Scalar chiral ground states of spin ladders with four-spin exchanges

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We show that scalar chiral order can be induced by four-spin exchanges in the two-leg spin ladder, using the spin-chirality duality transformation and matrix-product ansatz. Scalar-chiral-ordered states are found to be exact ground states in a family of spin ladder models. In this scalar chiral phase, there is a finite energy gap above the doubly degenerate ground states and a $Z_2\times Z_2\times Z_2$ symmetry is fully broken. It is also shown that the SU(4)-symmetric model, which is self-dual under the duality transformation, is on a multicritical point surrounded by the staggered dimer phase, the staggered scalar chiral phase, and the gapless phase.

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I. INTRODUCTION

Recently, four-spin exchange interactions have been attracting interest in spin ladder models and spin-orbital models, because these interactions in fact appear in many systems[1–8] and can induce exotic ground states.[9–17] Various types of four-spin interactions appear associated with diverse mechanisms, e.g., cyclic-exchange processes,[18–20] Coulomb repulsion between doubly degenerate orbitals,[1] and spin-phonon couplings. Though two-leg antiferromagnetic Heisenberg spin ladders show a rung-singlet ground state and have an energy gap between the ground state and excitations,[21] it was revealed that four-spin exchanges can induce a gapped staggered dimer (or spin-Peierls) phase[9–11] and a gapless phase[11–15] around the SU(4)-symmetric point and that a gapped phase with a dominant vector chirality correlation also appears.[16,17]

Very recently, Läuchli *et al.*[17] numerically found a new scalar chiral phase in the two-leg spin-1/2 ladder with four-spin cyclic exchange. Scalar chiral states, in which both the time-reversal and parity symmetries are broken, had been discussed in the context of anyon superconductivity[22,23] and the anomalous Hall effect,[24,25] but realization of scalar chiral order in SU(2)-symmetric systems had been difficult and a challenging problem. A scalar chiral order due to the four-spin cyclic exchange was first proposed on the triangular lattice for magnetism of solid ^{3}He films,[26,27] though finite-size system analysis could not find evidence for such ordering, instead showing spin-liquid ground states.[28,29] Study of spin ladders with four-spin exchanges is expected to clarify the possibility of exotic magnetism induced by the four-spin interactions.

In this paper, we study the two-leg spin-1/2 ladder with four-spin exchanges, whose Hamiltonian is given in the next subsection, and give a rigorous example of a scalar chiral ground state. In our analysis, the spin-chirality duality transformation we introduced in Ref. 16 plays an important role. A new class of models that have an exact ground state with scalar chiral order are constructed by the means of matrix-product ansatz. The scalar chiral phase has the following nature: (i) the ground states are doubly degenerate, (ii) there is a finite energy gap between ground states and excited states, (iii) the ground states have long-range staggered scalar chiral order and exponentially decaying spin correlations, and (iv) a $Z_2\times Z_2\times Z_2$ symmetry is fully broken. It is also found that the phase boundary of the scalar chiral phase touches the SU(4)-symmetric point and the SU(4)-symmetric model is on a multicritical point surrounded by the staggered dimer phase, the staggered scalar chiral phase, and the gapless phase.

This paper is organized as follows. The model Hamiltonian is given in the next subsection. In Sec. II, we summarize the duality transformation and its application to the present model. In Sec. III, it is shown that a scalar chiral state is the exact ground state in a parameter region of the model Hamiltonian. The nature of the scalar chiral phase is discussed. In Sec. IV, we discuss the phase diagram around the SU(4)-symmetric point and find that the SU(4) model is on a multicritical point. Finally in Sec. V we conclude with a discussion. The appendix contains a unitary description of the duality transformation.

Model Hamiltonian

The Hamiltonian of the two-leg spin-1/2 ladder with extended four-spin exchange interactions is defined as

$$\begin{aligned}\mathcal{H}=&J_1\sum_l (\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1}+\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1})+J_{\rm r}\sum_l \mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}\\&+J_{\rm d}\sum_l (\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l+1}+\mathbf{s}_{2,l}\cdot\mathbf{s}_{1,l+1})\\&+J_{\rm rr}\sum_l (\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l})(\mathbf{s}_{1,l+1}\cdot\mathbf{s}_{2,l+1})\\&+J_{\rm ll}\sum_l (\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1})(\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1})\\&+J_{\rm dd}\sum_l (\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l+1})(\mathbf{s}_{2,l}\cdot\mathbf{s}_{1,l+1}).\end{aligned}\quad(1)$$

This Hamiltonian includes a variety of models: (I) *Four-spin cyclic-exchange model*. When four-spin exchange constants satisfy $J_{\rm rr}=J_{\rm ll}=-J_{\rm dd}$, the four-spin terms describe the cyclic-exchange interaction.[2] (II) $SU(2)\times SU(2)$ *model*. When the parameters satisfy $J_r=J_d=J_{rr}=J_{dd}=0$, the Hamiltonian has an $SU(2)\times SU(2)$ symmetry. This model was studied extensively as the $SU(2)\times SU(2)$ spin-orbital model. It was revealed that when $J_{ll}>0$ the ground state has a staggered dimer (or spin-Peierls) order[9,10] for $4J_l>J_{ll}$ and is gapless[12–15] for $-J_{ll}\leqslant 4J_l\leqslant J_{ll}$. (III) *SU(4) model*. As a special case of model II, the Hamiltonian has an SU(4) symmetry[11,30] at $4J_l=J_{ll}$, which was exactly solved by the Bethe ansatz.[31]

