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Coulomb effects in granular materials at not very low temperatures

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We consider effects of Coulomb interaction in a granular normal metal at not very low temperatures suppressing weak localization effects. In this limit calculations with the initial electron Hamiltonian are reduced to integrations over a phase variable with an effective action, which can be considered as a bosonization for the granular metal. Conditions of the applicability of the effective action are considered in detail and importance of winding numbers for the phase variables is emphasized. Explicit calculations are carried out for the conductivity and the tunneling density of states in the limits of large $g \gg 1$ and small $g \ll 1$ tunneling conductances. It is demonstrated for any dimension of the array of the grains that at small g the conductivity and the tunneling density of states decay with temperature exponentially. At large g the conductivity also decays with decreasing the temperature and its temperature dependence is logarithmic independent of dimensionality and presence of a magnetic field. The tunneling density of states for $g \gg 1$ is anomalous in any dimension but the anomaly is stronger than logarithmic in low dimensions and is similar to that for disordered systems. The formulas derived are compared with existing experiments. The logarithmic behavior of the conductivity at large g obtained in our model can explain numerous experiments on systems with a granular structure including some high-T_c materials.

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I. INTRODUCTION

The study of disordered systems attracts a lot of attention. Even if the electron-electron interaction is neglected the problem is not simple. Nevertheless, by now a rather complete understanding for noninteracting systems exists. It is well known that systems with one-dimensional geometry must be insulators for any week disorder. Two-dimensional systems are also believed to be localized, whereas there should occur the Anderson metal-insulator transition in three-dimensional disordered systems. This scenario has been formulated on the basis of scaling arguments long ago[1] and has been checked by numerous methods including diagrammatic expansions[2] and nonlinear σ-model calculations[3] (for a review, see, e.g., Ref. 4).

At weak disorder, one obtains so called weak localization corrections which are divergent in one-dimension (1D) and 2D (in 2D they are logarithmic). The infrared divergency is cut off either by the sample size or external frequency, or by the inverse inelastic time. Within the scaling theory the concrete mechanism of this cutoff does not matter. Due to the existence of the cutoffs flow diagrams stop at some point when changing parameters such as the sample size or temperature. So, within this picture by increasing the temperature one cannot expect anything except cutting the weak localization corrections.

If an electron-electron interaction is added additional corrections discovered by Altshuler and Aronov[5] become important. They are also logarithmically divergent in 2D but remain convergent in 3D. For the systems with the electron-electron interaction one can also derive a proper σ model[6] and demonstrate the renormalizability. Within such an approach the role played by the temperature is, as for the noninteracting models, to cut the diverging infrared corrections. After they are cut off the system is assumed to become a conventional normal metal. Although granular systems with the electron-electron interaction were not considered explicitly, it is generally expected that there should not be a qualitative difference between them and ''just disordered'' materials. This is definitely true for systems without the electron-electron interaction.

Therefore, a recent experimental observation[7] came as a surprise. In the experiment[7] measurements of conductivity of granular Al-Ge thick films were performed and several interesting effects have been discovered. These films consisted of Al grains embedded in an amorphous Ge matrix. The size of an Al grain was about 120 Å and the grains were at low temperatures in a superconducting state.

Destroying the superconducting pairing by a magnetic field up to 17 T the authors could study, in particular, properties of the normal state. Some features of the normal state related to a negative magnetoresistance due to superconducting fluctuations have been discussed recently[8] but another unusual observation remained unexplained.

The authors of Ref. 7 found in some samples a peculiar temperature dependence of the conductivity. Samples that had a high room temperature resistivity (this corresponds to a weak tunneling between the grains) showed an exponential decay of the conductivity as a function of temperature. This behavior is typical for insulators and has been interpreted in Ref. 7 in this way. Samples with larger intergranular couplings did not show any exponential decay but the resistivity did not saturate at low temperatures and the authors described its temperature dependence by a power law

$$R = AT^{-\alpha}, \tag{1.1}$$

with $\alpha = 0.117$. Apparently, with such a small value of α a logarithmic temperature dependence

$$R=A(1-\alpha \ln T) \tag{1.2}$$

[obtained by expansion of Eq. (1.1) in α] could describe the experimental data as well.

Both the power law dependence (1.1) and the logarithmic dependence (1.2) were not expected from the theories of disordered metals. Although the logarithmic dependence is natural of 2D films, it is relevant to emphasize that the array of the grains was three dimensional and the behavior described by Eqs. (1.2) was observed at high magnetic fields. This definitely excludes an interpretation of the logarithmic or power law behavior in terms of weak localization corrections typical for 2D.[2,5,6]

In this paper we try to explain the transition from the exponential temperature behavior at small g to the logarithmic dependence of Eq. (1.2) assuming that the temperature is not very low, which actually corresponds to the experiment[7] where the lowest temperature was around 0.3 K. We consider a model of granular metals at not very low temperature and demonstrate that changing the dimensionless tunneling conductance g one can have either exponential temperature dependence of the resistivity at small $g\lesssim 1$ or the logarithmic behavior (1.2) at large $g\gtrsim 1$. It will be shown that the result is applicable for any dimensionality of the array of grains, which contrasts usual logarithmic corrections due to interference effects typical for 2D only.

Study of the regime of not very low temperatures of the granular metals has started recently in Ref. 9 where it was demonstrated that the regime of the temperatures

$$T\gg g\delta, \tag{1.3}$$

where δ is the mean level spacing, is clearly different from the regime of lower temperatures. In this regime, well known weak localization effects are suppressed but, nevertheless, the system may exhibit a nontrivial behavior.

Inequality (1.3) is written for $g\gg 1$. In this limit, one can study the model by summation of diagrams containing impurity lines, as it has been suggested in Ref. 9. Nontrivial effects arise from a renormalization of the Coulomb interaction by impurities. Actually, a nontrivial behavior originates from the diagrams of the Altshuler-Aronov type[5] modified for the granular system. Therefore, for large $g\gg 1$ the theory presented in this paper is equivalent to the summation of the Altshuler-Aronov diagrams. However, we use an effective phase functional, which gives us an opportunity to consider not only the first order in the Coulomb interaction but to go into higher orders by writing renormalization group equations.

If the tunneling conductance is not very large, $g\lesssim 1$, the inequality (1.3) should be replaced by the inequality

$$T\gg \delta. \tag{1.4}$$

In the limit $g\lesssim 1$, it is not possible to get any results by summation of conventional diagrams and one should use the phase functional. A very important result of Ref. 9 is that in the limit of not very low temperatures, Eqs. (1.3), (1.4), the phase functional has a form similar to that suggested by Ambegaokar, Eckern, and Schön (AES) long ago[10] for a description of quantum dissipation[11] in metals.

At lower temperatures $T\lesssim g\delta$ one must take into account low energy diffusion modes, and the system should be described by a more complicated nonlinear σ model containing both the phases ϕ and matrices Q. Of course, in the limit $T\to 0$ the AES functional is no longer applicable and the notion of "quantum dissipation" loses its sense. In other words, the "quantum dissipation" corresponds to a limit of not very low temperatures for the system under consideration.

It is relevant to mention that the model of a granular metal may roughly describe disordered systems with a low electron concentration. One can imagine that in such systems electrons spend a long time in some kind of traps or puddles and tunnel between them with a small rate. Then, such a situation resembles the granular systems. A short account of the ideas of the present work has been presented in a recent paper.[12]

The article is organized as follows. In Sec. II we formulate the model and derive a free energy functional containing phases. We discuss how one should integrate over the phases accounting for winding numbers. In Sec. III we consider conductivity in the limit of large conductances $g\gg 1$ and obtain the temperature behavior (1.2). For calculations we use both the perturbation theory and renormalization group techniques. In Sec. IV we consider the tunneling density of states in the limit $g\gg 1$. The final result for a two-dimensional array of grains, Eq. (4.9), has the same form as the corresponding formula for a disordered "homogeneous" metal. In Sec. V we consider the limit of small $g\ll 1$ and show how one can carry out summation over winding numbers. Physical quantities such as the conductivity and the tunneling density of states are shown to be exponentially small in temperature. In Sec. VI we discuss the results and make a comparison with the experiment.

II. CHOICE OF THE MODEL. PHASE FUNCTIONAL.

In the present work, we consider a simplified model where metal grains form a regular lattice. The grains are disordered due to impurities or irregular boundaries. Each grain is separated from its nearest neighbors by tunneling barriers. We assume that the main contribution to the macroscopic resistivity of the granular system comes from the intergranular tunneling.

The Hamiltonian describing the model is chosen as

$$\hat{H}=\hat{H}_0+\hat{H}_t+\hat{H}_c, \tag{2.1}$$

where $\hat{H}_0$ is the one-electron Hamiltonian of isolated grains including disorder within the grains

$$\hat{H}_0=\int \psi^+(\mathbf{r})\left(-\frac{\boldsymbol{\nabla}^2}{2m}+U(\mathbf{r})\right)\psi(\mathbf{r})d\mathbf{r},$$

where $U(\mathbf{r})$ is a random potential.

The tunneling of the electrons between the grains is given by

$$\hat{H}_t=\sum_{\mathbf{i},\mathbf{j},\alpha,\alpha'} t_{\mathbf{ij}}\hat{\psi}^+_{\alpha\mathbf{i}}\hat{\psi}_{\alpha'\mathbf{j}}, \tag{2.2}$$

where the summation is performed over the states α, α' of each grain (spin is conserved) and over neighboring grains i and j. The possibility to tunnel from the state α to an arbitrary state α' of other grains introduces an additional disorder resulting in a finite tunnel conductance.

The term $\hat{H}_c$ in Eq. (2.1) describes the charging energy

$$\hat{H}_c = \frac{e^2}{2} \sum_{\mathbf{ij}} \hat{N}_{\mathbf{i}} C_{\mathbf{ij}}^{-1} \hat{N}_{\mathbf{j}}. \tag{2.3}$$

In Eq. (2.3),

$$\hat{N}_{\mathbf{i}} = \int \hat{\psi}^{+}(\mathbf{r}_{\mathbf{i}}) \hat{\psi}(\mathbf{r}_{\mathbf{i}}) d\mathbf{r}_{\mathbf{i}} - \overline{N} \tag{2.4}$$

is the excess number of electrons in the **i**th grain, $\overline{N}$ is the dimensionless local potential, and $C_{\mathbf{ij}}$ is the capacitance matrix. The integration over $\mathbf{r}$ in Eqs. (2.1), (2.4) is performed over the grain with the coordinate **i** of the center and includes summation over the spin.

Equation (2.3) describes the long-range part of the Coulomb interaction in the limit of weak disorder inside the grains and has been used in many works (for a review, see[13]). In principle, one could consider also a superconducting or magnetic part of the interaction within the dot but we assume that the grains are in the normal state. The long range part of the Coulomb interaction, Eq. (2.3), describes the classical charging energy. This interaction can lead to the Coulomb blockade and to insulating macroscopic properties of the system of the grains.

Calculations with the Hamiltonian $\hat{H}$, Eqs. (2.1, 2.3), can be replaced in a standard way by computation of a functional integral over anticommuting fields $\psi(\mathbf{r}_{\mathbf{i}}, \tau)$.

