{\rm imp}=\frac{(B_{\rm ext}-B_n)^2}{16\pi^2}k^2\xi^2K_0(k\eta). \quad (11)$$

In order to simplify the discussion, for evaluating this energy as the simple sum of the contributions of each defect it was assumed that the field distributions of different impurities do not overlap significantly. A more accurate procedure is however possible for the search of more quantitative results.

This situation in which the system is in QHE regime should be compared with the case in which the applied homogeneous field $B_{\rm ext}$ completely penetrates the sample. The energy density required for a homogeneous deformation $B_{\rm ext}-B_n$ over the integral filling field B_n is calculated as follows. From the action for a free stationary magnetic field

$$\Gamma[\mathbf{A}]=-\int d\mathbf{x}\frac{(\nabla\times\mathbf{A})^2}{8\pi}\int dt,$$

and using the definition of effective action over a background (A3), we obtain

$$\tilde{\Gamma}[\mathbf{a}]\equiv\Gamma[\mathbf{A}_{\rm ext}]-\Gamma[\mathbf{A}_n]-\int d\mathbf{x}\frac{\delta\Gamma[\mathbf{A}]}{\delta\mathbf{A}(\mathbf{x})}\bigg|_{\mathbf{A}=\mathbf{A}_n}\cdot\mathbf{a}(\mathbf{x})$$

$$=\left[-\frac{1}{8\pi}B_{\rm ext}^2+\frac{1}{8\pi}B_n^2+\frac{1}{4\pi}B_n(B_{\rm ext}-B_n)\right]\int d\mathbf{x}\int dt=$$

$$-\frac{(B_{\rm ext}-B_n)^2}{8\pi}\int d\mathbf{x}\int dt.$$

Recalling that, as shown in Appendix A, the energy density is minus this expression per unit time and volume, and taking into account the contribution $\mathcal{U}(B_{\rm ext})$ from the electron sector, whose derivation is sketched in Appendix C (see also Ref. 9), given by Eq. (C2) and shown as the thick line in Fig. 2, the desired energy density for the case of complete external field penetration is given by

$$\mathcal{U}_{\rm homog}(B_{\rm ext})=\frac{(B_{\rm ext}-B_n)^2}{8\pi}+\mathcal{U}(B_{\rm ext}). \quad (12)$$

For the QHE regime to be energetically favorable it is clear we must have $\mathcal{U}_{\rm QHE}(B_{\rm ext})<\mathcal{U}_{\rm homog}(B_{\rm ext})$ and this is equivalent to

$$\left(\frac{k^2\xi^2K_0(k\eta)}{2\pi}-1\right)\frac{(B_{\rm ext}-B_n)^2}{8\pi}<\mathcal{U}(B_{\rm ext}). \quad (13)$$

The function $\mathcal{U}(B)$ is non-negative, so it is easily seen that the QHE regime is always preferred whenever $\xi<k^{-1}\sqrt{2\pi/K_0(k\eta)}$, which means that the Hall conductivity shows a perfect steplike dependence. However, in the limit of a very small ξ the impurities will strongly interact and the

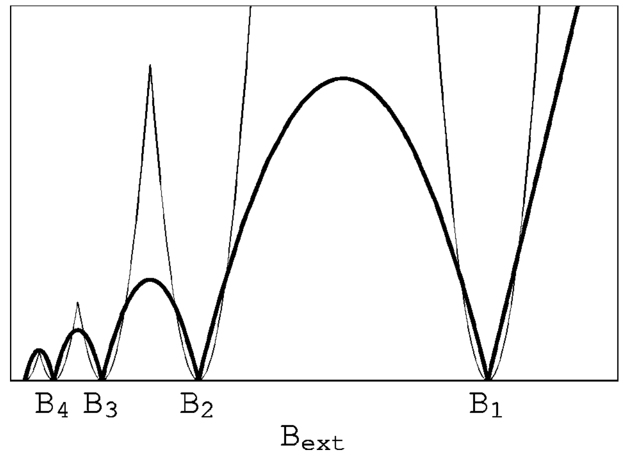

FIG. 2. Comparison of both members of inequality (13). The thinner line corresponds to the left member.

picture would no longer be applicable. In any case, this dirty limit should eventually destroy the QHE regime.