II. DUALITY

A. Duality transformation

Let us begin with the spin-chirality duality transformation, which we developed in Ref. 16. We introduced the duality transformation defining new spin-1/2 pseudospin operators

$$\mathbf{S}_l\equiv\frac{1}{2}(\mathbf{s}_{1,l}+\mathbf{s}_{2,l})-\mathbf{s}_{1,l}\times\mathbf{s}_{2,l}, \tag{2}$$

$$\mathbf{T}_l\equiv\frac{1}{2}(\mathbf{s}_{1,l}+\mathbf{s}_{2,l})+\mathbf{s}_{1,l}\times\mathbf{s}_{2,l}, \tag{3}$$

which obey the commutation relations of spins and satisfy $(S_l^\alpha)^2=(T_l^\alpha)^2=1/4$ for $\alpha=x, y,$ or z. In the same way, the original spins $\mathbf{s}_{1,l}$ and $\mathbf{s}_{2,l}$ are expressed in terms of $\mathbf{S}_l$ and $\mathbf{T}_l$ in the forms $\mathbf{s}_{1,l}=\frac{1}{2}(\mathbf{S}_l+\mathbf{T}_l)+\mathbf{S}_l\times\mathbf{T}_l$ and $\mathbf{s}_{2,l}=\frac{1}{2}(\mathbf{S}_l+\mathbf{T}_l)-\mathbf{S}_l\times\mathbf{T}_l$. In the appendix, we show that this transformation derives from a unitary operator $U_{\pi/2}$ in the form $\mathbf{S}_l=U_{\pi/2}\mathbf{s}_{1,l}U^\dagger_{\pi/2}$ and $\mathbf{T}_l=U_{\pi/2}\mathbf{s}_{2,l}U^\dagger_{\pi/2}$. Since the following relations hold,

$$\mathbf{s}_{1,l}+\mathbf{s}_{2,l}=\mathbf{S}_l+\mathbf{T}_l,$$
$$\mathbf{s}_{1,l}-\mathbf{s}_{2,l}=2\mathbf{S}_l\times\mathbf{T}_l,$$
$$-2\mathbf{s}_{1,l}\times\mathbf{s}_{2,l}=\mathbf{S}_l-\mathbf{T}_l,$$

this transformation exchanges the Néel-type spin and vector chirality degrees of freedom on the same rung. As an example of the duality, one can show that the transformation of the dimer (or spin-Peierls) operator

$$\mathcal{O}_D(l)=\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1}-\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1} \tag{4}$$

leads to the scalar chiral operator

$$\mathcal{O}_{SC}(l)=(\mathbf{s}_{1,l}+\mathbf{s}_{2,l})\cdot(\mathbf{s}_{1,l+1}\times\mathbf{s}_{2,l+1})+(\mathbf{s}_{1,l}\times\mathbf{s}_{2,l})\cdot(\mathbf{s}_{1,l+1}+\mathbf{s}_{2,l+1}). \tag{5}$$

It is known that a group of two-leg ladders with four-spin interactions shows the staggered dimer order in the ground state.[9–11] The above duality relation hence shows that their dual models have the staggered "scalar chiral" order in the ground state. We will discuss these models in Secs. III and IV.

We now consider the transformation of the spin states on rungs. Since the total spin on each rung is conserved under the transformation, the fully polarized spin state $|\uparrow\rangle_{1,l}|\uparrow\rangle_{2,l}$ is transformed to $|\uparrow\rangle_{S,l}|\uparrow\rangle_{T,l}$, where $|\alpha\rangle_{\mu,l}$ ($\alpha=\uparrow,\downarrow$ and $\mu=1,2$) denotes the spin state operated by $\mathbf{s}_{\mu,l}$, and $|\alpha\rangle_{S,l}$ ($|\alpha\rangle_{T,l}$) is the pseudospin state operated by $\mathbf{S}_l$ ($\mathbf{T}_l$), respectively. By applying S_l^- and T_l^- to $|\uparrow\rangle_{S,l}|\uparrow\rangle_{T,l}$, all pseudospin states in dual space can be constructed in the forms

$$|\uparrow\rangle_{S,l}|\uparrow\rangle_{T,l}=|\uparrow\rangle_{1,l}|\uparrow\rangle_{2,l},$$
$$|\uparrow\rangle_{S,l}|\downarrow\rangle_{T,l}=\frac{e^{-i\pi/4}}{\sqrt{2}}(|\uparrow\rangle_{1,l}|\downarrow\rangle_{2,l}+i|\downarrow\rangle_{1,l}|\uparrow\rangle_{2,l}),$$
$$|\downarrow\rangle_{S,l}|\uparrow\rangle_{T,l}=\frac{e^{i\pi/4}}{\sqrt{2}}(|\uparrow\rangle_{1,l}|\downarrow\rangle_{2,l}-i|\downarrow\rangle_{1,l}|\uparrow\rangle_{2,l}),$$
$$|\downarrow\rangle_{S,l}|\downarrow\rangle_{T,l}=|\downarrow\rangle_{1,l}|\downarrow\rangle_{2,l}. \tag{6}$$

Using eigenstates for the total spin on each rung, one finds that this transformation corresponds to a gauge transformation of the singlet bond state $|s\rangle_l\to -i|s\rangle_l$, while it keeps the triplet states invariant. See also Appendix for further arguments.

B. Duality in the model Hamiltonian

We now apply the transformation to the model (1) and consider the duality relation in parameter space. Under this duality transformation, the total form of the Hamiltonian (1) remains invariant, but the couplings change. The couplings of the dual Hamiltonian $\tilde{\mathcal{H}}$ are given by

$$\tilde{J}_r=J_r,\quad \tilde{J}_{rr}=J_{rr},$$
$$\tilde{J}_l=\frac{1}{2}(J_l+J_d)+\frac{1}{8}(J_{ll}-J_{dd}),$$
$$\tilde{J}_d=\frac{1}{2}(J_l+J_d)-\frac{1}{8}(J_{ll}-J_{dd}),$$
$$\tilde{J}_{ll}=2(J_l-J_d)+\frac{1}{2}(J_{ll}+J_{dd}),$$
$$\tilde{J}_{dd}=-2(J_l-J_d)+\frac{1}{2}(J_{ll}+J_{dd}). \tag{7}$$