Although the model described by Eqs. (2.1, 2.3) contains only the long range part of the Coulomb interaction, it is still very complicated, because at very low temperatures interference becomes very important and one has to consider an interplay of localization and interaction effects. One could do this either using diagrammatic expansions[5] or writing a nonlinear σ-model.[6] Both methods allow one to consider the limit of large tunneling conductances g and the results are strongly dependent on the dimensionality. However, the behavior, Eq. (1.1) or (1.2) was not predicted for 3D in any of these works.

The model, Eqs. (2.1), (2.3), becomes simpler if the temperature T is not very low such that low energy diffusion modes are damped. As it was discussed in a recent publication,[9] the granular metal can be well described at temperatures $T \gg g\delta$, where δ is the mean level spacing in a single grain, by the Ambegaokar-Eckern-Schön (AES)[10] functional of the free energy. If $g \lesssim 1$, this condition should be replaced by $T \gg \delta$. The limit of not very low temperatures not only simplifies the consideration but is interesting on its own because it leads to an unusual behavior of physical quantities and is easily accessible experimentally. In particular, we will see that changing the tunneling conductance g one may have a crossover from the exponential temperature dependence of the resistivity to the logarithmic behavior (1.2).

We calculate the conductivity $\sigma(\omega)$ using the Kubo formula and making an analytical continuation from Matsubara frequencies $i\Omega_n = 2\pi i n T$ to real frequencies ω.[14] Within this formalism the conductivity $\sigma(\omega)$ can be written in the form

$$\sigma(\omega) = \frac{i a^{d-2}}{\omega} \left[\sum_{\mathbf{j}} \int d\mathbf{r}_{\mathbf{j}} \int_0^{\beta} d\tau e^{i\Omega_n \tau} K_{\mathbf{aa}}(\mathbf{r}_{\mathbf{i}} - \mathbf{r}_{\mathbf{j}}, \tau) \right]_{\Omega_n \to -i\omega + \delta}, \tag{2.5}$$

where **a** is a vector connecting the centers of neighboring grains **i** and $\mathbf{i}+\mathbf{a}$, $a = |\mathbf{a}|$, and d is the dimensionality of the array.

The function $K_{\mathbf{aa}'}$ in Eq. (2.5) can be written as

$$K_{\mathbf{aa}'}(\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}; \tau) = \Pi_{\mathbf{aa}'}(\mathbf{r}_i, \mathbf{r}_j; \tau) - e^2 \delta_{\mathbf{aa}'} \delta_{\mathbf{ij}} \delta(\tau) \langle \hat{H}_{t\mathbf{i}}^{\mathbf{a}}(\tau) + \text{H.c.} \rangle, \tag{2.6}$$

where

$$\Pi_{\mathbf{aa}'}(\mathbf{r}_{\mathbf{i}}, \mathbf{r}_{\mathbf{j}}; \tau) = -\langle T_{\tau} \hat{J}_{\mathbf{a}}(\mathbf{r}_{\mathbf{i}}, \tau) \hat{J}_{\mathbf{a}'}(\mathbf{r}_{\mathbf{j}}, 0) \rangle, \tag{2.7}$$

and

$$\hat{H}_{t\mathbf{i}}^{\mathbf{a}}(\tau) = \sum_{\alpha, \alpha'} t_{\mathbf{i}+\mathbf{a}, \mathbf{i}} \hat{\psi}_{\alpha, \mathbf{i}+\mathbf{a}}^{+}(\tau) \hat{\psi}_{\alpha', \mathbf{i}}(\tau). \tag{2.8}$$

The abbreviation H.c. means a Hermitian conjugation. The tunneling current operator $\hat{J}_{\mathbf{a}}(\mathbf{r}_{\mathbf{i}}, \tau)$ entering Eq. (2.7) takes the standard form

$$\hat{J}_{\mathbf{a}}(\mathbf{r}_{\mathbf{i}}) = ie(\hat{H}_{t\mathbf{i}}^{\mathbf{a}} - \text{H.c.}). \tag{2.9}$$

The first term in Eq. (2.6) corresponds to a "paramagnetic" contribution whereas the second one is of a "diamagnetic" origin.

In order to reduce the calculation of physical quantities to a computation of correlation functions with the AES action we decouple the interaction term, Eq. (2.3), by integration over an additional $V_{\mathbf{i}}(\tau)$

$$\begin{aligned} \exp\left(-\frac{e^2}{2} \sum_{\mathbf{ij}} \hat{N}_{\mathbf{i}} C_{\mathbf{ij}}^{-1} \hat{N}_{\mathbf{j}} \right) &= \int \exp\left(-i \sum_{i} \int [\psi^*(\mathbf{r}_{\mathbf{i}}, \tau) \psi(\mathbf{r}_{\mathbf{i}}, \tau) d\mathbf{r}_{\mathbf{i}} - \overline{N}] V_{\mathbf{i}}(\tau) d\tau \right) \\ &\quad \times \exp\left(-\frac{1}{2e^2} \sum_{\mathbf{ij}} \int d\tau V_{\mathbf{i}}(\tau) C_{\mathbf{ij}} V_{\mathbf{j}}(\tau) \right) DV \end{aligned} \tag{2.10}$$

and then, following Refs. 15,9, remove this field from $\hat{H}_0$ by the gauge transformation

$$\psi(\mathbf{r}_{\mathbf{i}}, \tau) \to e^{-i\varphi_{\mathbf{i}}(\tau)} \psi_{\mathbf{i}}(\mathbf{r}_{\mathbf{i}}, \tau), \quad \dot{\varphi}_{\mathbf{i}}(\tau) = V_{\mathbf{i}}(\tau). \tag{2.11}$$

This is not a trivial procedure, because the new fields $\psi_{\alpha}(\tau)$ should obey, as before, the fermionic boundary condition

$$\psi(\mathbf{r_i},\tau)=-\psi(\mathbf{r_i},\tau+\beta),\quad \beta=1/T. \tag{2.12}$$

Let us forget for a while about the tunneling between the grains and explain how one can proceed for the Hamiltonian H_0 of a single grain.

Due to the boundary condition, Eq. (2.12), the field $V_{\mathbf{i}}(\tau)$ cannot be completely removed from $\hat{H}_0$ and this should be done approximately. The field $V_{\mathbf{i}}(\tau)$ can be represented as a sum of a static $V_{0\mathbf{i}}$ part and periodic function $\tilde{V}_{\mathbf{i}}(\tau)$

$$V_{\mathbf{i}}(\tau)=V_{0\mathbf{i}}+\tilde{V}_{\mathbf{i}}(\tau),\quad \int_0^{\beta}\tilde{V}_{\mathbf{i}}(\tau)d\tau=0. \tag{2.13}$$

The static part V_0 can still be arbitrary and we rewrite it as

$$V_{0\mathbf{i}}=2\pi Tk_{\mathbf{i}}+\rho_{\mathbf{i}},\quad -\pi T<\rho_{\mathbf{i}}<\pi T, \tag{2.14}$$

where $k_{\mathbf{i}}=0,\pm1,\pm2,\dots$.

If we neglected $\rho_{\mathbf{i}}$ we would be able to remove $V_{\mathbf{i}}(\tau)$ from H_0. In this case we would use, instead of the phases $\varphi(\tau)$ from Eq. (2.11), the phases $\tilde{\phi}(\tau)$ defined as

$$\tilde{\phi}_{\mathbf{i}}(\tau)=\phi_{\mathbf{i}}(\tau)+2\pi Tk_{\mathbf{i}}\tau, \tag{2.15}$$

where $-\infty<\phi_{\mathbf{i}}(\tau)<\infty$, $\phi_{\mathbf{i}}(0)=\phi_{\mathbf{i}}(\beta)$. It is clear that making the gauge transformation, Eq. (2.11), with the phases $\tilde{\phi}_{\mathbf{i}}(\tau)$ instead of $\varphi_{\mathbf{i}}(\tau)$, preserves the antiperiodicity of the $\psi(\tau)$, Eq. (2.12).

The variable $\rho_{\mathbf{i}}$ cannot generally be neglected. This term is not important only in the limit $T\gg\delta$. In order to estimate its contribution we can calculate the partition function $Z^{(\rho)}$ of a single grain (normalized by the partition function of the system without interaction). Carrying out the summation over the Matsubara frequencies ε_n and making the gauge transformation with the phases $\tilde{\phi}(\tau)$, Eq. (2.15), we obtain

$$Z^{(\rho)}=\exp f(\rho), \tag{2.16}$$

$$f(\rho)=\sum_{\alpha}\left[\ln\cosh\frac{\xi_\alpha+i\rho}{2T}-\ln\cosh\frac{\xi_\alpha}{2T}\right]-i\overline{N}\rho,$$

where ξ_α is the energy of the state α (the Fermi energy ε_F is subtracted).

The sum over α in Eq. (2.16) extends over all states of the grain. However, the contribution of the states far from the Fermi energy must be compensated by the local potentials $\overline{N}$. As usual, an essential contribution comes from energies of the order T in the vicinity of the Fermi energy. The linear in ρ term in the function $f(\rho)$, Eq. (2.16), must be absent due to the choice of $\overline{N}$.

A typical separation of the energy levels ξ_α is of the order of the mean level spacing δ. At not very low temperatures $T\gg\delta$, the sum over α can be replaced by a proper integral over a continuous variable ξ

$$\sum_{\alpha}\rightarrow\frac{1}{\delta}\int d\xi$$

and we write the function $f(\rho)$ in the form

$$f(\rho)=\frac{1}{\delta}\int_{-R}^{R}\left[\ln\cosh\frac{\xi+i\rho}{2T}-\ln\cosh\frac{\xi}{2T}\right]d\xi, \tag{2.17}$$

where R is an energy in the range $T\ll R\ll\varepsilon_F$. If ρ satisfies the inequality (2.14) we can deform the contour of integration when integrating with the first term in Eq. (2.17) and integrate first along the straight line $(-R,-R-i\rho)$, then along $(-R-i\rho,\ R-i\rho)$ and, at last, along $(R-i\rho,\ R)$. The integral over the second segment cancels the second term in the integrand in Eq. (2.17) and we come to the result

$$f(\rho)=-\frac{\rho^2}{2\delta T}. \tag{2.18}$$

Using Eqs. (2.16), (2.18) we can understand easily that the main contribution in integrals over ρ comes from $\rho\sim(T\delta)^{1/2}\ll T$. Therefore, in the limit $T\gg\delta$ the variable ρ can safely be put to zero in all calculations and we come to a phase free energy functional containing only the phases $\tilde{\phi}(\tau)$, Eq. (2.15). We see from this argument that introducing the "winding numbers" $k_{\mathbf{i}}$, Eqs. (2.15), (2.14), is unavoidable. The opposite limit $T\ll\delta$ is more difficult and is not considered in the present paper.