A plot of both members of the inequality as functions of external applied field $B_{\rm ext}$ is shown in Fig. 2 for the case when $\xi>k^{-1}\sqrt{2\pi/K_0(k\eta)}$. It can be seen the existence of wide regions around integral filling values B_n for which the QHE regime is favorable. So, in this case, the inequality (13) predicts a plateau width

$$(\Delta B)_n=B_n\left\{\frac{2n(n^2+\alpha)}{(n^2+\alpha)^2-n^2}\right\}.$$

In this equation we have introduced α as given by the expression $(mac^2/2e^2)[k^2\xi^2K_0(k\eta)/2\pi-1]$, which increases with a decrease in the impurity density. This result would clearly express the QHE stability if there are no alternative states of the system showing less energy.

The plateaus predicted by the above expression have been shown in Fig. 3 for successively increasing values of the quantity α. It is seen that the plateaus become wider as the

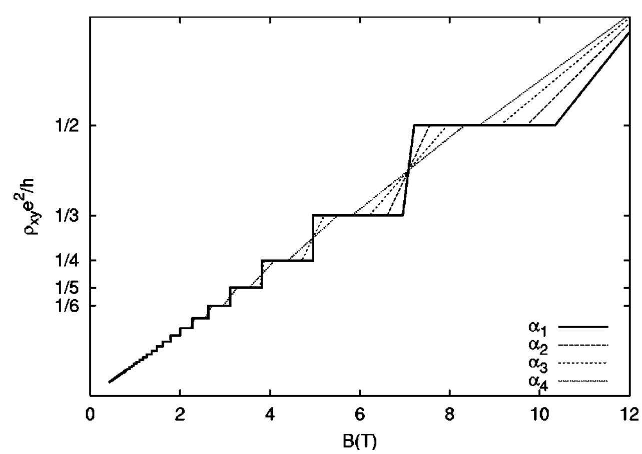

FIG. 3. Predicted plateaus for increasing values of the quantity α, which corresponds to a decreasing in the impurity density. Physical parameters are taken as in Ref. 13: $n_e=4.1\times10^{11}$ cm^{-2}, $a=146$ Å, and $1/\xi^2$ is varied in the order of 10^{7-8} cm^{-2}.

impurity density increases, which is in agreement with what is found in Ref. 16. We have used the same values for the physical parameters as those of the experiments described in Ref. 13: the electron density n_e was fixed at $4.1 \times 10^{11}\mathrm{cm}^{-2}$, the interplanar distance a was taken as 146 Å, and the electron effective mass m as 0.07 times the free-electron mass. The density of impurities was varied in the order of 10^{7-8} cm^{-2} for obtaining what is shown in Fig. 3.

V. CONCLUSIONS

It has been illustrated that the electromagnetic response of a superlattice of 2DEG free of impurities is able to transform an added distribution of impurities or defects in a set of charge reservoirs. The excess charge over that required by an integral filling of Landau levels is then accumulated in the impurities. It can be understood from the present analysis that it is the dynamically acquired capacity of accumulating charge of the impurities the main element to account for this effect.

A next important step of the present study would be to investigate a similar model but in the case of planar samples. Analogous results can be expected in this case. The main additional complication seems to be the determination of the fields associated with a single defect, since for that situation these fields spray out in the 3D space.

An additional task for this planar case, would be the inclusion of the temperature. We expect this study to provide a foundation of the successful phenomenological model of Ingraham and Wilkes.[17] In addition, the present discussion can also be viewed as giving a microscopic explanation of the model of Toyoda *et al.*[6] These authors assumed the existence of a kind of particle reservoir being in equilibrium with the 2DEG and with a limited capacity to account for the plateau widths found in experiments. The fact that the total energy (11) becomes greater than the energy of the homogeneous magnetic field distribution can be expected to play the role of the limiting capacity stopping the particle accumulation and accounting for the plateau widths.