To see the mapping in the parameter space, we rewrite the Hamiltonian (1) in the form

$$\mathcal{H}=J_r\sum_l \mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}+J_{rr}\sum_l(\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l})(\mathbf{s}_{1,l+1}\cdot\mathbf{s}_{2,l+1})+W\sum_l(\mathbf{s}_{1,l}+\mathbf{s}_{2,l})\cdot(\mathbf{s}_{1,l+1}+\mathbf{s}_{2,l+1})+X\sum_l\{(\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1})\times(\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1})+(\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l+1})(\mathbf{s}_{1,l+1}\cdot\mathbf{s}_{2,l})\}+Y\sum_l\{(\mathbf{s}_{1,l}-\mathbf{s}_{2,l})\cdot(\mathbf{s}_{1,l+1}-\mathbf{s}_{2,l+1})+4(\mathbf{s}_{1,l}\times\mathbf{s}_{2,l})\cdot(\mathbf{s}_{1,l+1}\times\mathbf{s}_{2,l+1})\}+Z\sum_l\{(\mathbf{s}_{1,l}-\mathbf{s}_{2,l})\cdot(\mathbf{s}_{1,l+1}-\mathbf{s}_{2,l+1})$$

$$-4(\mathbf{s}_{1,l}\times\mathbf{s}_{2,l})\cdot(\mathbf{s}_{1,l+1}\times\mathbf{s}_{2,l+1})\}, \tag{8}$$

where

$$W=\frac{1}{2}(J_{\mathrm{l}}+J_{\mathrm{d}}), \quad X=\frac{1}{2}(J_{\mathrm{ll}}+J_{\mathrm{dd}}),$$

$$Y=\frac{1}{16}(J_{\mathrm{ll}}-J_{\mathrm{dd}})+\frac{1}{4}(J_{\mathrm{l}}-J_{\mathrm{d}}),$$

$$Z=-\frac{1}{16}(J_{\mathrm{ll}}-J_{\mathrm{dd}})+\frac{1}{4}(J_{\mathrm{l}}-J_{\mathrm{d}}). \tag{9}$$

Straightforward calculations show that the duality transformation maps the parameters $(J_{\mathrm{r}},J_{\mathrm{rr}},W,X,Y,Z)$ to $(J_{\mathrm{r}},J_{\mathrm{rr}},W,X,Y,-Z)$; the transformation changes only the coupling Z of the last term to $-Z$, but leaves the other terms invariant. The Hamiltonian (1) is, thus, self-dual at the surface defined by $Z=0$, i.e.,

$$J_{\mathrm{ll}}-J_{\mathrm{dd}}-4(J_{\mathrm{l}}-J_{\mathrm{d}})=0. \tag{10}$$

The parameter space of the Hamiltonian has six dimensions in total, and the self-dual surface divides the parameter space into two regions $Z>0$ and $Z<0$. It should be remarked that the SU(4)-symmetric model $(4J_{\mathrm{l}}=J_{\mathrm{ll}},\ J_{\mathrm{r}}=J_{\mathrm{d}}=J_{\mathrm{rr}}=J_{\mathrm{dd}}=0)$ exists on the self-dual surface. Duality around this specific model will be discussed in Sec. IV.

C. U(1) summetry in the self-dual models

Here we describe a U(1) symmetry in the self-dual models. Consider a continuous transformation with the following unitary operator

$$U_\theta=\prod_l \exp[i\theta(\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}-\tfrac{1}{4})].$$

This is a continuous extension of the duality transformation (see Appendix) and it continuously transforms the dimer operator to the scalar chiral one. One can show that the Hamiltonian is invariant under this transformation as $U_\theta\mathcal{H}U_\theta^\dagger=\mathcal{H}$ for arbitrary θ if $Z=0$. Thus the self-dual models are isotropic under the continuous rotation with the generator $\Sigma_l\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}$, whereas the Z term of the Hamiltonian (8) lowers the symmetry.

III. MODELS WITH EXACT SCALAR CHIRAL GROUND STATES

In this section, we discuss an exact scalar chiral ground state of the Hamiltonian (1) with the periodic boundary condition. To obtain the ground state Ψ_0, we use matrix-product (MP) states. We start from the following ansatz:

$$\Psi_0(u)=\mathrm{tr}\{\tilde{\mathbf{g}}_1(u)\tilde{\mathbf{g}}_2(-u)\cdots\tilde{\mathbf{g}}_{2N-1}(u)\tilde{\mathbf{g}}_{2N}(-u)\}, \tag{11}$$

where u is a real variable and

$$\tilde{\mathbf{g}}_l(u)=\frac{1}{2}\begin{pmatrix} iu|s\rangle_l+|t_0\rangle_l & -\sqrt{2}|t_{+1}\rangle_l \\ \sqrt{2}|t_{-1}\rangle_l & iu|s\rangle_l-|t_0\rangle_l \end{pmatrix}$$

$$=\frac{1}{2}\{iu\mathbf{1}|s\rangle_l-\sqrt{2}\sigma^+|t_{+1}\rangle_l+\sqrt{2}\sigma^-|t_{-1}\rangle_l+\sigma_z|t_0\rangle_l\}. \tag{12}$$

Here $|s\rangle_l$ and $|t_\mu\rangle_l$ are, respectively, the singlet and triplet states of the lth rung, $2N$ is the total number of rungs, $\mathbf{1}$ is the 2×2 unit matrix, and σ_μ are the Pauli matrices. This form of the MP state can be obtained by the duality transformation of the MP state discussed by Kolezhuk and Mikeska,[10]

$$\Psi_{\mathrm{KM}}(u)=\mathrm{tr}\{\mathbf{g}_1(u)\mathbf{g}_2(-u)\cdots\mathbf{g}_{2N-1}(u)\mathbf{g}_{2N}(-u)\}, \tag{13}$$

where

$$\mathbf{g}_l(u)=\frac{1}{2}\{u\mathbf{1}|s\rangle_l-\sqrt{2}\sigma^+|t_{+1}\rangle_l+\sqrt{2}\sigma^-|t_{-1}\rangle_l+\sigma_z|t_0\rangle_l\}. \tag{14}$$

It was shown that for several models this MP state $\Psi_{\mathrm{KM}}(u)$ is the exact ground state with a staggered dimer order. At $u=0$ and $u=\infty$, the two states $\Psi_0(u)$ and $\Psi_{\mathrm{KM}}(u)$ are equivalent, which means that each of $\Psi_0(0)$ and $\Psi_0(\infty)$ is self-dual. $\Psi_0(0)$ and $\Psi_0(\infty)$ are, respectively, an Affleck-Kennedy-Lieb-Tasaki (AKLT) state[32] and a rung-singlet state. For $0<u<\infty$, however, $\Psi_0(u)$ and $\Psi_{\mathrm{KM}}(u)$ are orthogonal in the limit $N\to\infty$.