Thus, in the limit $T\gg\delta$, we are able to remove the effective voltage $V_{\mathbf{i}}(\tau)$ from the single grain Hamiltonian H_0, which means removing the Coulomb interaction. However, the phases enter now the tunneling Hamiltonian H_t. Expanding in the tunneling term H_t, Eq. (2.2), up to the second order we obtain the AES (Ref. 10) action S in the standard form

$$S=S_c+S_t, \tag{2.19}$$

where S_c describes the charging energy

$$S_c=\frac{1}{2e^2}\sum_{\mathbf{ij}}\int_0^{\beta}d\tau C_{\mathbf{ij}}\frac{d\tilde{\phi}_{\mathbf{i}}(\tau)}{d\tau}\frac{d\tilde{\phi}_{\mathbf{j}}(\tau)}{d\tau} \tag{2.20}$$

and S_t stands for tunneling between the grains

$$S_t=\pi g\sum_{|\mathbf{i}-\mathbf{j}|=a}\int_0^{\beta}d\tau d\tau'\,\alpha(\tau-\tau')\sin^2\left(\frac{\tilde{\phi}_{\mathbf{ij}}(\tau)-\tilde{\phi}_{\mathbf{ij}}(\tau')}{2}\right). \tag{2.21}$$

The function $\alpha(\tau)$ in Eq. (2.21) has the form

$$\alpha(\tau)=T^2\{\mathrm{Re}[\sin(\pi T\tau+i\delta)]^{-1}\}^2.$$

In Eqs. (2.19)–(2.21), **i** and **j** stand for coordinates of grains and a is the diameter of a grain. The dimensionless conductance g is given by

$$g=2\pi\nu_0^2t_{\mathbf{ij}}^2, \tag{2.22}$$

where $t_{\mathbf{ij}}$ is the tunneling amplitude from grain **i** to grain **j** (spin is included) and ν_0 is the density of states of noninteracting electrons.

Calculation of different averages with the functional S, Eqs. (2.19)–(2.21), implies integration over $\phi_{\mathbf{i}}(\tau)$ and summation over $k_{\mathbf{i}}$. At large $g\gg1$, one can put all $k_{\mathbf{i}}=0$. However, at $g\lesssim1$ one should sum over all $k_{\mathbf{i}}$ and neglecting the contribution of the nonzero winding numbers leads to incorrect results. We emphasize that the AES free energy functional can be derived under the assumption that the temperatures are not very low, Eqs. (1.3), (1.4), and is applicable in this limit only. At lower temperatures one would have to take into account interference effects[9] or the discreteness of the levels in single grains.

Using the Kubo formulas (2.5)–(2.9), we can express also the conductivity in terms of the phases $\tilde{\phi}_{\mathbf{i}}(\tau)$. After simple manipulations we represent the conductivity $\sigma(\omega)$ as

$$\sigma(\omega)=\frac{ia^{2-d}}{\omega}\left[\int_0^\beta d\tau e^{i\Omega_n\tau}K(\tau)\right]_{\Omega_n\to -i\omega+\delta}, \quad (2.23)$$

$$K(\tau)=\langle X_2^{\mathbf{a}}(\tau)\rangle-\sum_{\mathbf{i}}\langle X_{10}^{\mathbf{a}}(\tau)X_{1\mathbf{i}}^{\mathbf{a}}(0)\rangle,$$

$$X_2^{\mathbf{a}}(\tau)=e^2\pi g\int_0^\beta d\tau'[\delta(\tau)-\delta(\tau'-\tau)]\alpha(\tau')$$

$$\times\cos[\tilde{\phi}_{\mathbf{i},\mathbf{i}+\mathbf{a}}(\tau')-\tilde{\phi}_{\mathbf{i},\mathbf{i}+\mathbf{a}}(0)],$$

$$X_{1\mathbf{i}}^{\mathbf{a}}(\tau)=e\pi g\int_0^\beta \alpha(\tau-\tau')\sin[\tilde{\phi}_{\mathbf{i},\mathbf{i}+\mathbf{a}}(\tau')-\tilde{\phi}_{\mathbf{i},\mathbf{i}+\mathbf{a}}(\tau)]d\tau'.$$

In Eqs. (2.23), $\tilde{\phi}_{\mathbf{ij}}(\tau)=\tilde{\phi}_{\mathbf{i}}(\tau)-\tilde{\phi}_{\mathbf{j}}(\tau)$ for **i** and **j** standing for neighboring grains and

$$\langle\cdots\rangle=\int(\cdots)\exp(-S)D\tilde{\phi}\left(\int\exp(-S)D\tilde{\phi}\right)^{-1}, \quad (2.24)$$

where $D\tilde{\phi}$ implies both the functional integration over $\phi(\tau)$ and summation over the winding numbers $k_{\mathbf{i}}$.

Equations (2.19)–(2.24) represent the conductivity $\sigma(\omega)$ in a closed form in terms of a functional integral. The contribution of the function $X_2(\tau)$ originates from the diamagnetic term [the second term in Eq. (2.6)], whereas the correlation function $\langle X_1X_1\rangle$ comes from the paramagnetic term Π, Eq. (2.7) [the first term in Eq. (2.6)]. Although the model described by Eqs. (2.19)–(2.24) is simpler than the initial model (2.1)–(2.3), explicit formulas can be written only in limiting cases.

If the temperature T is very high, $T\gg E_c\sim e^2C_{ij}^{-1}$, where E_c is the electrostatic energy of adding one electron to a grain, fluctuations of the phases $\tilde{\phi}$ are negligible and one can set $\tilde{\phi}=0$ in the expressions for X_1 and X_2 in Eqs. (2.23). Then, we obtain easily the conductivity

$$\sigma_0=e^2ga^{2-d}. \quad (2.25)$$

Equation (2.25) describes the classical conductivity of the granular metal without the Coulomb interaction and shows that at temperatures exceeding E_c charging effects are not important.

In the opposite limit $T\ll E_c$, transport in the granulated system has much more interesting characteristics. This inequality can be compatible with the inequality $T\gg\max\{g\delta,\delta\}$, used for the derivation of Eqs. (2.19)–(2.24), because $E_c\gg\delta$ for 2D and 3D grains (the charging energy E_c is inversely proportional to size of the grain, while δ is inversely proportional to its volume).

At temperatures $T\lesssim E_c$, calculations are possible only in the limiting cases $g\gg 1$ and $g\ll 1$ and this will be done in the subsequent sections. The same action S, Eqs. (2.19)–(2.21), was used in Ref. 16, and a metal-insulator transition has been predicted in a 2D array of tunnel junctions. However, the authors of Ref. 16 did not calculate the conductivity but discussed properties of the partition function. For large g they did not account for phase fluctuations properly which, as we show here, are responsible for the behavior (1.2). Moreover, we find a transition in any dimensionality.

III. CONDUCTIVITY IN THE METALLIC REGIME AT $G\gg 1$

A. Perturbation theory

In the limit of large conductances $g\gg 1$, the tunneling term (2.21) suppresses large fluctuations of ϕ. It is clear that all nonzero winding numbers $k_{\mathbf{i}}$ can be neglected. Accounting for nonzero $k_{\mathbf{i}}$ (as well as variations of $\overline{N_{\mathbf{i}}}$) would lead to contributions of order $\exp(-g)$, which can be neglected in any expansion in $1/g$. At the same time, the phase fluctuations can change considerably the classical result (2.25), even in the limit $g\gg 1$. Let us understand first the role of the fluctuations within a perturbation theory in $1/g$. The zeroth order of the perturbation theory (all phases $\tilde{\phi}$ are set to zero), gives for the conductivity the classical result σ_0, Eq. (2.25). In order to consider higher orders we expand the action S, Eqs. (2.19)–(2.21), in ϕ.

The quadratic part S_2 of the action S will serve as the bare action in the perturbation theory we want to develop now. Keeping terms of the second order in ϕ in Eqs. (2.19)–(2.21) and performing Fourier transformation in both coordinates of the grains and the imaginary time we reduce the action S to the form

$$S_0=T\sum_{\mathbf{q},n}\phi_{\mathbf{q},n}G_{\mathbf{q},n}^{-1}\phi_{-\mathbf{q},-n}, \quad (3.1)$$

$$G_{\mathbf{q},n}^{-1}=\omega_n^2/[4E(\mathbf{q})]+2g|\omega_n|\sum_{\mathbf{a}}(1-\cos\mathbf{q}\cdot\mathbf{a}), \quad (3.2)$$

where $E(\mathbf{q})=e^2/[2C(\mathbf{q})]$ and $C(\mathbf{q})$ is the Fourier transform of the capacitance matrix $C_{\mathbf{ij}}$ (**q** are quasimomenta for the array of the grains). One should sum in Eq. (3.1) over d unit lattice vectors **a**, where d is the dimensionality of the array.

If we kept only quadratic in ϕ terms in the action S but did not expand the function X_2, Eq. (2.23), we would reduce the correlation function $\langle X_{2\mathbf{a}}(\Omega_n)\rangle$ to the form

$$\langle X_2^{\mathbf{a}}(\Omega_n)\rangle_0=\pi e^2g\int_0^\beta\alpha(\tau)(1-e^{i\Omega_n\tau})e^{-\tilde{G}_{\mathbf{a}}(\tau)}d\tau, \quad (3.3)$$

$$\tilde{G}_{\mathbf{a}}(\tau)=4Ta^d\sum_{\omega_n>0}\int\frac{d\mathbf{q}}{(2\pi)^d}G_{\mathbf{q}n}\sin^2\frac{\mathbf{q}\cdot\mathbf{a}}{2}\sin^2\frac{\omega_n\tau}{2}, \tag{3.4}$$

where $\langle\cdots\rangle_0$ means averaging over the phases ϕ with the action S_0, Eq. (3.1), and $\omega_n=2\pi n$. What remains to be done in order to calculate the contribution $X_2^{(0)}$ to the conductivity is to compute the integral in Eq. (3.3) for the Matsubara frequencies Ω_n and make the analytical continuation $\Omega_n\to-i\omega+\delta$. As it is clear from Eqs. (3.2), (3.4), the function $\tilde{G}_{\mathbf{a}}(\tau)$ contains large logarithms $\ln(gE_c\tau)$ and essential τ are of the order $1/T$. Therefore, we may calculate the integral for $\tilde{G}_{\mathbf{a}}(\tau)$ with a logarithmic accuracy. Neglecting the ω_n^2 term in $G_{\mathbf{q},n}^{-1}$, Eq. (3.2), we reduce Eq. (3.4) to the form

$$\tilde{G}_{\mathbf{a}}(\tau)=\frac{T}{dg}\sum_{\omega_n>0}^{\omega_c}\frac{1-\cos(\omega_n\tau)}{\omega_n}. \tag{3.5}$$

In Eq. (3.5) one should sum over positive Matsubara frequencies up to the cutoff $\omega_c\sim gE_c$. Equation (3.5) shows a remarkable independence of the result on the structure of the lattice. The only information about the lattice is the parameter d entering Eq. (3.5). For the cubic lattice considered here, d is the dimensionality of the array. However, for an arbitrary lattice the parameter d is equal to the one half of the coordination number. What is also important, there are no "infrared" divergencies in the integral over $\mathbf{q}$ in any dimensionality including 2D and 1D. This is specific for the conductivity. We will see later that the tunnelling density of states ν is sensitive to the dimensionality in the same approximation due to infrared divergencies.