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APPENDIX A: EFFECTIVE ACTION OVER A BACKGROUND FIELD AND ITS INTERPRETATION IN TERMS OF ENERGY DENSITY

In the following we include, for the more interested reader, the standard definition of the effective action in field theory and its energetic interpretation. For more details see Ref. 18.

Consider a field theory with elementary fields $A_\mu(x)$ and classical action $I[A]$. Introduce external sources $J_\mu(x)$ and couple them to the fields of the theory by adding to the classical action a term $\int dx J_\mu(x) A_\mu(x)$. The complete vacuum-vacuum transition amplitude $Z[J]$ for this theory in presence of the sources J_μ is given by the path integral

$$Z[J]=\int [\mathcal{D}A]\exp\left\{ iI[A]+i\int dx J_\mu(x)A_\mu(x)\right\}.$$

Denote by $iW[J]\equiv \ln Z[J]$ the connected vacuum-vacuum transition amplitude. In the presence of sources J_μ, the fields A_μ have vacuum expectation values

$$A^J_\mu(x)\equiv \frac{1}{Z[J]}\int [\mathcal{D}A]A_\mu(x)\exp\left\{ iI[A] +i\int dx J_\mu(x)A_\mu(x)\right\} = \frac{1}{iZ[J]}\frac{\delta Z[J]}{\delta J_\mu(x)}=\frac{\delta W[J]}{\delta J_\mu}.$$

The quantum effective action is defined as the Legendre transform of $W[J]$:

$$\Gamma[A]=W[J^A]-\int dx J^A_\mu(x)A_\mu(x), \tag{A1}$$

where J^A_μ are the sources which correspond to fields expectation values A_μ, and obviously satisfy

$$\frac{\delta W[J^A]}{\delta J^A_\mu}=A_\mu(x).$$

From these expressions it can be shown that A^J_μ solve

$$\frac{\delta \Gamma[A^J]}{\delta A^J_\mu(x)}=-J_\mu(x). \tag{A2}$$

Normally, the sources J_μ are ficticious and the equations of motion are given by Eq. (A2) with $J_\mu=0$. We see that Γ plays the rôle of the action, in the sense that solutions are those that minimize Γ, but with quantum corrections taken into account.

If the sources J_μ are actually present, it can be defined the field $a_\mu(x)$ over the background field solution $A^J_\mu(x)$ and an effective action

$$\tilde{\Gamma}[a]\equiv \Gamma[A^J+a]-\Gamma[A^J]-\int dx \left.\frac{\delta\Gamma[A]}{\delta A_\mu(x)}\right|_{A=A^J} a_\mu(x), \tag{A3}$$

which satisfies

$$\tilde{\Gamma}[a]|_{a=0}=\left.\frac{\delta\tilde{\Gamma}[a]}{\delta a_\mu(x)}\right|_{a=0}=0.$$

Let us show now the connection between the effective action and the energy density.[18] Suppose a current $J_\mu(x)$ is turned on adiabatically from zero at $t=-\infty$, and remains at a constant value $\mathcal{J}_\mu(\mathbf{x})$ for a large time T, and finally goes to zero smoothly as $t\to\infty$. In this process the system goes from the vacuum to a state with energy $E[\mathcal{J}]$—which is a functional of $\mathcal{J}(\mathbf{x})$—and then again to the vacuum. The amplitude of transition $Z[J]$, which is the overlap between the initial and final vacuum state vectors, is given by the phase $\exp(-iE[\mathcal{J}]T)$, which is equivalent to state

$$W[J]=-E[\mathcal{J}]T. \tag{A4}$$

The fact that the system in presence of the sources $\mathcal{J}_\mu(\mathbf{x})$ is in a state with energy $E[\mathcal{J}]$, which is reasonable to assume is the state with lower energy, implies that in this state

$$\langle H\rangle-\int d\mathbf{x}\mathcal{J}_\mu(\mathbf{x})A^J_\mu(\mathbf{x})=E[\mathcal{J}],$$

where H is the Hamiltonian of the system in absence of sources and $\langle H\rangle$ its expectation value. But it is easily checked this is the very equation that would be obtained from minimizing the expectation value of the Hamiltonian H, with the constraints that the state be normalized and fields A_μ have stationary expectation values $A_\mu(\mathbf{x})$, upon which $E[\mathcal{J}]$ and $\mathcal{J}_\mu(\mathbf{x})$ play the role of Lagrange multipliers, respectively. Clearly, for this to occur J should be fixed to the source J^A necessary for producing the field A.