Since $\Psi_{\mathrm{KM}}(u)$ $(0<u<\infty)$ has the staggered dimer order, $\Psi_0(u)$ has the staggered scalar chiral order because of the duality relation. Using the technique developed by Klümper *et al.*,[33,34] one can evaluate the scalar chiral correlation in $\Psi_0(u)$,

$$\langle\mathcal{O}_{\mathrm{SC}}(l)\mathcal{O}_{\mathrm{SC}}(m)\rangle=(-1)^{l-m}\left[\frac{12u}{(u^2+3)^2}\right]^2, \tag{15}$$

for $|l-m|>1$ in the limit $N\to\infty$ and show spontaneous breakdown of the chiral symmetry

$$\langle\mathcal{O}_{\mathrm{SC}}(l)\rangle=(-1)^l\frac{12u}{(u^2+3)^2}.$$

One can also evaluate that $\Psi_0(u)$ has no dimer correlation

$$\langle\mathcal{O}_{\mathrm{D}}(l)\mathcal{O}_{\mathrm{D}}(m)\rangle=0.$$

In the same way, the spin and vector chiral correlations are obtained as

$$\langle s_{1,l}^\alpha s_{1,m}^\alpha\rangle=\langle s_{1,l}^\alpha s_{2,m}^\alpha\rangle=-\frac{1}{(u^2+3)(u^2-1)}\left(\frac{u^2-1}{u^2+3}\right)^{|l-m|}, \tag{16}$$

$$\langle(\mathbf{s}_{1,l}\times\mathbf{s}_{2,l})^\alpha(\mathbf{s}_{1,m}\times\mathbf{s}_{2,m})^\alpha\rangle=\frac{u^2}{(u^2+3)(u^2-1)}\left(\frac{1-u^2}{u^2+3}\right)^{|l-m|}, \tag{17}$$

for $\alpha = x$, y, or z. The spin and vector chiral correlation lengths are equal and given by $\xi_s^{-1} = \xi_{vc}^{-1} = \ln\{(u^2+3)/|u^2-1|\}$, whereas the scalar chiral correlation does not have any exponentially decaying term.

By the duality transformation of the MP-solvable models presented by Kolezhuk and Mikeska,[10] we find that the MP state (11) is an exact ground state of the following three classes of models. One can prove that the state Ψ_0 is a ground state in these models, reducing a local Hamiltonian $h_{l,l+1}$ on the lth and $(l+1)$th rungs to a positive semidefinite form $(h_{l,l+1} - E_0) \geqslant 0$, where $\mathcal{H} = \Sigma_l h_{l,l+1}$, and showing that Ψ_0 has zero eigenenergy in the reduced Hamiltonian as $(h_{l,l+1} - E_0)\boldsymbol{g}_l\boldsymbol{g}_{l+1} = 0$. The reader who is interested in the method of proofs should refer to Ref. 10.

(A) *Scalar chiral models*. For a family of models

$$J_{\rm r} = \frac{8J(2-3y)}{3(4-3y)}, \quad J_{\rm l} = \frac{4J(1-y)}{4-3y},$$

$$J_{\rm d} = \frac{J(8-9y)}{3(4-3y)}, \quad J_{\rm ll} = \frac{16J}{3(4-3y)},$$

$$J_{\rm rr} = 0, \quad J_{\rm dd} = \frac{-4Jy}{4-3y}, \tag{18}$$

with $0 < y < 1$ and $J > 0$, the ground states are doubly degenerate and given by $\Psi_0(1)$ and $\Psi_0(-1)$. The ground-state energy per rung is $E_0 = -3J/4$. This model is dual to the "checkerboard-dimer model" given in Ref. 10, which has a staggered dimer order, and hence from the duality relation the present model belongs to the scalar chiral phase. Excitations of the staggered dimer phase were studied by variational trial states,[10] numerical calculations,[11] and field-theoretical analyses.[13–15] These studies concluded that excitations have a finite energy gap. In fact, extending the arguments by Knabe,[35,36] we can prove the finiteness of the energy gap in the checkerboard-dimer model dual to the model (18) with $y = 2/3$ and in a finite region around this point. From the duality we conclude that, in the scalar chiral phase, there is a finite energy gap between the ground states and excited states. The analysis in the dual model[10] indicates that, at $y = 1$, the present system enters the fully polarized ferromagnetic phase through a first-order transition. Furthermore, we can extend the parameter space which has the exact scalar chiral ground state. For $y = 2/3$, the dual Hamiltonian has an SU(2)× SU(2) symmetry and the Hamiltonian is written as a product of projection operators $(4J/3)\Sigma_l(\mathbf{s}_{1,l} \cdot \mathbf{s}_{1,l+1} + 3/4)(\mathbf{s}_{2,l} \cdot \mathbf{s}_{2,l+1} + 3/4)$. Then one can construct the model with the exact scalar chiral ground state by generalization of projection operators[37] and the duality transformation.

(B) *Model at a phase boundary between the scalar chiral and staggered dimer phases*. At $y = 0$ of the model (18), i.e.,

$$J_{\rm r} = J_{\rm ll} = \frac{4J}{3}, \quad J_{\rm l} = J,$$

$$J_{\rm d} = \frac{2J}{3}, \quad J_{\rm rr} = J_{\rm dd} = 0, \tag{19}$$

with $J > 0$, the ground states are given by $\Psi_0(u)$ with arbitrary u and highly degenerate. Note that this model is equivalent to the "multicritical model" in Ref. 10 and self-dual under the duality transformation. The scalar chiral model (18), thus, connects with the checkerboard-dimer model at this special parameter point. At this phase boundary, both one magnon and a pair of scattering solitons have energy gaps in the staggered dimer state, as discussed by Kolezhuk and Mikeska.[10] However, because of the U(1) symmetry in this model, the generator $\mathbf{s}_{1,l} \cdot \mathbf{s}_{2,l}$ can create gapless collective (Goldstone) modes, which are singlet bound states of two magnons. In fact, one can show that the following trial state becomes gapless at $p = 0$, π $(\zeta = -1)$ and $p = \pi/2$ $(\zeta = 1)$:

$$|\Psi_0(p)\rangle_{sb}^{\zeta} = \sum_l e^{2ipl}\mathrm{Tr}\left\{\prod_{i=1}^{l-1} \tilde{\boldsymbol{g}}_{2i-1}(1)\tilde{\boldsymbol{g}}_{2i}(-1)\mathbf{g}_l^{sb,\zeta}(1) \times \tilde{\boldsymbol{g}}_{2l+2}(-1)\prod_{i=l+2}^{N} \tilde{\boldsymbol{g}}_{2i-1}(1)\tilde{\boldsymbol{g}}_{2i}(-1)\right\},$$

$$\boldsymbol{g}_l^{sb,\zeta}(1) = \left\{\sum_\alpha \sigma^\alpha\tilde{\boldsymbol{g}}_{2l-1}(1)\sigma^\alpha\tilde{\boldsymbol{g}}_{2l}(-1)\right\}\tilde{\boldsymbol{g}}_{2l+1}(1) + \zeta\tilde{\boldsymbol{g}}_{2l-1}(1)\left\{\sum_\alpha \sigma^\alpha\tilde{\boldsymbol{g}}_{2l}(-1)\sigma^\alpha\tilde{\boldsymbol{g}}_{2l+1}(1)\right\}.$$