As we are performing the calculations with the logarithmic accuracy, we may replace τ by $1/T$ in the function $\tilde{G}_{\mathbf{a}}$ and calculate the remaining integral over τ in Eq. (3.3) ignoring the dependence of the function $\tilde{G}_{\mathbf{a}}$ on τ. Then, we obtain

$$\langle X_2(\omega)\rangle_0=-i\omega e^2g\left(\frac{T}{gE_c}\right)^{\alpha}, \tag{3.6}$$

where

$$\alpha=(2\pi gd)^{-1}. \tag{3.7}$$

This form of the correlation function $\langle X_2(\omega)\rangle$ would lead to the power law dependence of the conductivity on temperature of the form of Eq. (1.1). A similar result has been obtained for the voltage dependence of the conductance of a single junction in a model with an electromagnetic environment.[17,18] If Eq. (1.1) were the final result of our calculations we could argue that, in the model under consideration, fluctuations of the phases and, hence, of the voltages in the grains are equivalent to fluctuations in the electromagnetic environment of the works.[17,18] In other words, each grain would be considered as surrounded by an effective medium of other grains and the voltage fluctuations of the medium would lead to the power law (1.1).

However, Eq. (3.6), is not the final result yet because we have to calculate also the contribution coming from higher order terms of the expansion of the action S, Eqs. (2.19)–(2.21), in ϕ as well as the contribution of the correlation function $\langle X_1X_1\rangle$, Eq. (2.23). Taking into account these contributions can be performed writing an expansion for the conductivity in powers of $1/g$. The first three terms $\sigma^{(1)}$ of the expansion of the conductivity in $1/g$ coming from the function $\langle X_2(\omega)\rangle_0$, Eq. (3.6) can be written as

$$\sigma^{(1)}/\sigma_0=1-\alpha\ln\left(\frac{gE_c}{T}\right)+\frac{\alpha^2}{2}\ln^2\left(\frac{gE_c}{T}\right) \tag{3.8}$$

and we want to find now contributions of the order up to α^2 coming from $\langle X_1X_1\rangle$ and those originating from S_4, where S_4 contains terms of order ϕ^4 in the action S, Eqs. (2.19)–(2.21).

As concerns a contribution coming from the correlation function $\langle X_1X_1\rangle$, the first nonvanishing term is of the order α^2 and it does not contain powers of $\ln(gE_c/T)$. So, we neglect the function $\langle X_1X_1\rangle$ and concentrate on the contribution to the function $\langle X_2(\omega)\rangle$ coming from the anharmonic part S_4 of the action S. It is clear from a power counting that higher order terms of the expansion of S lead to contributions containing higher powers of α and we do not consider them now.

The lowest order contribution coming from S_4 is obtained by averaging with the action S_0, Eq. (3.1), of a product of a term ϕ^2 taken from the expansion of X_2, Eq. (2.23), and S_4. A proper expression can be written as

$$\begin{aligned}\Bigg\langle\sum_{\mathbf{i}_1,\mathbf{a}_1}\int_0^{\beta}\int_0^{\beta}\int_0^{\beta}d\tau d\tau_1 d\tau_1'(e^{i\Omega_n\tau}-1)\alpha(\tau)\alpha(\tau_1-\tau_1')\\ \times[\phi_{\mathbf{i}+\mathbf{a}_0,\mathbf{i}}(\tau)-\phi_{\mathbf{i}+\mathbf{a}_0,\mathbf{i}}(0)]^2\\ \times[\phi_{\mathbf{i}_1+\mathbf{a},\mathbf{i}_1}(\tau_1)-\phi_{\mathbf{i}_1+\mathbf{a},\mathbf{i}_1}(\tau_1')]^4\Bigg\rangle_0.\end{aligned} \tag{3.9}$$

After Fourier transforming the phases in both coordinates and time we can average easily with the action S_0, Eq. (3.1). Then, Eq. (3.9) is reduced to the form

$$\begin{aligned}12T^3\sum_{\mathbf{a},\boldsymbol{\omega},\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}G_{\mathbf{q}_1,n_1}^2G_{\mathbf{q}_2,n_2}|e^{i\mathbf{q}_1\mathbf{a}_0}-1|^2\prod_{i=1,2}|e^{i\mathbf{q}_i\mathbf{a}}-1|^2\\ \times\int_0^{\beta}(e^{i\Omega_n\tau}-1)|e^{-i\omega_{n1}\tau}-1|^2\alpha(\tau)d\tau\\ \times\int_0^{\beta}|1-e^{i\omega_{n1}\tau}|^2|1-e^{i\omega_{n2}\tau}|^2\alpha(\tau)d\tau.\end{aligned} \tag{3.10}$$

Integrals over τ can easily be calculated by changing to the variables $z=\exp(2\pi iT\tau)$. The lattice integrations are trivial as well and we come to the contribution to the conductivity $\sigma^{(2)}$ coming from Eq. (3.9)

$$\sigma^{(2)}/\sigma_0=-\alpha^2\sum_{n_1>n_2>0}\frac{1}{n_1n_2}=-\frac{\alpha^2}{2}\ln^2\left(\frac{gE_c}{T}\right). \tag{3.11}$$

Adding the contributions $\sigma^{(1)}$ and $\sigma^{(2)}$ we obtain for the conductivity σ in the limit $\omega \to 0$

$$\sigma = \sigma_0 \left[1 - \alpha \ln\left(\frac{gE_c}{T} \right) \right]. \tag{3.12}$$

We see that the terms of the order $\alpha^2 \ln^2(gE_c/T)$ in $\sigma^{(1)}$ and $\sigma^{(2)}$ cancel each other and the accuracy of Eq. (3.12) exceeds $\alpha \ln(gE_c/T)$. We emphasize again that, in this approximation, there is no dependence on the structure of the lattice of the grains except that the dimensionality (one half of the coordination number) d enters α, Eq. (3.7). However, this property holds only for contributions of the type $\alpha^n \ln^n(gE_c/T)$. Terms with lower powers of logarithms depend on the structure of the lattice in a more complicated way. The cancellation of the terms of the order $\alpha^2 \ln^2(gE_c/T)$ when calculating σ, Eq. (3.12), is not accidental and we want to demonstrate this within a renormalization group (RG) scheme.

B. Renormalization group

In order to sum up the logarithmic corrections to the conductivity we use RG arguments suggested for a one-dimensional inverse square *XY* model long ago[19] and used later in a number of works.[20–24] We assume that the tunnel conductance is large, $g \gg 1$ and therefore we use the phases ϕ neglecting the winding numbers. As the starting functional we take the tunneling action S_t

$$S_t = \pi g \sum_{|\mathbf{i}-\mathbf{j}|=a} \int_0^\beta \int_0^\beta d\tau d\tau' \, \alpha(\tau - \tau') \times \sin^2\left(\frac{\phi_{\mathbf{ij}}(\tau) - \phi_{\mathbf{ij}}(\tau')}{2} \right). \tag{3.13}$$

This action contains the conductance g, which determines the conductivity σ. The charging part S_c is not important for the renormalization group because it determines only the upper cutoff of integrations over frequencies. In the limit $T \to 0$ the function α is proportional to $(\tau - \tau')^{-2}$ and the action is dimensionless.

Following standard RG arguments we want to find how the form of the action S_t changes under changing cutoffs. Generally speaking, it is not guaranteed that after integrating over the phases ϕ in an interval of frequencies one comes to the same function $\sin^2\phi$ in the action. The form of the functional may change, which would lead to a functional renormalization group. In the present case appearance of terms $\sin^2 2\phi$, $\sin^2 4\phi$, etc., is not excluded and, indeed, they are generated in many loop approximations of the RG. Fortunately, the one loop approximation is simpler and the renormalization in this order results in a change of the effective coupling constant g only.

To derive the RG equation we represent the phase ϕ in the form

$$\phi_{\mathbf{ij}\boldsymbol{\omega}} = \phi^{(0)}_{\mathbf{ij}\boldsymbol{\omega}} + \overline{\phi}_{\mathbf{ij}\boldsymbol{\omega}}. \tag{3.14}$$

The function $\phi_{\mathbf{ij}\boldsymbol{\omega}}$ is not equal to zero in an interval of the frequencies $0 < \omega < \omega_c$, while the function $\phi^{(0)}_{\mathbf{ij}\boldsymbol{\omega}}$ is finite in the interval $\lambda \omega_c < \omega < \omega_c$, where λ is in the interval $0 < \lambda < 1$. Integrating in the expression for the partition function

$$Z = \int \exp(-S_t) D\phi$$

over the function $\phi^{(0)}_{\mathbf{ij}\boldsymbol{\omega}}$ we come to a new action $\overline{S}$ with the cutoff $\lambda \omega_c$. Substituting Eq. (3.14) into Eq. (3.13) we expand the action S_t up to terms quadratic in $\phi^{(0)}_{\mathbf{ij}\boldsymbol{\omega}}$. Integrating over $\phi^{(0)}_{\mathbf{ij}\boldsymbol{\omega}}$ is straightforward and we obtain with the logarithmic accuracy a renormalized effective action $\tilde{S}_t$

$$\tilde{S}_t = 2\pi g \sum_{|\mathbf{i}-\mathbf{j}|=1} \int_0^\beta \int_0^\beta d\tau d\tau' \, \alpha(\tau - \tau') \times \sin^2\left(\frac{\phi_{\mathbf{ij}}(\tau) - \phi_{\mathbf{ij}}(\tau')}{2} \right) \left(1 - \frac{\xi}{2\pi g d} \right), \tag{3.15}$$

where $\xi = -\ln \lambda$.

We see from Eq. (3.15) that the form of the action is reproduced for any dimensionality d of the lattice of the grains. This allows us to write immediately the following renormalization group equation

$$\frac{\partial g(\xi)}{\partial \xi} = -\frac{1}{2\pi d}. \tag{3.16}$$

The solution of Eq. (3.16) is simple. Neglecting the Coulomb interaction in the action S_t, Eq. (3.13), is justified only for energies smaller than gE_c and this energy is the upper cutoff. Then the renormalized conductance $g(T)$ takes the form

$$g(T) = g - \frac{1}{2\pi d} \ln \frac{gE_c}{T} \tag{3.17}$$

and we come to Eq. (3.12) for conductivity. Both the quantities depend on the temperature logarithmically.

Equation (3.17) is obtained in the one loop approximation and should be valid so long as the effective conductance $g(T)$ remains much larger than 1. Therefore, Eq. (3.12) is also more than a result of the perturbation theory and is valid for $\sigma/\sigma_0 \gg 1/g$. This gives the necessary condition for the applicability of Eqs. (3.12), (3.17)

$$1 - (2\pi d g)^{-1} \ln(gE_c/T) \gg 1/g. \tag{3.18}$$

However, the condition (3.18) is not sufficient for the applicability of Eqs. (3.12), (3.17) because the AES action may be used only in the limit

$$T \gg g\delta. \tag{3.19}$$

If the conductance g or the size of the grains are not very large the condition (3.19) can become stronger than Eq. (3.18). Then, at lower temperatures one should take into account interference effects and, depending on the dimensionality d of the array, both metal and insulating states are possible. In contrast, Eqs. (3.12), (3.17), are valid in *any* dimensionality.

Actually, in the main approximation the action S_t contains only the phase differences $\phi_{\mathbf{ij}}(\tau)$ that can be considered as variables on sites of a lattice dual to the original one. These sites are not coupled in this approximation with each other and this explains why we obtained the same results as those for one contact. Taking into account the charging energy S_c, Eq. (2.20), couples the sites and the structure of the lattice may become important in next orders of the renormalization group equations.