Using now Eq. (A4) and recalling the definition of the effective action (A1) we obtain

$$\begin{aligned}\langle H\rangle&=-\frac{1}{T}W[J^A]+\int d\mathbf{x}\mathcal{J}^A_\mu(\mathbf{x})A_\mu(\mathbf{x})\\&=-\frac{1}{T}\left\{W[J^A]-\int dxJ^A_\mu(x)A_\mu(x)\right\}=-\frac{1}{T}\Gamma[A].\end{aligned}$$

Dividing by the volume we obtain the energy density. So, we conclude that minus the effective action per unit time and volume, evaluated in a given stationary field configuration, equals the minimum of the energy density among all the states which are consistent with this stationary field expectation configuration.

From this interpretation and the definition and properties of the effective action over a background $\tilde{\Gamma}[a]$, it should be clear that minus $\tilde{\Gamma}[a]$ per unit time and volume, evaluated in a stationary configuration $a_\mu(\mathbf{x})$ equals the energy density to produce this static deformation with respect to the configuration $a_\mu=0$.

APPENDIX B: 2D HALL EFFECT RESPONSE ALWAYS DESCRIBED BY THE TOPOLOGICAL CHERN-SIMONS ACTION

The set of equations (1) can be translated into a more physically appealing representation by expressing them in terms of the electric and magnetic field intensities

$$\begin{aligned}E_i&=i(\partial_i a_0-\partial_0 a_i),\\B_i&=\epsilon^{ijk}\partial_j a_k.\end{aligned} \tag{B1}$$

Using these relations and the gauge conditions $\nabla\cdot\mathbf{a}=0$ and $\mathbf{n}\cdot\mathbf{a}=0$, Eqs. (1) can be written in the following form:

$$\nabla\cdot\left(\mathbf{E}+4\pi\frac{\chi_e}{a}\mathbb{P}\cdot\mathbf{E}\right)=4\pi\frac{\sigma_H}{a}\mathbf{n}\cdot\mathbf{B}, \tag{B2}$$

$$\nabla\times\mathbf{B}=4\pi\frac{\sigma_H}{a}\mathbf{E}\times\mathbf{n}+\frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{E}+4\pi\frac{\chi_e}{a}\mathbb{P}\cdot\mathbf{E}\right), \tag{B3}$$

$$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}=0, \tag{B4}$$

$$\nabla\cdot\mathbf{B}=0, \tag{B5}$$

where $\mathbb{P}$ is the projector "on the 2DEG" dyadic 3D tensor, the spatial part of Eq. (2), and $\mathbf{n}$ is a unit vector normal to the plane.

Equations (B4) and (B5) are analogues of the vacuum equations for Faraday induction and the absence of magnetic charge. In addition, in Eqs. (B2) and (B4), the χ_e-dependent terms show that the electron gas behaves as a dielectric surface which polarizes itself linearly only due to the tangential components of the electric field. The Hall currents appear now explicitly in the right-hand side (RHS) of Eq. (B4). The only unusual contribution is the magnetic field dependent charge density appearing in the RHS of Eq. (B2). However, as mentioned above, and discussed in Ref. 19, this term arises from the general condition of charge conservation. Therefore, its presence should not be considered as an unusual outcome. Rather, such charge densities are determined by assuming the existence of a Hall conductivity in a planar medium. This fact will be discussed below.