(C) *Model with two second-order phase boundaries*. For a family of models

$$J_{\rm r} = -J_{\rm rr} = \frac{J}{6}(u^2-1)(u^2+3),$$

$$J_{\rm l} = \frac{J}{48}(3u^2+5)(u^2+3), \quad J_{\rm d} = \frac{J}{3}u^2,$$

$$J_{\rm ll} = \frac{J}{12}(5u^2+3)(u^2+3), \quad J_{\rm dd} = \frac{J}{6}(u^4-6u^2-3), \tag{20}$$

with arbitrary u, the ground states are $\Psi_0(u)$ and $\Psi_0(-u)$, and the ground-state energy per rung is $E_0 = -J(7u^4+22u^2+19)/64$. The model at $u = 1$ is equivalent to the model (18) at $y = 2/3$. The arguments for the dual model[10] lead to the conclusion that the model (20) undergoes a phase transition to the Haldane phase at $u = 0$ and to the rung-singlet phase at $u = \infty$. Both of the transitions are of second order and accompanied with vanishing of energy gaps for solitons.

In total, five phases appear in the MP-solvable models discussed above and in Ref. 10. Some of the phase transitions between them actually happen in the parameter space of the solvable models. These are summarized in Fig. 1.

The nature of the scalar chiral phase is summarized as follows: (1) the ground states are doubly degenerate, (2)

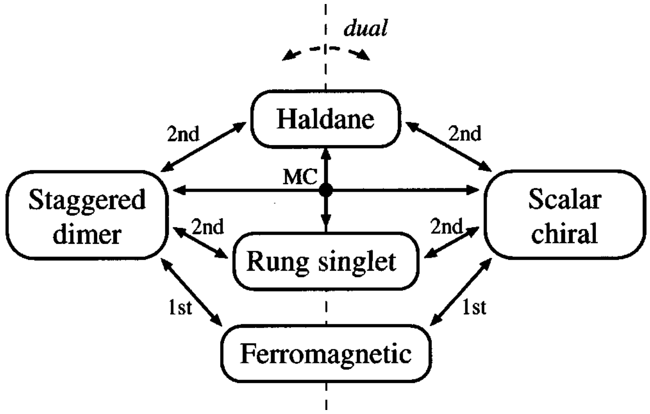

FIG. 1. Schematic picture of five phases and phase transitions that appear in MP-solvable models. Phase transitions occur along arrows. The number attached to each arrow denotes the order of the phase transition. Black circle (MC) denotes the multicritical point (19). The scalar chiral phase is dual to the staggered dimer phase, while Haldane, rung-singlet, and ferromagnetic phases are self-dual.

there is a finite energy gap between the ground states and excited states, and (3) the ground states have long-range staggered scalar chiral order and exponentially decaying spin correlations.

It is also easy to show that the string order, which was originally found in the Haldane state,[38,39] exists in the scalar chiral state $\Psi_0(u)$. The expectation value is given by

$$\left\langle (s_{1,j}^{\alpha}+s_{2,j}^{\alpha})\prod_{l=j}^{k-1}\exp\{i\pi(s_{1,l}^{\alpha}+s_{2,l}^{\alpha})\}(s_{1,k}^{\alpha}+s_{2,k}^{\alpha})\right\rangle$$
$$=4/(u^2+3)^2 \tag{21}$$

for $\alpha=x$, y or z. We note that the staggered dimer state $\Psi_{\mathrm{KM}}(u)$ also has exactly the same expectation value of the string order because the string operator is invariant under the duality transformation. This string order implies that a hidden $Z_2\times Z_2$ symmetry is spontaneously broken in the ground state.[40] This hidden symmetry was found[40,41] by applying a nonlocal unitary transformation U and one can find that this symmetry exists also in the Hamiltonian (1). It should be noted that this $Z_2\times Z_2$ symmetry is independent of the Z_2 chiral symmetry associated with the scalar chirality, since the scalar chiral operator after the nonlocal unitary transformation $U\mathcal{O}_{\mathrm{SC}}U^{-1}$ has the corresponding $Z_2\times Z_2$ symmetry. In finite systems with open boundary conditions, the scalar chiral MP ground states in fact have eightfold degeneracy associated with boundary spins and chirality. Thus, the hidden $Z_2\times Z_2$ symmetry, as well as the Z_2 chiral symmetry, is spontaneously broken in the scalar chiral phase. Recently, a different useful quantity $z_{2N}=\langle\exp[(2\pi i/2N)\Sigma_{l=1}^{2N}l(s_{1,l}^z+s_{2,l}^z)]\rangle$ was proposed,[42] which detects the average number n of valence bonds between neighboring rungs as $\lim_{N\to\infty}z_{2N}=(-1)^n$. In $\Psi_0(u)$, the expectation value is estimated as $\lim_{N\to\infty}z_{2N}=-1$ for finite u. This is consistent with the above valence bond picture, because z_{2N} is also invariant under the duality transformation, and $n=1$ in the staggered dimer state.

IV. AROUND THE SU(4)-SYMMETRIC POINT

Using the duality relation, we discuss the phase diagram around the SU(4)-symmetric point and show that this SU(4)-symmetric point is a multicritical point.