One can check that the contribution coming from the correlator of the functions X_1 in Eqs. (2.23) contains additional powers of $1/g$ and can be neglected in the main approximation. It is very important that the correlator $\langle X_1 X_1\rangle$ in Eqs. (2.23) representing the "paramagnetic contribution" contains a summation over j, which corresponds to the zero quasimomentum of the function K. Keeping in X_1 linear terms in ϕ would give zero even if anharmonic terms in S would have been taken into account. If we carried out the computation for a single grain the contribution from the $\langle X_1 X_1\rangle$ would not be smaller than that coming from $\langle X_2\rangle$.

It is very important to note that we use the linear response theory for the calculation of the conductivity assuming that the external electric field is homogeneous. This contrasts calculation of the conductance for a single contact. In our calculations we do not obtain a contribution to the conductivity corresponding to the inelastic cotunneling known for single dots.[25] This is natural because in the problem considered the inelastic cotunnelling can occur only through the entire system. This would lead to contributions exponentially small in the size of the system.

IV. TUNNELING DENSITY OF STATES AT $G \gg 1$

The tunneling density of states (DOS) $\nu_{\mathbf{i}}(\varepsilon)$ in the grain **i** can be introduced in a standard way through the retarded single particle Green function $G^R_{\mathbf{ii}}(\varepsilon)$ with both the coordinates in the grain **i**. As we use here the imaginary time representation we calculate first the temperature Green function $\mathcal{G}_{\mathbf{ii}}(\tau)$ at Matsubara frequencies $\varepsilon_n = \pi T(2n+1)$. This leads to a function $\tilde{\nu}_{\mathbf{i}}(\varepsilon_n)$

$$\tilde{\nu}_{\mathbf{i}}(\varepsilon_n) = -\pi^{-1}\int d\tau e^{i\varepsilon_n\tau}\mathcal{G}_{\mathbf{ii}}(\tau). \tag{4.1}$$

The DOS $\nu_{\mathbf{i}}(\varepsilon)$ can be found by the analytical continuation

$$\nu_{\mathbf{i}}(\varepsilon) = \mathrm{Im}[\tilde{\nu}_{\mathbf{i}}(\varepsilon_n)|_{\varepsilon_n \to -i\varepsilon+\delta}]. \tag{4.2}$$

Following the same procedure as the one used in the previous sections we perform the gauge transformation (2.11), reducing the calculation to integration over the phases $\phi_{\mathbf{i}}(\tau)$. As a result, we obtain for the function $\tilde{\nu}_i(\varepsilon_n)$

$$\tilde{\nu}_{\mathbf{i}}(\varepsilon_n) = \nu_0 T\int_0^\beta d\tau \frac{e^{i\varepsilon_n\tau}}{\sin \pi T\tau}\langle \exp\{-i[\tilde{\phi}_{\mathbf{i}}(\tau) - \tilde{\phi}_{\mathbf{i}}(0)]\}\rangle, \tag{4.3}$$

where the symbol $\langle \cdots \rangle$ means as before averaging with the action S, Eqs. (2.19)–(2.21) and integration over $\tilde{\phi}_{\mathbf{i}}(\tau)$ includes summation over the winding numbers k_i. The principal value of the integral over τ is implied in Eq. (4.3).

Calculation of DOS is quite different from that for the conductivity because Eq. (4.3) contains the phases $\tilde{\phi}_{\mathbf{i}}(\tau)$ corresponding to the grain **i** but not the phase differences $\tilde{\phi}_{\mathbf{ij}}(\tau)$. This can lead to a nontrivial dependence of the final result on the dimensionality d of the lattice.

In the limit of large $g \gg 1$ we expand, as when calculating the conductivity, the functional S, Eqs. (2.19)–(2.21), in $\phi_{\mathbf{ij}}(\tau)$. If $d=3$ one should expand in $\phi(\tau)$ both the action S and the exponential in Eq. (4.3). The reason is the same as for calculation of the conductivity: all terms of the expansion give additional logarithms and there is no reason to keep in the action only quadratic terms and at the same time not to expand the exponential in Eq. (4.3). Therefore in the 3D case one can expect logarithmic corrections to the tunneling density of states with coefficients depending on the structure of the lattice. Making analytical continuation onto real energies ε we can write the tunneling density of states $\nu_3(\varepsilon)$ in 3D in the main approximation as

$$\nu_3(\varepsilon) = \nu_0 T\int_0^\beta \frac{e^{i\varepsilon_n\tau}}{\sin \pi T\tau}[1 - G(\tau)]d\tau, \tag{4.4}$$

where

$$G(\tau) = 2Ta^d \sum_{\omega_n>0} \int \frac{d^d\mathbf{q}}{(2\pi)^d} G_{\mathbf{q}n} \sin^2\frac{\omega_n\tau}{2} \tag{4.5}$$

and $G_{\mathbf{q}n}$ is given by Eq. (3.2).

Calculating the sum over ω_n with the logarithmic accuracy and making the analytical continuation we obtain

$$\nu_3(\varepsilon) = \nu_0\left[1 - \frac{A}{4\pi g}\ln\left(\frac{gE_c}{\max(T,\varepsilon)}\right)\right], \tag{4.6}$$

where

$$A = \int \frac{a^3 d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\sum_{\mathbf{a}} (1 - \cos \mathbf{q}\cdot\mathbf{a})}. \tag{4.7}$$

Equation (4.6) shows that the density of states $\nu_3(\varepsilon)$ of the 3D array of the grains has a logarithmic dependence on temperature like the conductivity σ, Eq. (3.12). However, in contrast to the latter, Eq. (4.7) is valid only if the logarithmic term is much smaller than 1.

The situation is more interesting in one and two dimensional systems. In this cases the integral over **q** in Eq. (4.7) formally diverges, which means that one should take into account the ω^2 term in the function $G_{\mathbf{q}n}$, Eq. (3.2). This term cuts the infrared divergency in the integral over **q** but one obtains in the density of states a stronger singularity in ε, T than the one in 3D, Eq. (4.6).

Fortunately, this makes the calculation even easier and allows us to obtain explicit results in the nonperturbative regime when the DOS considerably deviates from ν_0. This simplification is due to the fact that the strongest singularities come from the expansion in $\phi_{\mathbf{i}}(\tau)$ of the exponential in Eq.

(4.3). The anomalous contributions originating from small q arise in the expansion of $\phi_{\mathbf{i}}$ but cancel each other in the contributions coming from expansions in $\phi_{\mathbf{ij}}(\tau)=\phi_{\mathbf{i}}(\tau)-\phi_{\mathbf{j}}(\tau)$ in the action S_t, Eq. (2.21). Therefore, expanding in $\phi_{\mathbf{ij}}(\tau)$ one obtains usual logarithmic contributions of the type $[g\ln(gE_c/T)]^n$, which allows us to keep in the action S only quadratic in $\phi_{\mathbf{ij}}(\tau)$ terms.

Then the integration over $\phi(\tau)$ in Eq. (4.3) can easily be carried out and we obtain for the low dimensions

$$\nu(\varepsilon)=\nu_0 T\int_0^\beta \frac{e^{i\varepsilon_n\tau}}{\sin\pi T\tau}\exp[-G(\tau)]d\tau. \tag{4.8}$$

It is clear that the singularity in the exponent in Eq. (4.8) is stronger than logarithmic and this justifies the approximation used in the derivation.

In the most interesting case of a 2D array of the granules both the summation over the frequencies and integration over the momenta $\mathbf{q}$ give logarithms and we come to the final result

$$\nu_2(\varepsilon)=\nu_0\exp\left[-\frac{1}{16\pi^2 g}\ln^2\left(\frac{gE_c}{\max(\varepsilon,T)}\right)\right]. \tag{4.9}$$

Equation (4.9) is valid down to $\max(\varepsilon,T)\sim g\delta$ when the description in terms of the phase functional S, Eqs. (2.19)–(2.21), is still applicable. Equation (4.9) perfectly agrees with the corresponding result obtained long ago for disordered films using a replica σ model.[6] This result was reproduced for disordered films in a number of subsequent publications[26–29] using different approaches. The strong anomaly in the exponent in Eq. (4.9) is due to the fact that the one particle Green function is not gauge invariant.[6] The singularity is formed by almost pure gauge fluctuations of the electric fields. Gauge-invariant characteristics such as conductivity are not influenced by such fluctuations and therefore are less anomalous.

The applicability of Eq. (4.9) not only for disordered films but also for the granular systems at not very low temperatures shows that the result is very robust. In contrast, the dependence of the conductivity on temperature, Eqs. (3.12)–(3.17), cannot be used for very low temperatures or disordered "homogeneous systems." The formal reason for this difference is quite clear: the main contribution to the conductivity comes from momenta $q\sim a^{-1}$, whereas the main contribution to the density of states comes from small $q\ll a^{-1}$. The latter limit is not sensitive to the structure of the system at short distances.

The coefficient in front of $\ln^2$ is somewhat different than that of the recent work.[29] This is because we assumed that $E(\mathbf{q})$ remains finite in the limit of $q\to 0$ (there is a screening in the system). In the integral over $\mathbf{q}$ in Eqs. (4.5), (3.2), essential $(qa)^2$ were of the order of ω/gE_c. In contrast, if one starts with a nonscreened 2D Coulomb interaction $V_0=2\pi e^2/q$, the essential q are proportional to ω and this increases the coefficient by the factor of 2.

The first term of the expansion of the exponential in Eq. (4.9) is just the Altshuler-Aronov correction generalized to the case of the granular metal.[5] Of course, the same is true for the correction to the conductivity (3.12). This is quite natural because at not very low temperatures weak localization corrections are suppressed but the Altshuler-Aronov corrections still give important contributions. Actually, the function $|\omega_n|G_{\mathbf{q}}$, Eq. (3.2), is just the Coulomb propagator screened by the electron-electron interaction. In other words, the theory developed in the previous sections for $g\gg 1$ starting from the AES action is another way of calculation of the Altshuler-Aronov corrections for the granular systems. These calculations could be performed diagrammatically, although establishing the nonperturbative results for both the conductivity (3.17) and the DOS (4.9) would be considerably more difficult. Clearly, the action S described by Eqs. (2.19)–(2.21) may not be used at zero temperature and the dissipation resulting from this action is not a zero temperature effect.

If the tunnel conductance g becomes of the order of 1 or smaller diagrammatic expansions are no longer helpful and calculations without the phase functional, Eqs. (2.19)–(2.21), are hardly possible. In this regime a proper account of nonzero winding number $k_{\mathbf{i}}$ is very important. In the next section we show how calculations can be carried out in the limit of weak coupling constants $g\ll 1$.

V. WEAK COUPLING BETWEEN THE GRAINS

A. Phase correlation function

Study of the granular system described by the action S, Eqs. (2.19)–(2.23), for an arbitrary conductance g is difficult. The logarithmic behavior (3.17) describes the conductivity of the granular system at sufficiently large $g\gtrsim 1$. Equations (4.6)–(4.9) are also applicable only in this limit. Not being able to consider the model for arbitrary g, we restrict ourselves with the limit of small $g\ll 1$.