In the set of equations (B2)–(B5), let us substitute the magnetic field dependent charge density by an undefined function ρ^{Hall}. All the other terms remain unaltered. After taking the divergence of Eq. (B4), and adding the result to Eq. (B2), the following conservation conditions follow for the polarization and for the Hall currents and densities

$$\begin{aligned}\frac{\partial}{\partial t}(\rho^{\mathrm{Hall}}+\rho^{\mathrm{pol}})+\nabla\cdot(\mathbf{J}^{\mathrm{Hall}}+\mathbf{J}^{\mathrm{pol}})&=0,\\\frac{\partial}{\partial t}\rho^{\mathrm{Hall}}+\nabla\cdot\mathbf{J}^{\mathrm{Hall}}&=0.\end{aligned} \tag{B6}$$

After substituting in Eq. (B6) the expression (4) for the Hall current, it follows that

$$\frac{\partial}{\partial t}\rho^{\mathrm{Hall}}=-\frac{\sigma_H}{a}\nabla\cdot(\mathbf{E}\times\mathbf{n})=-\frac{\sigma_H}{a}\mathbf{n}\cdot(\nabla\times\mathbf{E})=\frac{\partial}{\partial t}\left(\frac{\sigma_H}{ca}\mathbf{n}\cdot\mathbf{B}\right). \tag{B7}$$

This relation shows that the unknown quantity ρ^{Hall} should differ by a time independent function from the magnetic field dependent charge density that appears in Eq. (B2). Then, after assuming that before any perturbation (as, for example, incoming waves) the charge densities vanished, it follows that the magnetic field dependent surface charge densities (and then the whole Chern-Simons structure of the Hall current four-vector) is implied by the existence of a Hall conductivity.

APPENDIX C: DERIVATION OF $U(B)$

Consider a system of 2D noninteracting electrons in a perpendicular magnetic field B_{ext}. The energy levels of this system are the well-known Landau levels

$$\varepsilon_n=\left(n+\frac{1}{2}\right)\hbar\omega_c,$$

where $\omega_c=eB_{\text{ext}}/mc$ is the cyclotronic frequency. Each one of these levels have a degeneracy of eB_{ext}/hc per unit area, so the filling factor of Landau levels is given by

$$\nu=\frac{n_e hc}{eB_{\text{ext}}},$$

with n_e the electron density. This means that the first $[\nu]$ levels are completely occupied and the rest of the electrons, $n_e-[\nu]eB_{\text{ext}}/hc$ per unit area, are occupying the $([\nu]+1)$-th one. Here $[\nu]$ denotes the integer part of ν. The energy per unit area is obviously given by

$$\frac{eB_{\text{ext}}}{hc}\sum_{i=1}^{[\nu]}\varepsilon_i+\left(n_e-[\nu]\frac{eB_{\text{ext}}}{hc}\right)\varepsilon_{[\nu]+1}$$
$$=\frac{n_e^2h^2}{4\pi m}\left\{-[\nu]([\nu]+1)\left(\frac{B_{\text{ext}}}{B_1}\right)^2+(2[\nu]+1)\frac{B_{\text{ext}}}{B_1}\right\},$$

with $B_1=n_e hc/e$ the field value for which $\nu=1$.

If we have now several parallel 2D systems, placed at a distance a from each other, the energy density per unit volume is clearly given by the last expression divided by a:

$$\frac{n_e^2h^2}{4\pi ma}\left\{-[\nu]([\nu]+1)\left(\frac{B_{\text{ext}}}{B_1}\right)^2+(2[\nu]+1)\frac{B_{\text{ext}}}{B_1}\right\}. \tag{C1}$$

This expression, evaluated at an integral-filling field $B_{\text{ext}}=B_n$, has the value $n_e^2h^2/4\pi ma$ irrespective of n. As we are interested in energy differences with respect to the integral filling situation we subtract this background from the last expression to obtain the desired contribution to the energy density

$$\mathcal{U}(B_{\text{ext}})=\mathcal{U}_0\left\{-[\nu]([\nu]+1)\left(\frac{B_{\text{ext}}}{B_1}\right)^2+\left[(2[\nu]+1)\frac{B_{\text{ext}}}{B_1}-1\right\},\right.$$

$$\mathcal{U}_0=\frac{n_e^2h^2}{4\pi ma}. \tag{C2}$$

This function is shown as the thick line in Fig. 2.

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