A. SU(2)×SU(2) spin ladders

We start from two SU(2)×SU(2) spin ladders. One SU(2)×SU(2) spin ladder is model II. Here we consider the case $J_{\mathrm{l}}\geqslant 0$. For $J_{\mathrm{l}}/J_{\mathrm{ll}}=1/4$, this model is SU(4) symmetric and exactly solvable by the Bethe ansatz, and the ground state is gapless critical.[31] For $J_{\mathrm{l}}/J_{\mathrm{ll}}>1/4$, the ground state has a finite gap and a staggered dimer order,[10–12] whereas for $0\leqslant J_{\mathrm{l}}/J_{\mathrm{ll}}\leqslant 1/4$ the ground state is gapless and critical.[11–15] The total Hamiltonian can be divided into the part of the SU(4) model $\mathcal{H}_0$ ($J_{\mathrm{ll}}=4J_{\mathrm{l}}$ with fixed J_{l}) and the perturbation $\mathcal{V}$ in the form

$$\mathcal{H}'=\mathcal{H}_0+\lambda\mathcal{V},$$

$$\mathcal{H}_0=J_{\mathrm{l}}\sum_l\{(\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1}+\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1})$$
$$+4(\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1})(\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1})\},$$

$$\mathcal{V}=\sum_l(\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1})(\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1}), \tag{22}$$

where $\lambda\equiv -4J_{\mathrm{l}}+J_{\mathrm{ll}}$ and J_{l} is fixed. Renormalization group analysis concluded that if the parameter λ is negative, the perturbation $\mathcal{V}$ is relevant and leads to a generation of a staggered dimer order with a finite spin gap, and if the coupling parameter is positive, this perturbation is irrelevant and keeps the ground state gapless.[13–15]

Applying the spin-chirality duality transformation to Eq. (22), one obtains the dual Hamiltonian

$$\tilde{\mathcal{H}}'=\mathcal{H}_0+\lambda\tilde{\mathcal{V}},$$

$$\tilde{\mathcal{V}}=\frac{1}{8}\sum_l(\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1}+\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1})-\frac{1}{8}\sum_l(\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l+1}$$
$$+\mathbf{s}_{2,l}\cdot\mathbf{s}_{1,l+1})+\frac{1}{2}\sum_l(\mathbf{s}_{1,l}\cdot\mathbf{s}_{1,l+1})(\mathbf{s}_{2,l}\cdot\mathbf{s}_{2,l+1})$$
$$+\frac{1}{2}\sum_l(\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l+1})(\mathbf{s}_{2,l}\cdot\mathbf{s}_{1,l+1}). \tag{23}$$

Here $\mathcal{H}_0$ is self-dual, and $\tilde{\mathcal{V}}$ is the perturbation dual to $\mathcal{V}$. The couplings of $\tilde{\mathcal{H}}'$, in total, are given as

$$\tilde{J}_{\mathrm{r}}=\tilde{J}_{\mathrm{rr}}=0,$$

$$\tilde{J}_{\mathrm{l}}=\frac{1}{2}J_{\mathrm{l}}+\frac{1}{8}J_{\mathrm{ll}},\quad \tilde{J}_{\mathrm{d}}=\frac{1}{2}J_{\mathrm{l}}-\frac{1}{8}J_{\mathrm{ll}},$$

$$\tilde{J}_{\mathrm{ll}}=2J_{\mathrm{l}}+\frac{1}{2}J_{\mathrm{ll}},\quad \tilde{J}_{\mathrm{dd}}=-2J_l+\frac{1}{2}J_{\mathrm{ll}}. \tag{24}$$

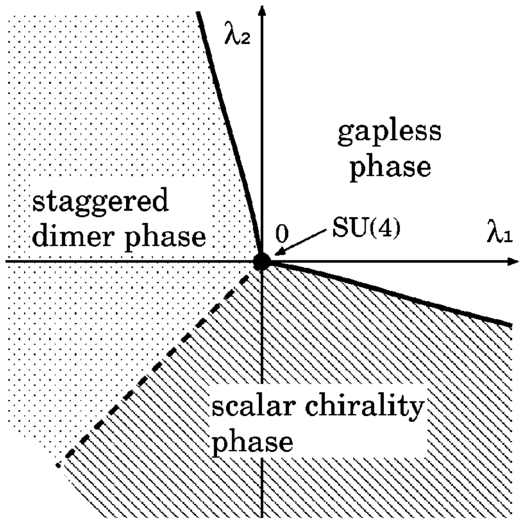

FIG. 2. Schematic possible phase diagram around the SU(4)-symmetric point. The phase transition on the solid line is of second order and that on the dashed line is either first order or second order. Other phases might be inserted around the phase boundaries.

From this transformation, one finds that the Hamiltonian (23) has a hidden $SU(2)\times SU(2)$ symmetry, where generators are given by $\Sigma_l S_l^\alpha$ and $\Sigma_l T_l^\alpha$ for $\alpha = x$, y, or z. The duality transformation leads to the case where, if the coupling parameter λ is negative, the perturbation $\tilde{\mathcal{V}}$ is relevant and induces a staggered scalar chiral order with a finite spin gap, and if the coupling parameter is positive, this perturbation is irrelevant and keeps the ground state gapless. When $\lambda = -\frac{8}{3}J_1$ (i.e., $J_{ll} = \frac{4}{3}J_l$), the model $\tilde{\mathcal{H}}'$ equals to the scalar chiral model (18) with $y = 2/3$ and as we have shown in Sec. III the exact ground state has an energy gap and the scalar chiral order.

B. Phase diagram around the SU(4) point

We now discuss the phase diagram around the SU(4)-symmetric point, and the two $SU(2)\times SU(2)$ models given above, considering the following generalized Hamiltonian:

$$\mathcal{H}'' = \mathcal{H}_0 + \lambda_1 \mathcal{V} + \lambda_2 \tilde{\mathcal{V}}. \tag{25}$$

This Hamiltonian contains two kinds of perturbation to the SU(4)-symmetric model. Because of the duality, phase boundaries are symmetric with the line $\lambda_1 = \lambda_2$. The nature of phases in $\lambda_1 > \lambda_2$ is related to that in $\lambda_1 < \lambda_2$ by the duality transformation. The above consideration leads to a conclusion that the SU(4)-symmetric point is a multicritical point and surrounded by the staggered dimer phase, the staggered scalar chiral one, and the critical one. If the scalar chiral phase touches with the staggered dimer phase, the phase boundary between two phases must exist exactly on the self-dual line $\lambda_1 = \lambda_2$ (see Fig. 2). Because of the U(1) symmetry, a rigorous theorem[46] concludes that in general both orders disappear on the self-dual line and hence the transition between the scalar chiral and dimer phases is second order, but, if uniform susceptibility of $\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}$ diverges, both orders can exist on this line. Note that the latter actually happens in the model (19). One plausible phase diagram is shown in Fig. 2. Recently we have numerically studied this phase diagram and obtained results consistent with the present conclusion.[43]