We will see that the temperature dependence of both the conductivity and the DOS becomes exponential in this limit. This means that increasing the tunneling amplitude at a fixed T we go from the almost metallic regime to an insulating one. This can be achieved experimentally changing the coupling between the grains while measuring at the same temperature.[7]

Calculation of physical quantities at small $g\ll 1$ can be performed expanding the functional integral in Eq. (2.24) in the tunneling part S_t, Eq. (2.21), of the action. When calculating conductivity the main contribution comes again from the function $\langle X_2(\tau)\rangle$ in Eq. (2.23). As concerns the tunneling density of states we can use as before Eq. (4.3). In the lowest order one can completely neglect S_t in the both formulas. Then, the calculation of the DOS (4.3) reduces to computation of the phase correlation function $\Pi(\tau)$

$$\Pi(\tau)=\langle\exp\{-i[\tilde{\phi}_{\mathbf{i}}(\tau)-\tilde{\phi}_{\mathbf{i}}(0)]\}\rangle_{S_c}, \tag{5.1}$$

where the phases $\tilde{\phi}_{\mathbf{i}}(\tau)$ are introduced in Eq. (2.15), S_c is given by Eq. (2.20) and the averaging should be performed with this functional. As concerns the conductivity, we need a slightly different correlation function $\tilde{\Pi}(\tau)$

$$\tilde{\Pi}(\tau)=\langle\exp\{-i[\tilde{\phi}_{\mathbf{ij}}(\tau)-\tilde{\phi}_{\mathbf{ij}}(0)]\}\rangle_{S_c}. \tag{5.2}$$

The phase correlation function $\Pi(\tau)$ is somewhat simpler and let us show in detail how to calculate it. A proper modification for $\tilde{\Pi}(\tau)$ is simple.

The computation of the average in Eq. (5.1) can be performed using two different methods. A more straightforward way of calculating is to use the definition of $\tilde{\phi}_i(\tau)$, Eq. (2.15), which allows us to represent the action S_c as

$$S_c=S_c[\phi]+S_c[k], \tag{5.3}$$

$$S_c[\phi]=\frac{T}{4}\sum_{n,\mathbf{i},\mathbf{j}}\phi_{\mathbf{i}n}\omega_n^2(B^{-1})_{\mathbf{ij}}\phi_{\mathbf{j}-n}, \tag{5.4}$$

$$S_c[k]=T\pi^2\sum_{\mathbf{ij}}k_{\mathbf{i}}(B^{-1})_{\mathbf{ij}}k_{\mathbf{j}}, \tag{5.5}$$

where

$$B_{\mathbf{ij}}=\frac{e^2}{2}(C^{-1})_{\mathbf{ij}}.$$

Writing Eqs. (5.4), (5.5) we neglected integration over the variables $\rho_{\mathbf{i}}$ from Eq. (2.14). As we have discussed in Sec. II, in the limit $T\gg\delta$ the main contribution comes from $\rho\sim(T\delta)^{1/2}$ and we can simply put in all expressions $\rho=0$. Using Eqs. (5.3)–(5.5) one can carry out integration over the phase ϕ and summation over the winding numbers separately. The phase correlation function $\Pi(\tau)$ can be written as

$$\Pi(\tau)=\langle\exp\{-i[\phi_{\mathbf{i}}(\tau)-\phi_{\mathbf{i}}(0)]\}\rangle_{\phi}\langle\exp(-2\pi i k_{\mathbf{i}}\tau T)\rangle_k. \tag{5.6}$$

Integrating over the phase $\phi_{\mathbf{i}}(\tau)$ we obtain for $0<\tau<\beta$.

$$\langle\exp\{-i[\phi_{\mathbf{i}}(\tau)-\phi_{\mathbf{i}}(0)]\}\rangle_{\phi}=\exp[-B_{\mathbf{ii}}(\tau-T\tau^2)]. \tag{5.7}$$

Equation (5.7) was used for the function $\Pi(\tau)$ in many previous works.[16,15] However, Eq. (5.7) is not the final result because the function $\Pi(\tau)$, Eq. (5.6), contains the second average and, in addition to integrating over $\phi_{\mathbf{i}}(\tau)$, one must sum over the winding numbers $k_{\mathbf{i}}$.

Calculation of the second average in Eq. (5.6) can be performed using the Poisson formula

$$\sum_{k=-\infty}^{\infty}f(2\pi k)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{imx}f(x)dx$$

for any function $f(x)$. As a result, we rewrite the second average in Eq. (5.6) as

$$\langle\exp(-2\pi i k_{\mathbf{i}}\tau T)\rangle_k=Z_x^{-1}\sum_{\{m_{\mathbf{i}}\}}\int\exp\left(-\frac{T}{4}\sum_{\mathbf{jl}}x_{\mathbf{j}}(B^{-1})_{\mathbf{jl}}x_{\mathbf{l}}+i\sum_{\mathbf{j}}x_{\mathbf{j}}(m_{\mathbf{j}}-\tau T\delta_{\mathbf{ij}})\right)\mathcal{D}x, \tag{5.8}$$

where Z_x is the normalization factor that can be obtained putting $\tau=0$ in the integral in the second line of Eq. (5.8) and $\mathcal{D}x=\Pi_{\mathbf{i}}dx_{\mathbf{i}}$. Summation over all integers $m_{\mathbf{i}}$ for all grains is implied. The integration over $x_{\mathbf{i}}$ should be performed in the infinite limits and the integral in Eq. (5.8) can be easily calculated. Substituting the result of the integration into Eq. (5.6) and using Eq. (5.7) we find for the phase correlation function $\Pi(\tau)$

$$\Pi(\tau)=Z^{-1}\exp(-B_{\mathbf{ii}}\tau)\times\sum_{\{m_{\mathbf{j}}\}}\exp\left(-\sum_{\mathbf{k}}2\tau m_{\mathbf{k}}B_{\mathbf{ki}}-\beta\sum_{\mathbf{k,l}}B_{\mathbf{kl}}m_{\mathbf{k}}m_{\mathbf{l}}\right), \tag{5.9}$$

where all $m_{\mathbf{j}}$ are integers and Z is the normalization coefficient $[\Pi(0)=1]$. The quantity $B_{\mathbf{ii}}$ in Eq. (5.9) is the charging energy of an extra electron in a grain **i** in an otherwise neutral system without excitations. The necessary periodicity in τ of the function $\Pi(\tau)$ with the period β is evident from Eq. (5.9). The final result for the phase correlation function $\Pi(\tau)$, Eq. (5.9), is essentially different from Eq. (5.7) obtained by neglecting the contribution of nonzero winding numbers. The exponent contains only linear in τ terms, which contrasts Eq. (5.7).

Actually, the form of Eq. (5.9) is absolutely natural and can be obtained using the standard quantum mechanical formalism instead of calculating the functional integrals in Eq. (5.1). Within this formalism velocities in the action in the functional integral should be replaced by the corresponding momentum operators in the Hamiltonian. So, instead of having the derivatives $\partial\phi/\partial\tau$ in the action S_c, Eq. (2.20), one would have to write the operators $\partial/\partial\phi$. The presence of winding numbers in the action introduces periodicity. As a result, the corresponding angle variables ϕ in the Hamiltonian formulation should be taken in the interval $[0,2\pi]$ and all wave functions of the Hamiltonian must be periodic in ϕ with the period 2π.

As the result, the calculation of the phase correlation function $\Pi(\tau)$, Eq. (5.1), reduces to calculation of quantum mechanical averages with the effective Hamiltonian $\hat{H}_{\mathrm{eff}}$

$$\hat{H}_{\mathrm{eff}}=\sum_{\mathbf{ij}}B_{\mathbf{ij}}\hat{\rho}_{\mathbf{i}}\hat{\rho}_{\mathbf{j}},\quad \hat{\rho}_{\mathbf{i}}=-i\partial/\partial\phi_{\mathbf{i}}, \tag{5.10}$$

where the "angles" $\phi_{\mathbf{i}}$ vary between 0 and 2π. The correlation function $\Pi(\tau)$ can be written in the form

$$\Pi(\tau)=\langle e^{-i(\hat{\phi}_{\mathbf{i}}(\tau)-\hat{\phi}_{\mathbf{i}}(0))}\rangle_{\hat{H}_{\mathrm{eff}}},\quad \hat{\phi}_{\mathbf{i}}(\tau)=e^{\hat{H}_{\mathrm{eff}}\tau}\phi_{\mathbf{i}}e^{-\hat{H}_{\mathrm{eff}}\tau}. \tag{5.11}$$

In order to calculate the average with $\hat{H}_{\mathrm{eff}}$ in Eq. (5.11) one should find eigenfunctions $\Psi_{\{n\}}$ of the Hamiltonian $\hat{H}_{\mathrm{eff}}$, Eq. (5.10). These eigenfunctions have the simple form

$$\Psi_{\{n\}}=\prod_{\mathbf{i}}\exp(in_{\mathbf{i}}\phi_{\mathbf{i}}), \tag{5.12}$$

where $n_{\mathbf{i}}$ are integers. Calculating the matrix elements entering Eq. (5.11) and performing summation over all states with the weight $\exp(-E_{\{n\}}/T)$, where $E_{\{n\}}$ are eigenenergies of the Hamiltonian $\hat{H}_{\mathrm{eff}}$, Eq. (5.10), we come to Eq. (5.9).

The operators $\hat{\rho}_{\mathbf{i}}$ and $\phi_{\mathbf{i}}$ are conjugate to each other with the commutator

$$[\hat{\rho}_{\mathbf{i}},\phi_{\mathbf{i}}]=-i.$$

Therefore we can alternatively write the operators $\hat{\phi}_{\mathbf{i}}$ as $\hat{\phi}_{\mathbf{i}}=\partial/\partial\rho_{\mathbf{i}}$. The eigenvalues of the operator $\rho_{\mathbf{i}}$ are integers. The operator $\exp(\pm i\hat{\phi})$ acts as

$$\exp(\pm i\hat{\phi})f(\rho)=\exp(\mp\partial/\partial\rho)f(\rho)=f(\rho\pm 1). \tag{5.13}$$

This gives another convenient way of calculation of the quantum mechanical averages.

The use of the Hamiltonian formalism for calculation of functional integrals over the phase ϕ of the superconducting order parameter has been suggested in an earlier work on granulated superconductors.[30] Within this approach the effective Hamiltonian $\hat{H}_{\mathrm{eff}}$, Eq. (5.10), was derived for the operator ρ of the number of Cooper pairs and the phase correlation function $\Pi(\tau)$, Eq. (5.9), has been obtained. As concerns the normal metals the correct form of the phase correlation function $\Pi(\tau)$, Eq. (5.9), has been written for the first time in our previous paper.[12]

The present consideration demonstrates explicitly that accounting for the winding numbers leads to the charge quantization. The function $\tilde{\Pi}(\tau)$, Eq. (5.2), can be calculated in the same way and can be written as

$$\tilde{\Pi}(\tau)=Z^{-1}\exp(-B_{\mathbf{i}}^{\mathbf{a}}\tau)\times\sum_{\{m_{\mathbf{k}}\}}\exp\left(-\sum_{\mathbf{k}}2\tau m_{\mathbf{k}}(B_{\mathbf{k},\mathbf{i}+\mathbf{a}}-B_{\mathbf{ki}})-\beta\sum_{\mathbf{j},\mathbf{l}}B_{\mathbf{jl}}m_{\mathbf{j}}m_{\mathbf{l}}\right), \tag{5.14}$$

where $B_{\mathbf{i}}^{\mathbf{a}}=B_{\mathbf{ii}}+B_{\mathbf{i}+\mathbf{a},\mathbf{i}+\mathbf{a}}-B_{\mathbf{i},\mathbf{i}+\mathbf{a}}-B_{\mathbf{i}+\mathbf{a},\mathbf{i}}$ is the energy of an electron-hole excitation between neighboring grains $\mathbf{a}=\mathbf{j}-\mathbf{i}$. As the array of the grains is regular we omit the subscripts $\mathbf{i},\mathbf{j}$ when writing the correlation functions $\Pi(\tau)$ and $\tilde{\Pi}(\tau)$. The functions $\Pi(\tau)$ and $\tilde{\Pi}(\tau)$ allow us to calculate the tunneling density of states and the conductivity using Eqs. (4.3) and (2.23).

B. Tunneling density of states at $g\ll 1$

Substituting Eq. (5.9) into Eq. (4.3) we have to calculate the remaining integral over τ and perform the analytical continuation $\varepsilon_n\to -i\varepsilon+\delta$. It is convenient to shift the contour of the integration and integrate first along the line $(0,i\infty)$, then along $(i\infty,i\infty+\beta)$, and finally along $(i\infty+\beta,\beta)$. The integral over the second segment vanishes and we reduce the function $\tilde{\nu}_{\mathbf{i}}(\varepsilon_n)$, Eq. (4.3), to the form

$$\tilde{\nu}_{\mathbf{i}}(\varepsilon)=i\nu_0\left[1-2T\int_0^\infty dt\frac{\exp(-\varepsilon_n t)}{\sinh(\pi Tt)}\sin(tB_{\mathbf{ii}})\times Z^{-1}\sum_{\{m_{\mathbf{i}}\}}\exp\left(-\beta\sum_{\mathbf{kl}}m_{\mathbf{k}}B_{\mathbf{kl}}m_{\mathbf{l}}-2it\sum_{\mathbf{k}}m_{\mathbf{k}}B_{\mathbf{ki}}\right)\right]. \tag{5.15}$$

Now the analytical continuation can be done easily. Taking the imaginary part of the function $\tilde{\nu}$ according to Eq. (4.2) and using the integral

$$\int_0^\infty\frac{\sin au}{\sinh\pi u}du=\frac{1}{2}\tanh\frac{a}{2}$$

we obtain for the density of states $\nu(\varepsilon)$

$$\frac{\nu(\varepsilon)}{\nu_0}=Z^{-1}\sum_{\{m_{\mathbf{k}}\}}\exp\left(-\frac{\sum_{\mathbf{kl}}m_{\mathbf{k}}B_{\mathbf{kl}}m_{\mathbf{l}}}{T}\right)\times\left[n\left(\frac{\varepsilon+B_{\mathbf{ii}}-2\sum_{\mathbf{k}}m_{\mathbf{k}}B_{\mathbf{ki}}}{T}\right)+n\left(\frac{-\varepsilon+B_{\mathbf{ii}}+2\sum_{\mathbf{k}}m_{\mathbf{k}}B_{\mathbf{ki}}}{T}\right)\right], \tag{5.16}$$

where

$$Z=\sum_{\{m_{\mathbf{k}}\}}\exp\left(-\frac{\sum_{\mathbf{kl}}m_{\mathbf{k}}B_{\mathbf{kl}}m_{\mathbf{l}}}{T}\right) \tag{5.17}$$

and

$$n(x)=\frac{1}{e^x+1}$$

is the Fermi distribution function. In Eqs. (5.16), (5.17) summation over all positive and negative integer $m_{\mathbf{k}}$ should be performed.

The function $\nu(\varepsilon)$ is even in the energy ε and approaches 1 in the limits $T\to\infty$ or $|\varepsilon|\to\infty$. At low temperatures $T\ll B_{\mathbf{ii}},|\varepsilon|$ and $|\varepsilon|<B_{\mathbf{ii}}$ the main contribution comes from the ground state configuration when all $m_{\mathbf{i}}=0$. In this limit we obtain

$$\frac{\nu(\varepsilon)}{\nu_0}=2\exp\left(-\frac{B}{T}\right)\cosh\frac{\varepsilon}{T}, \tag{5.18}$$

where $B=B_{\mathbf{ii}}$. Equation (5.18) demonstrates that there is a gap in the density of states and this gap is equal to the single electron charging energy of the grain. Of course, Eq. (5.16) corresponds to a fixed chemical potential in the grain, which means that the grain is not completely isolated and there are processes that keep the chemical potential fixed.

One can easily generalize Eq. (5.16) to the case when an additional voltage is present in each grain. Writing the effective chemical potential $\mu_{\mathbf{i}}$ of a grain $\mathbf{i}$ as

$$\mu_{\mathbf{i}}=\mu+2\sum_{\mathbf{j}} N_{\mathbf{j}}B_{\mathbf{ij}}$$

we can generalize Eq. (5.16) by replacing everywhere $m_{\mathbf{k}} \rightarrow m_{\mathbf{k}}+N_{\mathbf{k}}$. If $N_{\mathbf{k}}$ is randomly distributed over the grains one should carry out an additional averaging over this variable. This is beyond the scope of the present paper.

To the best of our knowledge, Eq. (5.16) has not been written before, although it clearly follows from the "orthodox theory" of the Coulomb blockade.[31] The latter can be seen from the fact that the one particle part $\hat{H}_0$ of the Hamiltonian, Eq. (2.1) commutes with the term $\hat{H}_c$, Eq. (2.3), describing the charging energy. This means that in the absence of tunneling between the grains the charging energy should be simply added to the Fermi energy for noninteracting particles. Equation (5.16) clearly corresponds to this picture. For an arbitrary configuration $\{m_{\mathbf{k}}\}$ of charges on the grains the energy of adding one electron (hole) to a grain is equal to

$$B\pm 2\sum_{\mathbf{k}} m_{\mathbf{k}}B_{\mathbf{ki}}.$$

We see that just this energy enters the Fermi distribution functions in Eq. (5.16) shifting the Fermi energy. The exponential in Eq. (5.16) is the weight for the configuration of the charges in the system and two Fermi functions describe contributions of electrons and holes. Although the direct derivation "in the electron language" would be simpler, the present calculation demonstrates explicitly how the phase functional S_c, Eq. (2.20), works. We emphasize that without the summation over the winding numbers the correct result could not be (and has not been) obtained.

C. Conductivity at $g \ll 1$

Calculation for the conductivity can be performed in the same way as for the tunneling density of states. It is important that in the limit $g \ll 1$ the main contribution comes from the term X_2 in Eq. (2.23). The contribution coming from the term X_1 is of higher order in g and we neglect it. Then, the Fourier transformed response function $K(\Omega_n)$, Eq. (2.23), takes the form

$$K(\Omega_n)=2\pi e^2 gT^2\int_0^{\beta} d\tau \frac{1-\exp(i\Omega_n\tau)}{\sin^2\pi T\tau}\tilde{\Pi}(\tau), \quad (5.19)$$

where, again, the principal value of the integral should be taken.

Shifting the contour of integration over τ in the same way as it has been done when calculating the tunnelling density of states we can perform the analytical continuation $\Omega_n -i\omega+\delta$. In the limit of small frequencies ω we expand the integral in ω. The first nonvanishing term is linear in ω and we reduce the formulas for the conductivity σ using Eqs. (2.23), (2.25) to the form

$$\sigma=\sigma_0\left(1-2\pi T\int_0^{\infty} dt \frac{Tt}{\sinh^2(\pi Tt)}\sin(tB_{\mathbf{i}}^{\mathbf{a}})\mathcal{C}_{\mathbf{i}}^{\mathbf{a}}(t)\right), \quad (5.20)$$

where

$$\mathcal{C}_{\mathbf{i}}^{\mathbf{a}}(t)=Z^{-1}\sum_{\{m_{\mathbf{k}}\}} \exp\left(-\beta\sum_{\mathbf{kl}} m_{\mathbf{k}}B_{\mathbf{kl}}m_{\mathbf{l}}\right) \times\cos\left(2t\sum_{\mathbf{k}} m_{\mathbf{k}}(B_{\mathbf{k,i+a}}-B_{\mathbf{ki}})\right), \quad (5.21)$$

Z is given by Eq. (5.17) and σ_0 is the classical conductivity, Eq. (2.25). Calculation of the integral over t in Eq. (5.20) can easily be performed using the formula

$$\int_0^{\infty} du \frac{u\sin(au)}{\sinh^2(\pi u)}=\frac{1}{2\pi}[1-f(a)], \quad (5.22)$$

$$f(a)=2\frac{(a-1)e^{-a}+e^{-2a}}{(1-e^{-a})^2},$$

and we obtain finally

$$\sigma=\sigma_0 Z^{-1}\sum_{\{m_{\mathbf{k}}\}} \exp\left(-\beta\sum_{\mathbf{kl}} m_{\mathbf{k}}B_{\mathbf{kl}}m_{\mathbf{l}}\right) \times f\left[\beta\left(B_{\mathbf{i}}^{\mathbf{a}}+2\sum_{\mathbf{k}} m_{\mathbf{k}}(B_{\mathbf{k,i+a}}-B_{\mathbf{ki}})\right)\right]. \quad (5.23)$$

We see from Eq. (5.23) that, in order to get an explicit expression for the conductivity, one should sum again over all charge configurations. [In Eq. (5.23) the conductivity is calculated between the grains $\mathbf{i}$ and $\mathbf{i}+\mathbf{a}$].

The limit of high temperatures T exceeding the charging energies B can be obtained using the property $f(0)\rightarrow 1$ when $a\rightarrow 0$. In this limit we come to $\sigma=\sigma_0$, which demonstrates that as soon as the Coulomb energy is not important the transport is described by the Drude formula.

In the opposite limit, $T\ll B$, the main contribution in Eq. (5.23) comes from charge configurations with the lowest charging energies. For the calculation of the conductivity we also need the asymptotics of the function $f(a)$ in the limit $a\rightarrow\infty$. In this limit we may write this function as

$$f(a)\simeq 2a\exp(-a). \quad (5.24)$$

As when calculating the tunneling density of states, we consider first the contribution of the ground state configuration with all $m_{\mathbf{i}}=0$. However, in contrast to the tunneling density of states, this configuration gives not necessarily the main contribution because it contains a large energy $B_{\mathbf{i}}^{\mathbf{a}}$ of a dipole consisting of an additional electron in the grain $\mathbf{i}$ and a hole in the grain $\mathbf{i}+\mathbf{a}$ (or vice versa). The energy of this dipole is equal to $2B$ if we neglect nondiagonal $B_{\mathbf{ij}}$. If nondiagonal components $B_{\mathbf{ij}}$ are not equal to zero the dipole energy is smaller but we assume that it is larger than the charging energy of one electron B. Physically, this contribution corresponds to transport in a completely neutral grains.

In order to contribute to the current an electron must jump from one grain to another. However, this costs an energy which is just the dipole energy $B_{\mathbf{i}}^{\mathbf{a}}$. So, the contribution of the ground state configuration can be estimated as $\exp(-B_{\mathbf{i}}^{\mathbf{a}}/T)$ and we want to show that a larger contribution exists.

The most efficient process contributing to the current is when an additional charge exists in a grain but all other grains are neutral. Then, jumping from grain to grain costs no energy. As such a configuration is not the ground state, the probability to have this state is proportional to $\exp(-B/T)$. However, the overall contribution to the conductivity is in this case larger because $B<B_{\mathbf{i}}^{\mathbf{a}}$.