V. DISCUSSION

In this paper we have shown a rigorous example of scalar chiral ground states in SU(2) spin ladders with four-spin exchanges. The exact duality relation is the keystone of our theory. Our results demonstrated that four-spin exchanges can actually induce the scalar chiral long-range order. The scalar chiral phase extends to a wide parameter region and touches with the SU(4)-symmetric point. Previously, a scalar chiral phase was numerically found in the four-spin cyclic-exchange model on the two-leg ladder.[17] In their phase diagram, the scalar chiral phase appears next to the staggered dimer phase and the phase boundary indeed exists on the self-dual point[16] $J_4/J = 1/2$. This situation in the vicinity of the self-dual point shows a resemblance to that around the self-dual line ($\lambda_1 = \lambda_2 < 0$) in Fig. 2. Our recent numerical study of the Hamiltonian (1) indicates that the scalar chiral phase we found in this paper extends to the four-spin cyclic exchange case and that two phases belong to the same one.

We have shown that the SU(4)-symmetric model is self-dual under the spin-chirality duality transformation. We here note that this statement holds for the SU(4) spin-orbital models on arbitrary lattices. Recently SU(4) spin-orbital models on two-dimensional lattices[44] and on ladders[45] have been studied and it was discussed that a plaquette ordering may appear in the ground state. On a four-site plaquette, the SU(4) singlet state is the unique ground state and therefore it must be self-dual under the duality transformation. We hence conclude that plaquette ordering is also self-dual.

Last, we discuss the universality classes of phase transitions. The phase transitions into the scalar chiral phase are naturally in the same universality class as the dual transitions into the staggered dimer phase. For example, since the phase transition between the rung-singlet phase and the staggered dimer phase belongs to the $c = 3/2$ $SU(2)_2$ criticality,[9] we conclude that the transition between the scalar chiral phase and the rung-singlet phase also belongs to the same one. Since the two-dimensional Ising model is related to the $c = 1/2$ criticality, this $c = 3/2$ criticality can be plausibly regarded as a consequence of the $Z_2\times Z_2\times Z_2$ symmetry breaking.

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APPENDIX: UNITARY OPERATOR FOR DUALITY TRANSFORMATION

In this appendix, we show that the duality transformation (2) and (3) corresponds to a unitary transformation of two spins on rungs. The unitary operator is given by

$$U_\theta=\prod_l \exp[-i\theta P_l(s)]$$

$$=\prod_l \exp[i\theta(\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}-\tfrac{1}{4})] \tag{A1}$$

with $\theta=\pi/2$, where $P_l(s)$ denotes the projection operator onto the singlet state on the lth rung. Since the generator is transformed as $\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}=(\mathbf{s}_{1,l}+\mathbf{s}_{2,l})^2/2-3/4$, this unitary conserves the total spin on each rung. Note that the generator of this unitary is a summation of SU(4) generators $s_1^\alpha s_2^\alpha(\alpha=x,y,z)$ and hence the SU(4) symmetric model is naturally invariant under this transformation.

Let us demonstrate the unitary transformation of spins. It is convenient to reduce the unitary operator to the form

$$U_\theta=\prod_l [1+(1-e^{-i\theta})(\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l}-\tfrac{1}{4})], \tag{A2}$$

where we have used the relation $[P_l(s)]^2 = P_l(s)$. Using the following commutation relations

$$[\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l},\ \ \mathbf{s}_{1,l}+\mathbf{s}_{2,l}]=0, \tag{A3}$$

$$[\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l},\ \ \mathbf{s}_{1,l}-\mathbf{s}_{2,l}]=2i\mathbf{s}_{1,l}\times\mathbf{s}_{2,l}, \tag{A4}$$

$$[\mathbf{s}_{1,l}\cdot\mathbf{s}_{2,l},\ \ \mathbf{s}_{1,l}\times\mathbf{s}_{2,l}]=-\frac{i}{2}(\mathbf{s}_{1,l}-\mathbf{s}_{2,l}), \tag{A5}$$

and the unitary relation $U_\theta U_\theta^\dagger=1$, one can perform the unitary transformation of spins in the forms

$$U_\theta(\mathbf{s}_{1,l}+\mathbf{s}_{2,l})U_\theta^\dagger=\mathbf{s}_{1,l}+\mathbf{s}_{2,l}, \tag{A6}$$

$$U_\theta(\mathbf{s}_{1,l}-\mathbf{s}_{2,l})U_\theta^\dagger=\cos\theta(\mathbf{s}_{1,l}-\mathbf{s}_{2,l})-2\sin\theta\,(\mathbf{s}_{1,l}\times\mathbf{s}_{2,l}), \tag{A7}$$

$$U_\theta(\mathbf{s}_{1,l}\times\mathbf{s}_{2,l})U_\theta^\dagger=\frac{1}{2}\sin\theta\,(\mathbf{s}_{1,l}-\mathbf{s}_{2,l})+\cos\theta\,(\mathbf{s}_{1,l}\times\mathbf{s}_{2,l}). \tag{A8}$$

When $\theta=\pi/2$, we obtain the original duality transformation

$$\mathbf{S}_l=U_{\pi/2}\mathbf{s}_{1,l}U_{\pi/2}^\dagger, \tag{A9}$$

$$\mathbf{T}_l=U_{\pi/2}\mathbf{s}_{2,l}U_{\pi/2}^\dagger. \tag{A10}$$

From the form of the unitary operator, it is clear that this unitary corresponds to a gauge transformation of the singlet bond state

$$U_\theta|s\rangle_l=e^{-i\theta}|s\rangle_l, \tag{A11}$$

$$U_\theta|t_m\rangle_l=|t_m\rangle_l,\quad (m=-1,0,1). \tag{A12}$$

When $\theta=\pi/2$, one obtains the relation (6) from

$$|\sigma\rangle_{S,l}|\sigma'\rangle_{T,l}=U_{\pi/2}|\sigma\rangle_{1,l}|\sigma'\rangle_{2,l} \tag{A13}$$

for $\sigma\ \ (\sigma')=\uparrow,\downarrow$.