This picture clearly follows from Eq. (5.23). Two configurations give the main contribution to σ in Eq. (5.23). We can put $m_{\mathbf{k}}=1$ at $\mathbf{k}=\mathbf{i}$ and $m_{\mathbf{k}}=0$ for all $\mathbf{k}\neq\mathbf{i}$ or $m_{\mathbf{k}}=-1$ for $\mathbf{k}=\mathbf{i}+\mathbf{a}$ and $m_{\mathbf{k}}=0$ for all $\mathbf{k}\neq\mathbf{i}+\mathbf{a}$. In both the cases the argument of the function f in Eq. (5.23) is equal to zero and we obtain for the conductivity at low temperatures

$$\sigma=2\sigma_0\exp(-B/T). \tag{5.25}$$

Equation (5.25) shows that in the limit of small couplings between the grains g the macroscopic conductivity is exponentially small in temperature. This is a typical example of an activation process.

Of course, the exponential behavior of the physical quantities, Eqs. (5.18), (5.25), was derived under the assumption that all the grains are mesoscopically equal (they have the same size and shape but may have small irregularities different for different grains). In real samples the shape and the size may vary and qualitative estimates show that instead of the activation law a dependence of the type $\exp(-A/\sqrt{T})$ can be more proper for this case.[32]

VI. DISCUSSION

We studied effects of the Coulomb interaction on the conductivity σ and tunneling density of states $\nu(\varepsilon)$ of granular metals. Calculations with the Hamiltonian for interacting electrons were reduced to calculation of functional integrals with a phase action of a form proposed by Ambegaokar, Eckern, and Schön.[10] This action has been derived recently for the granular systems microscopically,[9] which allowed us to clarify conditions for its applicability. These conditions, Eqs. (1.3), (1.4), correspond to the limit of not very low temperatures, such that weak localization effects are suppressed. In the limit of large tunneling conductances g the results obtained with AES functional correspond to Altshuler-Aronov corrections[5] and could be calculated diagrammatically (although this way of calculations would be more complicated). At smaller g nonzero winding numbers become very important and we developed a proper scheme of calculations.

Although the interference effects leading to localization corrections are neglected at such temperatures, interesting effects occur. In the limit of large g a logarithmic dependence of the conductivity σ on temperature, Eq. (3.12), is obtained. Eqs. (3.12), (3.17) are applicable in any dimension and at any magnetic field, which distinguishes it from the weak localization correction. The logarithm in Eqs. (3.12), (3.17) is not just a small correction and these formulas are applicable until the conductance becomes of order unity. (It is important that the conductivity and not the resistivity is linear in the logarithm of the temperature.)

In contrast to the conductivity σ, the tunneling density of states $\nu(\varepsilon)$ is dependent on the dimensionality d of the system in a nontrivial way in the limit $g\gg1$. In 3D it has a logarithmic correction, Eq. (4.6), but is described by a more complicated formula (4.9) in 2D. This formula is well known for disordered films[6,29] where it was obtained within the σ-model approach. We see that the same formula is valid for the granular metal at not very low temperatures and it can be obtained without σ models.

In the limit of a low coupling between the grains both the tunneling density of states and the conductivity are exponentially small, Eqs. (5.18), (5.25). This is due to a finite charging energy arising when the electron tunnels from one to another grain. Although the results in the limit of vanishing g are rather simple, it was important to derive them from the AES phase functional with a proper summation over the winding numbers. This way of calculations is very close to the one suggested previously for granular superconductors.[30] In some experiments[7] the dependence of the conductivity on temperature is described not by the activation type formulas but by the function $\exp(-a/\sqrt{T})$. This dependence may originate from fluctuations of the charging energy of the grains.[32]

Comparing Eqs. (3.12), (4.6), (4.9) obtained in the limit $g\gg1$ with Eqs. (5.18), (5.25) derived for $g\ll1$ we conclude that there must be a considerable change of the temperature behavior of the physical quantities when changing the coupling between the grains. Of course, this cannot be a sharp transition because we consider the limit of finite temperatures and cannot extrapolate the results to $T=0$. However, tuning the coupling experimentally at a given temperature one may see the change of the regimes that would look similar to a ‘‘metal-insulator’’ transition. We emphasize that this ‘‘transition’’ should occur in any dimension of the array of the grains.

It is not clear from the present consideration whether there should be a sharp transition from the metallic to the insulator state at a critical value g_c and this is a definitely interesting problem for a further investigation. This problem is closely related to the question whether the activation energy (Coulomb gap) turns to zero or has a jump at $g=g_c$.

The model of the granular metal may be used to describe disordered electron systems at low electron density. In such systems electrons can spend a considerable time in traps or ‘‘puddles’’ that can be due to strong fluctuations of a disorder potential. Considering such systems with the model of a granular metal may be a reasonable approximation. In this case, potential wells where electrons are trapped would correspond to the grains in the model of the granular metal.

The model considered in the present paper and the results obtained can be relevant to many experiments on different materials. Of course, specially prepared granular metals such as those considered in Refs. 7,32,33 should be the first object of the application of the theory developed. As the model and the results obtained are quite robust, one can expect that the corresponding phenomena have been observed in the granular materials.

This is really so and many experimental data can be explained in the framework of our model. First, let us make a comparison of our theory with experimental results of Ref. 7 on films made of Al grains embedded in an amorphous Ge matrix. At low temperature superconductivity of in Al grains was destroyed by a strong magnetic field. Depending on the coupling between the grains (extracted from the conductivity at room temperatures) the samples of the experiment[7] were macroscopically either in an insulating state with the temperature dependence of the resistivity $R\sim\exp(a/T^{1/2})$ or in a ‘‘metallic’’ one. However, the resistivity of the metallic state depended on temperature and the authors suggested Eq. (1.1) to describe this dependence. As the exponent α for the ‘‘metallic’’ sample was small we may argue that Eq. (1.2) should not be worse for fitting the experimental data. Then, we can estimate the exponent α without using fitting parameters.

The sample of the experiment[7] had the room temperature resistivity $R_0=7.3\times10^{-3}$ Ω cm. The diameter of the grains was 120 ± 20 Å, which allows, using the value $\hbar/e^2=4.1\times10^3\Omega$, to estimate the dimensionless tunnel conductivity as $g=0.7$. If we put $d=2$ in Eq. (3.7) we obtain $\alpha=0.116$, which would perfectly agree with the experimental value $\alpha=0.117$ from Eq. (1.1). However, everything is not so simple because the films used in Ref. 7 were rather thick and, at first glance, one should use $d=3$. This would change the result by 30% making the agreement less exciting. Nevertheless, the value of d in Eq. (3.7) corresponds rather to the half of the contacts of a single grain than to the real dimensionality. Therefore the experimental value of α indicates that either the grains are not closely packed such that the typical number of contacts per grain is 4 or our calculation is too rough to provide a quantitative agreement with the experiment [the value of α, Eq. (3.7), is based on the assumption $g\gg1$ but the experimental value of g is of order 1].

The resistivity of samples with a high room temperature resistivity (a weak coupling between the grains) behaved as $\exp(a/T^{1/2})$ rather than obeying the activation law, Eq. (5.25). According to Ref. 32 this can be attributed to a variation of the size of the grains or of the local potential. However, a model considered in Ref. 32 is rather special because the distance between the grains and the tunneling amplitude was assumed to be related to the charging energy in a certain way. The law $\exp(a/T^{1/2})$ is rather common for granular materials with weak coupling (see also, e.g., Ref. 33) and the reason for such an universality is still not clear. A random hopping conduction mechanism of Ref. 34 suggested for semiconductors can hardly be used for the granular metals.

A logarithmic dependence of the resistivity on temperature has been observed in other granular materials. In Ref. 33 a granular cermet consisting of NbN grains in a boron nitride insulating matrix was studied. Again, at small coupling between the grains the temperature dependence of the resistivity $\exp(a/T^{1/2})$ was observed in a very broad interval of temperatures. The resistivity of samples with a strong coupling between the grains was very well described by the law

$$R=R_0\ln(T_0/T) \tag{6.1}$$

which is close to Eqs. (1.1), (1.2) if the temperature interval is not very large such that the variation of the resistivity is small. However, the law (6.1), gave a good description for the temperature dependence of the resistivity in a very broad region and the changing of the resistivity was not small. The reason for the applicability of Eq. (6.1) in a so broad interval of temperatures is not clear because according to the results of the renormalization group analysis of Sec. III not the resistivity but the conductivity should obey Eq. (6.1). A more careful experimental study might clarify this question. Anyway, the logarithmic behavior of Refs. 7,33 remained unexplained at all and our work is the first attempt to construct a theory of this effect (an explanation in terms of weak localization corrections or the Kondo effect can be excluded immediately because the logarithmic temperature dependence was observed also in very strong magnetic fields and the systems were three dimensional).

The unusual logarithmic behavior of the type (6.1) has been observed not only in ‘‘standard’’ granular systems but also in high-T_c cuprates at very strong magnetic fields. The first observation of this dependence was done on underdoped $La_{2-x}Sr_xCuO_4$ crystals.[35] The superconductivity in this experiment was suppressed with pulsed magnetic fields of 61 T. It was found that both the in-plane resistivity ρ_{ab} and out-of-plane resistivity ρ_c diverged logarithmically with decreasing the temperature. This means again that a 3D effect was observed in a very strong magnetic field and traditional explanations such as localization or Kondo effect could not clarify the situation.

In a subsequent publication[36] a metal-insulator crossover was observed in the same material at a Sr concentration near optimum doping ($x\simeq0.16$). In underdoped samples both ρ_{ab} and ρ_c showed no evidence of saturation at low temperatures and diverged as the logarithm of the temperature. The authors called this state ‘‘insulator’’ in contrast to the state at high doping where the resistivity did not have a pronounced dependence on the temperature. It was conjectured in Ref. 36 that the logarithmic behavior they observed might be related to the one seen in the experiment[33] on granular NbN. This would demand a phase segregation throughout the underdoped regime of LSCO. However, as no explanation had been given for the logarithmic behavior in Ref. 33, no explanation has been given to the experiments[35,36] either. We hope that our results for the model of the granular materials may be applicable to the experiments on the $La_{2-x}Sr_xCuO_4$ crystals,[35,36] which would mean that the underdoped crystals have a granular structure and the logarithmic behavior is due to the Coulomb interaction. The transition to the metallic state of Refs. 35,36 would mean that at higher doping the granularity disappears.

The logarithmic dependence of the resistivity on temperature has also been observed in many other experiments. For example, in Ref. 37 this dependence was observed in granular Pb films. It was also observed in phase compounds of

$Nd_{2-x}Ce_xCuO_{4-y}$, Ref. 38. In each case the reason for such a behavior was not clear.

An interesting conclusion has been made recently about the structure of a 2D gas in GaAs/AlGaAs heterostructures where a metal-insulator transition was observed.[39] Measuring the local electronic compressibility the authors found thatthe metallic phase was homogeneous in space, which is natural. In contrast, the system becomes spatially inhomogeneous as it crosses into the insulating phase. The structure seen in the insulating state indicates that the electrons were located in "puddles." So, modeling the system in terms of a granular metal might be a reasonable approximation also in this case.

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