[1] K. L. Kugel' and D. I. Khomskii, Zh. Éksp. Teor. Fiz. **64**, 1429 (1973) [Sov. Phys. JETP **37**, 725 (1973)].
[2] M. Roger, J. H. Hetherington, and J. M. Delrieu, Rev. Mod. Phys. **55**, 1 (1983).
[3] S. Brehmer, H.-J. Mikeska, M. Muller, N. Nagaosa, and S. Uchida, Phys. Rev. B **60**, 329 (1999).
[4] M. Matsuda, K. Katsumata, R. S. Eccleston, S. Brehmer, and H.-J. Mikeska, Phys. Rev. B **62**, 8903 (2000); J. Appl. Phys. **87**, 6271 (2000).
[5] K. P. Schmidt, C. Knetter, and G. S. Uhrig, Europhys. Lett. **56**, 877 (2001).
[6] T. S. Nunner, P. Brune, T. Kopp, M. Windt, and M. Grüninger, Phys. Rev. B **66**, 180404 (2002).
[7] Y. Honda, Y. Kuramoto, and T. Watanabe, Phys. Rev. B **47**, 11 329 (1993).
[8] R. Coldea, S. M. Hayden, G. Aeppli, T. G. Perring, C. D. Frost, T. E. Mason, S.-W. Cheong, and Z. Fisk, Phys. Rev. Lett. **86**, 5377 (2001).
[9] A. A. Nersesyan and A. M. Tsvelik, Phys. Rev. Lett. **78**, 3939 (1997).
[10] A. K. Kolezhuk and H.-J. Mikeska, Phys. Rev. Lett. **80**, 2709 (1998); Int. J. Mod. Phys. B **12**, 2325 (1998).
[11] S. K. Pati, R. R. P. Singh, and D. I. Khomskii, Phys. Rev. Lett. **81**, 5406 (1998).
[12] Y. Yamashita, N. Shibata, and K. Ueda, J. Phys. Soc. Jpn. **69**, 242 (2000).
[13] P. Azaria, A. O. Gogolin, P. Lecheminant, and A. A. Nersesyan, Phys. Rev. Lett. **83**, 624 (1999).
[14] P. Azaria, E. Boulat, and P. Lecheminant, Phys. Rev. B **61**, 12 112 (2000).
[15] C. Itoi, S. Qin, and I. Affleck, Phys. Rev. B **61**, 6747 (2000).
[16] T. Hikihara, T. Momoi, and X. Hu, Phys. Rev. Lett. **90**, 087204 (2003).
[17] A. Läuchli, G. Schmid, and M. Troyer, Phys. Rev. B **67**, 100409(R) (2003).
[18] M. Takahashi, J. Phys. C **10**, 1289 (1977).
[19] H. J. Schmidt and Y. Kuramoto, Physica C **167**, 263 (1990).
[20] M. Roger, C. Bäuerle, Y. M. Bunkov, A. S. Chen, and H. Godfrin, Phys. Rev. Lett. **80**, 1308 (1998), and references therein.
[21] E. Dagotto and T. M. Rice, Science **271**, 618 (1996) and references therein.
[22] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11 413 (1989).
[23] R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988).
[24] K. Ohgushi, S. Murakami, and N. Nagaosa, Phys. Rev. B **62**, R6065 (2000).
[25] Y. Taguchi, Y. Oohara, H. Yoshizawa, N. Nagaosa, and Y. Tokura, Science **291**, 2573 (2001).

[26] K. Kubo and T. Momoi, Z. Phys. B: Condens. Matter **103**, 485 (1997).
[27] T. Momoi, K. Kubo, and K. Niki, Phys. Rev. Lett. **79**, 2081 (1997).
[28] G. Misguich, B. Bernu, C. Lhuillier, and C. Waldtmann, Phys. Rev. Lett. **81**, 1098 (1998).
[29] G. Misguich, C. Lhuillier, B. Bernu, and C. Waldtmann, Phys. Rev. B **60**, 1064 (1999).
[30] Y. Yamashita, N. Shibata, and K. Ueda, Phys. Rev. B **58**, 9114 (1998).
[31] B. Sutherland, Phys. Rev. B **12**, 3795 (1975).
[32] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987); Commun. Math. Phys. **115**, 477 (1988).
[33] A. Klümper, A. Schadschneider, and J. Zittartz, Z. Phys. B: Condens. Matter **87**, 281 (1992); Europhys. Lett. **24**, 293 (1993). A similar analysis was also performed in Ref. 34.
[34] M. Fannes, B. Nachtergale, and R. F. Werner, Europhys. Lett. **10**, 633 (1989); Commun. Math. Phys. **144**, 443 (1992).
[35] S. Knabe, J. Stat. Phys. **52**, 627 (1988).
[36] For the model (18) with $y=2/3$, the local Hamiltonian between the nearest-neighbor rungs can be written as a projection operator if one sets $J=3/4$ and hence one can apply the rigorous argument in Ref. 35 to the present system. We thereby prove that the energy gap in the thermodynamic limit is larger than $0.115\,474J$, where we have used the value of the energy gap $\Delta\epsilon$ for a finite system with eight rungs, evaluated as $\Delta\epsilon=0.289\,454J$.
[37] K. Itoh, J. Phys. Soc. Jpn. **68**, 322 (1999).
[38] M. den Nijs and K. Rommelse, Phys. Rev. B **40**, 4709 (1989).
[39] S. M. Girvin and D. P. Arovas, Phys. Scr. **T27**, 156 (1989).
[40] T. Kennedy and H. Tasaki, Phys. Rev. B **45**, 304 (1992); Commun. Math. Phys. **147**, 431 (1992).
[41] S. Takada and H. Watanabe, J. Phys. Soc. Jpn. **61**, 39 (1992).
[42] M. Nakamura and S. Todo, Phys. Rev. Lett. **89**, 077204 (2002); Suppl. Prog. Theor. Phys. **145**, 217 (2002).
[43] M. Nakamura, T. Hikihara, T. Momoi, and X. Hu (unpublished).
[44] Y. Q. Li, M. Ma, D. N. Shi, and F. C. Zhang, Phys. Rev. Lett. **81**, 3527 (1998).
[45] M. van den Bossche, P. Azaria, P. Lecheminant, and F. Mila, Phys. Rev. Lett. **86**, 4124 (2001).
[46] T. Momoi, J. Stat. Phys. **85**, 193 (1996).